On the spectral function of the Poisson-Voronoi cells. [*]

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Abstract

Denote by $\varphi(t) = \sum_{n \geq 1} e^{-\lambda_n t}$, $t > 0$, the spectral function related to the Dirichlet Laplacian for the typical cell $\mathcal{C}$ of a standard Poisson-Voronoi tessellation in $\mathbb{R}^d$, $d \geq 2$. We show that the expectation $\mathbf{E}\varphi(t)$, $t > 0$, is a functional of the convex hull of a standard d-dimensional Brownian bridge. This enables us to study the asymptotic behaviour of $\mathbf{E}\varphi(t)$, when $t \to 0^+, +\infty$. In particular, we prove that the law of the first eigenvalue λ_1 of $\mathcal{C}$ satisfies the asymptotic relation $\ln \mathbf{P}\{\lambda_1 \leq t\} \sim -2^d \omega_d j_{(d-2)/2}^d \cdot t^{-d/2}$ when $t \to 0^+$, where ω_d and $j_{(d-2)/2}$ are respectively the Lebesgue measure of the unit ball in $\mathbb{R}^d$ and the first zero of the Bessel function $J_{(d-2)/2}$.

Introduction and statement of the main results.

Consider $\Phi = \{x_n; n \geq 1\}$ a homogeneous Poisson point process in $\mathbb{R}^d$, $d \geq 2$, with the d-dimensional Lebesgue measure V_d for intensity measure. The set of cells

$$C(x) = \{y \in \mathbb{R}^d; ||y - x|| \leq ||y - x'||, x' \in \Phi\}, \quad x \in \Phi,$$

(which are almost surely bounded polyhedra) is the well-known *Poisson-Voronoi tessellation* of $\mathbb{R}^d$. Introduced by J. L. Meijering [18] and E. N. Gilbert [9] as a model of crystal aggregates, it provides now models for many natural phenomena as thermal conductivity [17], telecommunications [2], astrophysics [29] and ecology [23]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [26] and Okabe et al. [22].

In order to describe the statistical properties of the tessellation, the notion of *typical cell* $\mathcal{C}$ in the Palm sense is commonly used [20]. Consider the space $\mathcal{K}$ of convex compact sets of $\mathbb{R}^d$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set $B \subset \mathbb{R}^d$ such that $0 < V_d(B) < +\infty$. The typical cell $\mathcal{C}$ is defined by means of the identity [20]:

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_d(B)} \mathbf{E} \sum_{x \in B \cap \Phi} h(C(x) - x),$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the space of bounded measurable functions.

Consider now the cell

$$C(0) = \{y \in \mathbb{R}^d; ||y|| \leq ||y - x||, x \in \Phi\}$$

obtained when the origin is added to the point process Φ. It is well known [20] that $C(0)$ and $\mathcal{C}$ are equal in law. On the other hand, the typical cell can also be characterized by means of the empirical distributions. Indeed, let $\mathcal{V}_{d,R}$ be the set of all cells $C(x)$, $x \in \Phi$, included in the ball $B(R)$ centered at the origin and of radius $R > 0$. Let us define $N_R = \#\mathcal{V}_{d,R}$ and fix $h : \mathcal{K} \longrightarrow \mathbb{R}$ an arbitrary bounded measurable function. Then (see [12]):

$$\mathbf{E}h(\mathcal{C}) = \lim_{R \to +\infty} \frac{1}{N_R} \sum_{C(x) \in \mathcal{V}_{d,R}} h(C(x) - x), \quad \text{a.s..}$$

Explicit formulas for the distributions of the main geometric characteristics of the typical cell of the Poisson-Voronoi tessellation have been recently obtained [3], [4]. Nevertheless their expressions are intricate and in general, we do not have any precise idea about the asymptotic behaviour of these distributions.

In this paper, we are interested in the properties of the *fundamental frequencies* of the typical cell $\mathcal{C}$. More precisely let us consider $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_n \leq ...$ the (random) eigenvalues of the Laplacian under Dirichlet boundary conditions on $\mathcal{C}$ and denote by

$$\varphi(t) = \sum_{n \geq 1} e^{-\lambda_n t}, \quad t > 0,$$

the associated *spectral function*. Let us remark that the spectral function of deterministic sets has been largely studied in connection with their geometrical structure [8], [15], [30], [31].

Besides, let

$$W(t) \in \mathbb{R}^d, \quad t \in [0, 1], W(0) \equiv W(1) \equiv 0,$$

be a d-dimensional Brownian bridge on the interval $[0, 1]$ independent of the point process Φ. Let us denote by

$$\mathbf{W} = \{W(t); 0 \leq t \leq 1\} \subset \mathbb{R}^d \tag{1}$$

the associated path and by $\widehat{\mathbf{W}}$ its closed convex hull.

For a set $D \subset \mathbb{R}^d$, we define

$$B(D; x) = \cup_{y \in D} B(y, ||y - x||), \quad \widetilde{B}(D; x) = \cup_{y \in D} \widetilde{B}(y, ||y - x||),$$

where $B(y, r)$ (resp. $\widetilde{B}(y, R) = \{x \in \mathbb{R}^d; ||x - y|| < R\}$) is the closed (resp. open) ball centered at y and of radius $R > 0$. $V_d(D, x)$ will denote the d-dimensional Lebesgue measure of the set $\widetilde{B}(D; x)$, i.e.

$$V_d(D, x) = V_d(\widetilde{B}(D; x)).$$

The notation $\mathbf{P}$ (resp. $\overline{\mathbf{P}}$) is used for the probability associated to the point process Φ (resp. to the Brownian bridge W). Likewise the expectations $\mathbf{E}$ and $\overline{\mathbf{E}}$ refer respectively to Φ and W.

We show that the expectation $\mathbf{E}\varphi(t)$, $t > 0$, is a functional of the Lebesgue measure of the sets $B(\widehat{\mathbf{W}}; x)$, $x \in \mathbb{R}^d$.

Theorem 1 *For $d \geq 2$,*

$$\begin{aligned}
\mathbf{E}\varphi(t) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int \overline{\mathbf{E}} \exp(-(2t)^{\frac{d}{2}} V_d(\widehat{\mathbf{W}}, x)) dx \tag{2} \\
&= \frac{1}{(4\pi t)^{\frac{d}{2}}} \int \overline{\mathbf{E}} \exp(-V_d(\sqrt{2t}\widehat{\mathbf{W}}, x)) dx, \quad t > 0.
\end{aligned}$$

Formula (2) is the key point for the proof of our main result providing the logarithmic asymptotic behaviour of the distribution function (and of the Laplace transform) of the square of the fundamental frequency (e.g. the first eigenvalue λ_1 of the Dirichlet laplacian) of the Poisson-Voronoi typical cell.

Theorem 2 *Denote by μ_1 the first eigenvalue of the largest random ball centered at the origin and included in $C(0)$. We have*

$$
\begin{aligned}
\lim_{t \longrightarrow \infty} t^{-\frac{d}{d+2}} \ln \mathbf{E} e^{-t\lambda_1} &= \lim_{t \longrightarrow \infty} t^{-\frac{d}{d+2}} \ln \mathbf{E}\varphi(t) \\
&= \lim_{t \to +\infty} t^{-\frac{d}{d+2}} \ln \mathbf{E} e^{-t\mu_1} \\
&= -2^{\frac{3d}{d+2}} \omega_d^{\frac{2}{d+2}} \left(\frac{d+2}{2}\right) \left(\frac{j_{(d-2)/2}^2}{d}\right)^{\frac{d}{d+2}},
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
\lim_{t \to 0^+} t^{d/2} \ln \mathbf{P}\{\lambda_1 \le t\} &= \lim_{t \to 0^+} t^{d/2} \ln \mathbf{P}\{\mu_1 \le t\} \\
&= -2^d \omega_d j_{(d-2)/2}^d,
\end{aligned}
\tag{4}
$$

where $\omega_d = \sigma_d/d$ and $j_{(d-2)/2}$ are respectively the Lebesgue measure of the unit-ball of $\mathbb{R}^d$ and the first positive zero of the Bessel function $J_{(d-2)/2}$.

This non-trivial result is strongly connected with the Donsker-Varadhan theorem about the volume of the Wiener sausage [6] providing the logarithmic equivalent, when $t \to +\infty$, of the survival probability $\mathbf{P}\left\{\sqrt{2t}\widehat{\mathbf{W}} \subset [\cup_{x \in \Phi} B(x, \varepsilon)]^c\right\}$ of a Brownian path until time $t > 0$ in a random medium of Poissonian obstacles which are the balls centered at $x \in \Phi$ and of fixed radius $\varepsilon > 0$. More precisely, we will show as an intermediary result of our proof that this last logarithmic equivalent is precisely the same as the logarithmic equivalent of the probability $\mathbf{P}\{\sqrt{2t}\widehat{\mathbf{W}} \subset 2 \cdot C(0)\}$ that the Brownian bridge stays until time $t > 0$ in the homothetic $2 \cdot C(0)$ of the Poisson-Voronoi typical cell.

Besides, Theorem 2 shows that the asymptotic behaviour of the first eigenvalue of the typical cell $C(0)$ is the same as the first eigenvalue of the largest ball included in $C(0)$. This result is substantially close to the work of A. S. Sznitman who proved that the large deviations of the first eigenvalue of the complementary of a Poisson cloud of obstacles (e.g. balls) in a fixed box are controlled by the largest ball free of obstacles (see [28], p.182).

We can also exploit Theorem 1 by investigating the asymptotic estimation of $\mathbf{E}\varphi(t)$ when $t \to 0^+$. Let us recall that H. Weyl proved in 1911 [31] that for a bounded domain $U \subset \mathbb{R}^d$ with a piecewise-smooth boundary, the spectral function φ_U satisfies the asymptotic relation

$$
\varphi_U(t) \sim_{t \to 0^+} \frac{V_d(U)}{(4\pi t)^{d/2}}.
$$

For a bounded polygonal convex domain in $\mathbb{R}^2$, it was shown (see [15], [24], [30]) that when $t \to 0^+$,

$$
\varphi_U(t) = \frac{V_2(U)}{4\pi t} - \frac{V_1(U)}{4\sqrt{4\pi t}} + \frac{1}{24}(\pi \alpha^{-1}(U) - N_0(U) + 2) + O(e^{-c/t}),
\tag{5}
$$

where

(i) $V_1(U)$ is the perimeter of U;

(ii) $N_0(U)$ is the number of vertices of U;

(iii) $\alpha^{-1}(U) = \sum_{i=1}^{N_0(U)} 1/\alpha_i$ is the harmonic mean of the inside-facing angles $\alpha_1, \cdots, \alpha_{N_0(U)}$ at the vertices of U;

(iv) $c > 0$ is a positive constant independent of $t > 0$.

For a bounded convex non-degenerate polyhedron $U \subset \mathbb{R}^d$, $d \geq 3$, B. U. Fedosov [8] proved that we have when $t \to 0^+$,

$$\varphi_U(t) = \frac{V_d(U)}{(4\pi t)^{\frac{d}{2}}} - \frac{V_{d-1}(U)}{4(4\pi t)^{\frac{d-1}{2}}} + \frac{1}{8(4\pi t)^{\frac{d-2}{2}}} \sum_{i=1}^{N_{d-2}(U)} \frac{1}{3}\left(\frac{\omega_i}{\pi} - \frac{\pi}{\omega_i}\right) V_{d-2}(F_i) + O\left(\frac{1}{t^{\frac{d-3}{2}}}\right), \quad (6)$$

where

(i') $V_{d-1}(U)$ is the $(d-1)$-dimensional measure of the boundary of U;

(ii') F_i, $i = 1, \cdots, N_{d-2}(U)$ are the $(d-2)$-dimensional faces of U;

(iii') ω_i is the magnitude of the dihedral angle at the face F_i, $1 \leq i \leq N_{d-2}(U)$.

Returning now to the spectral function of the typical cell $\mathcal{C}$, we prove that for all $k \geq 1$ we have the asymptotic relation (when $t \to 0^+$)

$$\mathbf{E}\varphi(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\left\{\sum_{i=0}^{k-1} c_{d,i} t^{i/2} + O(t^{k/2})\right\}, \quad (7)$$

where the coefficients $c_{d,i}$ are expressed in terms of the covariances

$$c(u_1, \cdots, u_j) = \overline{\mathbf{E}}(M_{u_1} \cdots M_{u_j}), \quad u_1, \cdots, u_j \in \mathbb{S}^{d-1}, 1 \leq j \leq i,$$

and $M_u = \sup_{0 \leq s \leq 1}(u \cdot W(s))$, $u \in \mathbb{S}^{d-1}$, is the projection of the d-dimensional Brownian path $\mathbf{W}$ on the half-line $\mathbb{R}_+ u$ (see Remark 3).

The calculation of the coefficients $c_{2,i}$, $i \geq 0$, (resp. $c_{d,i}$, $i = 0, 1$) shows that they have the same geometrical meaning (up to the expectation) as in deterministic formulas (5) and (6).

This paper is structured as follows. The first section is devoted to some useful preliminaries on the convex hull of the Brownian bridge. We then prove in the second section Theorem 1 and we provide some easy consequences of it. The following two sections are focused respectively on the proof of Theorem 2 about the asymptotic behaviour of $\mathbf{E}\varphi(t)$, $t \to +\infty$, and on the asymptotic estimation of $\mathbf{E}\varphi(t)$, $t \to 0^+$. We finally enunciate some concluding remarks.

The principal results of this paper were announced in [13].

1 Preliminaries on the convex hull of the d-dimensional Brownian bridge.

We begin with the following elementary facts:

Proposition 1 *Consider a bounded closed set $C \subset \mathbb{R}^d$ and denote by $\widehat{C}$ its closed convex hull. Furthermore let $D \subset C$ be a countable, dense subset of C. We have*

$$B(C; x) = B(\widehat{C}; x) \tag{8}$$

$$V_d(D, x) = V_d(C, x) = V_d\left(\overline{B(D; x)}\right), \quad x \in \mathbb{R}^d. \tag{9}$$

Proof of (8). Let $u \in \mathbb{S}^{d-1}$ be a unit vector such that

$$(x + \mathbb{R}_+ u) \cap B(\widehat{C}; x) \neq \emptyset.$$

It can be easily seen that there exist a point $z \in (x + \mathbb{R}_+ u)$ and a support hyperplane H_u of $\widehat{C}$ perpendicular to u such that:

(i)
$$(x + \mathbb{R}_+ u) \cap B(\widehat{C}; x) = \overline{xz},$$
where $\overline{xz} = \{\lambda x + (1 - \lambda)z, 0 \le \lambda \le 1\}$ is the closed segment with bounding points x, z;

(ii)
$$\|y - z\| = \|y - x\|, \quad \forall y \in H_u \cap \partial\widehat{C}.$$

Moreover it is known ([25], corollary 18.3.1) that the intersection $H_u \cap \partial\widehat{C}$ must contain at least one point $y \in C$. Therefore

$$(x + \mathbb{R}_+ u) \cap B(\widehat{C}; x) \subset B(C; x) \quad \forall u \in \mathbb{S}^{d-1},$$

which implies (8).

Proof of (9). Fix $y \in \overline{B(D; x)}$. An elementary geometrical argument shows (the set D being bounded) that

$$\{\lambda x + (1 - \lambda)y; 0 < \lambda \le 1\} \subset B(D; x).$$

Integrating then in spherical coordinates (with x as center) we obtain the equality

$$V_d(D, x) = V_d(\overline{B(D; x)}).$$

Combining this with the obvious inclusions

$$\widetilde{B}(D; x) \subset \widetilde{B}(C; x) \subset B(C; x) \subset \overline{B(D; x)}, \quad x \in \mathbb{R}^d,$$

we obtain the result.

$\square$

Our next task is to give a useful estimate of the difference

$$V_d(C, x) - \omega_d \|x\|^d = V_d(B(C; x) \setminus B(\|x\|))$$

for sets $C \subset \mathbb{R}^d$ containing the origin. It follows from (8) that we may suppose that the set C is convex. Let us introduce a few notations. Fix $x \in \mathbb{R}^d \setminus \{0\}$ and define:

(i) $H = \{y \in \mathbb{R}^d; (y - x) \cdot x = 0\}$ the polar hyperplane of the point x;

(ii) $H^+ = \{y \in \mathbb{R}^d; (y - x) \cdot x \le 0\}$ the half-space associated with H and containing the origin;

(iii) $\mathbb{S}^+ = \mathbb{S}^{d-1} \cap (H^+ - x), \quad \mathbb{S}^- = \mathbb{S}^{d-1} \setminus \mathbb{S}^+;$

(iv) $H_{0,u}$ the hyperplane perpendicular to the vector $u \in \mathbb{S}^{d-1}$ and containing the origin;

(v) H_u the support hyperplane of C perpendicular to u and included in the half-space $H_{0,u}^+ = \{y \in \mathbb{R}^d; y \cdot u \geq 0\}$;

(vi) $\mathcal{A}(C,x) = \{u \in \mathbb{S}^-; H_u \cap (x + \mathbb{R}_+u)\} \neq \emptyset\}$ (notice that $H_u \cap (x + \mathbb{R}_+u) \neq \emptyset$ for all $u \in \mathbb{S}^+$);

(vii) $m(x,u) = d(x, H_u)$ the distance between x and H_u,
$\rho(x,u) = |x.u| = d(x, H_{0,u})$ the distance between x and $H_{0,u}$,
$h(u) = d(0, H_u)$ the distance between H_u and $H_{0,u}$.

Remark 1 For all $x \in \mathbb{R}^d \setminus \{0\}$, we have

$$\begin{cases} m(x,u) = & h(u) + \rho(x,u) & \text{if } u \in \mathbb{S}^+, \\ m(x,u) = & |h(u) - \rho(x,u)| & \text{if } u \in \mathbb{S}^-, \\ \mathcal{A}(C,x) = & \{u \in \mathbb{S}^-; h(u) - \rho(x,u) \geq 0\} & . \end{cases} \tag{10}$$

Proposition 2 For $x \in \mathbb{R}^d$, we have:

$$V_d(C,x) - \omega_d ||x||^d = \frac{2^d}{d} \sum_{j=1}^{d} \binom{d}{j} \int_{\mathbb{S}^+} h(u)^j \rho(x,u)^{d-j} d\nu_d(u) + \frac{2^d}{d} \int_{\mathbb{S}^-} [(h(u) - \rho(x,u)) \vee 0]^d d\nu_d(u), \tag{11}$$

and in particular,

$$V_d(C,0) = \frac{2^d}{d} \int_{\mathbb{S}^{d-1}} h(u)^d d\nu_d(u). \tag{12}$$

Proof. For $u \in \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d \setminus \{0\}$, three possibilities occur:

Case 1 $u \notin \mathbb{S}^+ \cup \mathcal{A}(C,x)$ and consequently

$$(x + \mathbb{R}_+u) \cap B(C;x) = (x + \mathbb{R}_+u) \cap B(||x||) = \{x\}.$$

Case 2 $u \in \mathbb{S}^+$ which implies that

$$(x + \mathbb{R}_+u) \cap B(C;x) = \overline{xz}, \quad (x + \mathbb{R}_+u) \cap B(||x||) = \overline{xz'},$$

with

$$||x - z|| = 2m(x,u), \quad ||x - z'|| = 2\rho(x,u).$$

Case 3 $u \in \mathcal{A}(C,x)$ which implies that

$$(x + \mathbb{R}_+u) \cap B(C;x) = \overline{xz}, \quad (x + \mathbb{R}_+u) \cap B(||x||) = \{x\},$$

with

$$||x - z|| = 2m(x,u).$$

Then integration in spherical coordinates (with x as center) gives that

$$\begin{aligned} V_d(C,x) - \omega_d ||x||^d &= \int_{\mathbb{S}^+} \left[\int_{2\rho(x,u)}^{2m(x,u)} r^{d-1} dr \right] d\nu_d(u) + \int_{\mathcal{A}(C,x)} \left[\int_0^{2m(x,u)} r^{d-1} dr \right] d\nu_d(u) \\ &= (2^d/d) \left[\int_{\mathbb{S}^+} (m(x,u)^d - \rho(x,u)^d) d\nu_d(u) + \int_{\mathcal{A}(C,x)} m(x,u)^d d\nu_d(u) \right]. \tag{13} \end{aligned}$$

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From (10) we get that

$$m(x,u)^d - \rho(x,u)^d = \sum_{j=1}^{d} \binom{d}{j} h(u)^j \rho(x,u)^{d-j}, \quad u \in \mathbb{S}^+,$$

and

$$(h(u) - \rho(x,u)) \vee 0 = \begin{cases} m(x,u) & \text{for } u \in \mathcal{A}(C,x) \\ 0 & \text{for } u \in \mathbb{S}^- \setminus \mathcal{A}(C,x). \end{cases}$$

Substituting these expressions in (13) we find the final result (11).

To prove (12), it suffices to notice that for all $u \in \mathbb{S}^{d-1}$, we have

$$\mathbb{R}_+ u \cap B(C;0) = \overline{0z}$$

with

$$||z|| = 2h(u),$$

so integrating in spherical coordinates we obtain the result.

$\square$

Suppose now that C is a random convex set containing the origin and invariant (in law) by rotations with the origin as center. Thus the random variables $h(u)$, $u \in \mathbb{S}^{d-1}$, are equal in law and we obtain:

Proposition 3 *Suppose that C satisfies the above conditions and that $\mathbf{E}\{h(u)^d\} < \infty$, $u \in \mathbb{S}^{d-1}$. Fixing $u_0 \in \mathbb{S}^{d-1}$, we have then*

$$\mathbf{E}\left(V_d(C,x) - \omega_d ||x||^d\right) = \sum_{j=1}^{d} I_{d,j} ||x||^{d-j} \mathbf{E}\left(h(u_0)^j\right) + \frac{2^d}{d} \int_{\mathbb{S}^-} \mathbf{E}\left([(h(u_0) - \rho(x,u)) \vee 0]^d\right) d\nu_d(u)$$

where

$$I_{d,j} = \frac{2^d}{d} \sigma_{d-1} \binom{d}{j} \int_0^1 t^{d-j} (1-t^2)^{\frac{d-3}{2}} dt = \frac{2^d}{d} \binom{d}{j} \frac{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{d+1-j}{2}\right)}{\Gamma\left(d - \frac{j}{2}\right)}, \quad 1 \leq j \leq d.$$

Proof. Taking the expectation in (11) the result follows from the direct evaluation of the integrals

$$\int_{\mathbb{S}^+} \rho(x,u)^{d-j} d\nu_d(u) = ||x||^{d-j} I_{d,j}. \square$$

Remark 2 Under the conditions stated in Proposition 3 we obtain that

$$\mathbf{E}\left(V_d(\varepsilon C,x) - \omega_d ||x||^d\right) \sim_{\varepsilon \longrightarrow 0} I_{d,1} \varepsilon ||x||^{d-1} \mathbf{E}h(u_0).$$

In particular, in dimension $d = 2$, it follows from the Cauchy formula giving the perimeter of a convex set that

$$\mathbf{E}\left(V_2(\varepsilon C,x) - \pi ||x||^2\right) \sim_{\varepsilon \longrightarrow 0} \varepsilon \frac{4||x||}{\pi} \mathbf{E}V_1(C),$$

where $V_1(C)$ denotes the perimeter of the convex set C.

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Choose now for C the closed convex hull $\widehat{\mathbf{W}} \subset \mathbb{R}^d$, of the sample path of the d-dimensional Brownian bridge on the interval $[0, 1]$. Recall that $\widehat{\mathbf{W}}$ is invariant by rotations with the origin as center. Hence the random variables $h(u)$, $u \in \mathbb{S}^{d-1}$, defined above coincide in law with the maximum M_0 of the 1-dimensional Brownian bridge. The law of M_0 is explicitely known, namely [27]

$$\overline{\mathbf{P}}\{M_0 \geq u\} = e^{-2u^2}. \tag{14}$$

Hence all the moments of M_0 are finite, and we have

$$\overline{\mathbf{E}}M_0{}^{2k} = \left(\frac{1}{2}\right)^k k!, \quad \overline{\mathbf{E}}M_0{}^{2k+1} = \frac{(2k+1)!}{8^k k!} \frac{\sqrt{\pi}}{2\sqrt{2}}, \quad k \in \mathbb{N}.$$

In particular,

$$\overline{\mathbf{E}}\left(V_d(\varepsilon\widehat{\mathbf{W}}, x) - \omega_d||x||^d\right) \sim_{\varepsilon \longrightarrow 0} I_{d,1}\varepsilon||x||^{d-1}\frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Now, denote by

$$M = \sup_{y \in \widehat{\mathbf{W}}} ||y|| = \sup_{0 \leq s \leq 1} ||W(s)||$$

the maximum of the radial part of the Brownian bridge

$$W(s) = (W_1(s), \cdots, W_d(s)), \quad 0 \leq s \leq 1.$$

The components $W_i(s)$, $0 \leq s \leq 1$, $i = 1, \cdots, d$, are independent one-dimensional Brownian bridges. Hence

$$\begin{aligned}
\overline{\mathbf{P}}\{M \geq s\} &\leq d\overline{\mathbf{P}}\{\sup_{0 \leq s \leq 1} |W_1(s)| \geq s/\sqrt{d}\} \\
&\leq 2d\overline{\mathbf{P}}\{M_0 \geq s/\sqrt{d}\} \\
&= 2de^{-2s^2/d}, \quad s \geq 0,
\end{aligned} \tag{15}$$

and

$$\overline{\mathbf{E}}M^k < +\infty, \quad \forall k \geq 0. \tag{16}$$

As a consequence we deduce the following result:

Proposition 4 *For all $k \in \mathbb{N}^*$ there exists a constant $0 < c_k < +\infty$ such that*

$$\overline{\mathbf{E}}|V_d(\varepsilon\widehat{\mathbf{W}}, x) - \omega_d||x||^d|^k \leq c_k \sum_{i=k}^{kd} \varepsilon^i||x||^{kd-i},$$

$\varepsilon > 0$, $x \in \mathbb{R}^d$.

Proof. From Proposition 2 we have

$$|V_d(\varepsilon\widehat{W}, x) - \omega_d||x||^d| \leq (2^d/d)\left[\sum_{j=1}^{d-1} \binom{d}{j}(\varepsilon M)^j \int_{\mathbb{S}^+} \rho(x, u)^{d-j} d\nu_d(u) + \omega_d(\varepsilon M)^d\right],$$

which by (16) implies the result.

$\square$

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2 Proof of Theorem 1 and consequences.

Proof of Theorem 1. Consider the spectral function

$$\varphi(t) = \sum_{n \geq 1} e^{-t\lambda_n}, \quad t > 0,$$

of the typical cell $\mathcal{C}$.

Let us recall first (see [10], [27]) that the spectral function $\varphi_U(t)$, $t > 0$, of any bounded domain $U \subset \mathbb{R}^d$ can be expressed in term of the Brownian bridge under the form:

$$\varphi_U(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_U \overline{\mathbf{P}}\{x + \sqrt{2t}\mathbf{W} \subset U\}dx. \tag{17}$$

Applying the formula above to the domain

$$C(0) = \{y \in \mathbb{R}^d; ||y|| \leq ||y - x||, x \in \Phi\} \stackrel{law}{=} \mathcal{C},$$

and taking the expectation we obtain (by Fubini theorem)

$$\mathbf{E}\varphi(t) \quad = \quad \frac{1}{(4\pi t)^{d/2}}\overline{\mathbf{E}}\int \mathbf{P}\{x + \sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\}dx$$

Observe that

$$-x + \sqrt{2t}\widehat{\mathbf{W}} \subset C(0) \Longleftrightarrow \Phi \cap \{-x + B(\sqrt{2t}\widehat{\mathbf{W}}; x)\} = \emptyset.$$

Therefore applying the property of the Poisson point process Φ we obtain

$$\mathbf{P}\{-x + \sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\} = \exp\{-V_d(\sqrt{2t}\widehat{\mathbf{W}}, x)\},$$

and consequently

$$\mathbf{E}\varphi(t) = \frac{1}{(4\pi t)^{d/2}} \int \overline{\mathbf{E}}\exp\left\{-V_d(\sqrt{2t}\widehat{\mathbf{W}}, x)\right\}dx.$$

The obvious identity

$$V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) = (2t)^{d/2}V_d(\widehat{\mathbf{W}}, x/\sqrt{2t}), \quad t > 0, x \in \mathbb{R}^d,$$

and an elementary change of variable provide the result.

$$\square$$

In particular, Theorem 1 provides an infinite expansion of $\mathbf{E}\varphi(t)$ for $d = 2$, which is valid for every $t > 0$.

Theorem 3 *For $d = 2$ there exists $t_0 > 0$ such that:*

$$\mathbf{E}\varphi(t) = \frac{1}{4\pi t} \sum_{n \geq 0} \frac{(-1)^n}{n!} \int \overline{\mathbf{E}}(V_2(\sqrt{2t}\widehat{\mathbf{W}}, x) - \pi||x||^2)^n e^{-\pi||x||^2}dx, \tag{18}$$

for all $0 < t < t_0$, the series being absolutely convergent.

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Proof. Regarding (2) it suffices to prove that there exists $t_0 > 0$ such that

$$\int \overline{\mathbf{E}} \exp(V_2(\sqrt{2t_0}\widehat{\mathbf{W}}, x) - 2\pi||x||^2) dx < +\infty.$$

Using the obvious inclusion

$$B(\sqrt{2t}\widehat{\mathbf{W}}; x) \subset B(2\sqrt{2t}M + ||x||),$$

we obtain the inequality

$$V_2(\sqrt{2t}\widehat{\mathbf{W}}, x) \le \pi(||x|| + 2\sqrt{2t}M)^2 \le \frac{3\pi}{2}||x||^2 + 24t\pi M^2. \tag{19}$$

Hence,

$$
\begin{aligned}
\int \overline{\mathbf{E}} \exp(V_2(\sqrt{2t}\widehat{\mathbf{W}}, x) - 2\pi||x||^2) dx \ &\le\ \int \overline{\mathbf{E}} \exp(V_2(\sqrt{2t}\widehat{\mathbf{W}}, x) - \frac{\pi}{2}||x||^2) dx \\
&\le\ \overline{\mathbf{E}} \exp(24t\pi M^2) \\
&=\ \int_0^{+\infty} e^s \overline{\mathbf{P}}\{24\pi M^2 \ge s/t\} ds + 1 \\
&\le\ 1 + 4\int_0^{+\infty} e^s e^{-s/(24\pi t)} ds,
\end{aligned}
$$

by using (15). Thus it suffices to take $t_0 < \frac{1}{24\pi}$. $\square$

In the two-dimensional case $d = 2$, the formula (2) provides straightforwardly an identification, in term of Brownian bridge, of the expectation of the distribution function of the eigenvalues

$$N(t) = \sum_{n \ge 1} 1_{\{\lambda_n \le t\}}, t > 0,$$

Indeed on one hand we have

$$\mathbf{E}\varphi(t) = t \int_0^{+\infty} e^{-ts} \mathbf{E} \sum_{n \ge 1} 1_{\{\lambda_n \le s\}} ds, \quad t > 0.$$

On the other hand an elementary computation yields

$$\frac{1}{2\pi} \int e^{-2tV_2(\widehat{\mathbf{W}}, x)} dx = \frac{t}{2\pi} \int_0^{+\infty} e^{-ts} \left(\int \overline{\mathbf{P}}\{2V_2(\widehat{\mathbf{W}}, x) \le s\} dx \right) ds.$$

So by injectivity of the Laplace transform and Theorem 1 we obtain

Theorem 4 *In dimension $d = 2$, the expectation of the distribution function $N(s), s > 0$, is of the form*

$$\mathbf{E}N(t) = \frac{1}{2\pi} \int \overline{\mathbf{P}}\{2V_2(\widehat{\mathbf{W}}, x) \le t\} dx.$$

3   Proof of Theorem 2.

The main object in this paragraph is to obtain the logarithmic equivalent of the Laplace transform of the distribution of the square of the fundamental frequency (that means the first eigenvalue λ_1 of the Dirichlet-Laplacian) of the Poisson-Voronoi typical cell. Using a Tauberian argument, this result leads us to the asymptotic evaluation when $t \to 0^+$ of the logarithm of the distribution function $\mathbf{P}\{\lambda_1 \leq t\}$.

Proof of (3). Let us first notice that

$$e^{-t\mu_1} \leq e^{-t\lambda_1} \leq \varphi(t), \quad t > 0,$$

Consequently, it suffices to prove that

$$\lim_{t \to +\infty} t^{-\frac{d}{d+2}} \ln \mathbf{E} e^{-t\mu_1} = -2^{\frac{3d}{d+2}} \omega_d^{\frac{2}{d+2}} \left(\frac{d+2}{2}\right) \left(\frac{j_{(d-2)/2}^2}{d}\right)^{\frac{d}{d+2}} \tag{20}$$

and

$$\limsup_{t \to +\infty} t^{-\frac{d}{d+2}} \ln \mathbf{E} \varphi(t) \leq -2^{\frac{3d}{d+2}} \omega_d^{\frac{2}{d+2}} \left(\frac{d+2}{2}\right) \left(\frac{j_{(d-2)/2}^2}{d}\right)^{\frac{d}{d+2}} \tag{21}$$

In order to obtain the asymptotic result (20), let us remark that the random radius R_m of the largest disc centered at the origin and included in $C(0)$ has the distribution

$$\mathbf{P}\{R_m \geq r\} = e^{-2^d \omega_d r^d}, \quad r \geq 0. \tag{22}$$

Thus using the fact that the first eigenvalue of a disc of radius $r > 0$ is equal to $j_{(d-2)/2}^2/r^2$, we deduce that

$$
\begin{aligned}
\mathbf{E} e^{-t\mu_1} &= \int_0^{+\infty} \exp\left\{-t\frac{j_{(d-2)/2}^2}{r^2} - 2^d \omega_d r^d\right\} d2^d \omega_d r^{d-1} dr \\
&= t^{d/(d+2)} \int_0^{+\infty} \exp\left\{-t^{d/d+2}\left(\frac{j_{(d-2)/2}^2}{r^2} - 2^d \omega_d r^d\right)\right\} d2^d \omega_d r^{d-1} dr.
\end{aligned}
$$

We then get from Laplace method that

$$\lim_{t \to +\infty} t^{-d/(d+2)} \ln \mathbf{E} e^{-t\mu_1} = -\min_{r>0}\left\{\frac{j_{(d-2)/2}^2}{r^2} + 2^d \omega_d r^d\right\} = -2^{\frac{3d}{d+2}} \omega_d^{\frac{2}{d+2}} \left(\frac{d+2}{2}\right) \left(\frac{j_{(d-2)/2}^2}{d}\right)^{\frac{d}{d+2}},$$

which proves the result (20).

The proof of (21) is far more delicate. Using Theorem 1, we first have

$$
\begin{aligned}
\mathbf{E} \varphi(t) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{W}, x)} dx \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \left[\int_{\{\|x\|\leq t\}} \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{W}, x)} + \int_{\{\|x\|>t\}} \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{W}, x)} dx\right]
\end{aligned}
$$

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The obvious following inequality

$$V_d(\widehat{W}, x) \geq \omega_d ||x||^d, \quad x \in \mathbb{R}^d,$$

then implies that

$$\int_{\{||x||>t\}} \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{W}, x)} dx \leq \int_{\{||x||>t\}} e^{-(2t)^{\frac{d}{2}} \omega_d ||x||^d} dx$$

$$= (2t)^{-\frac{d}{2}} e^{-2^{d/2} \omega_d t^{\frac{3d}{2}}}, \quad t > 0.$$

Therefore

$$\lim_{t \to +\infty} t^{-d/(d+2)} \ln \int_{\{||x||>t\}} \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{W}, x)} dx = -\infty.$$

Consequently in order to obtain (21), it suffices to prove that

$$\limsup_{t \to +\infty} t^{-d/(d+2)} \int_{\{||x|| \leq t\}} \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{W}, x)} \leq -2^{\frac{3d}{d+2}} \omega_d^{\frac{2}{d+2}} \left(\frac{d+2}{2} \right) \left(\frac{j_{(d-2)/2}^2}{d} \right)^{\frac{d}{d+2}}. \tag{23}$$

The key estimation which enables us to derive the above asymptotic behaviour is contained in the following lemma.

Lemma 1 *For every $t > 0$, and $x \in \mathbb{R}^d$,*

$$\overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{\mathbf{W}}, x)} \leq \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{\mathbf{W}}, 0)}, \tag{24}$$

or equivalently

$$\mathbf{E}\overline{\mathbf{P}}\{x + \sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\} \leq \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\}. \tag{25}$$

This last result will be proved at the end of the section. Let us notice that Lemma 1 implies the inequality

$$\int_{\{||x|| \leq t\}} \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{W}, x)} dx \leq \omega_d t^d \overline{\mathbf{E}} e^{-(2t)^{\frac{d}{2}} V_d(\widehat{W}, 0)} = \omega_d t^d \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\}, \quad t > 0.$$

Therefore in order to prove (23), it suffices to have

$$\limsup_{t \to +\infty} t^{-d/(d+2)} \ln \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\} \leq -2^{\frac{3d}{d+2}} \omega_d^{\frac{2}{d+2}} \left(\frac{d+2}{2} \right) \left(\frac{j_{(d-2)/2}^2}{d} \right)^{\frac{d}{d+2}}, \tag{26}$$

This last result is a consequence of the Donsker-Varadhan theorem about the volume of the Wiener sausage [7].

More precisely, let us denote by L the distance from the origin to its nearest neighbor. The distribution of L is given by the equality

$$\mathbf{P}\{L \geq t\} = \mathbf{P}\{R_m \geq t/2\} = e^{-\omega_d t^d}, \quad t > 0. \tag{27}$$

Consequently,

$$\mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\} = \int_0^{+\infty} \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\widehat{\mathbf{W}} \subset C(0)|L = s\} d\omega_d s^{d-1} e^{-\omega_d s^d} ds. \tag{28}$$

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Moreover, it is obvious from the definition of the Voronoi cell $C(0)$ that for every $c \in [1, 2)$,

$$C(0) \subset \left[\cup_{x \in \Phi} B\left(\frac{1}{c} \cdot x, \left(\frac{1}{c} - \frac{1}{2} \right) \cdot L \right) \right]^c. \tag{29}$$

This leads us to compare for the Brownian bridge the probability to stay in the cell $C(0)$ with the probability to avoid Poissonian obstacles. For every fixed $\varepsilon > 0$ and $c \in [1, 2)$, we obtain

$$\int_{2\varepsilon}^{+\infty} \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\widehat{\mathbf{W}} \subset C(0) | L = s\} d\omega_d s^{d-1} e^{-\omega_d s^d} ds$$

$$\leq \int_{2\varepsilon}^{+\infty} \mathbf{E}\overline{\mathbf{P}} \left\{ \sqrt{2t}\mathbf{W} \subset \left[\cup_{x \in \Phi} B\left(\frac{1}{c} \cdot x, \left(\frac{1}{c} - \frac{1}{2} \right) \varepsilon \right) \right]^c \middle| L = s \right\} d\omega_d s^{d-1} e^{-\omega_d s^d} ds$$

$$\leq \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\mathbf{W} \subset \left[\cup_{(c \cdot x) \in \Phi} B\left(x, \left(\frac{1}{c} - \frac{1}{2} \right) \varepsilon \right) \right]^c\}$$

$$= \overline{\mathbf{E}} \exp \left\{ -c^d V_d \left(\cup_{y \in \sqrt{2t}\widehat{\mathbf{w}}} B(y, (c^{-1} - 0.5)\varepsilon) \right) \right\}.$$

It is well known that the estimation of the Wiener sausage provided by Donsker and Varadhan [7] is still relevant for the path of a Brownian bridge. Consequently, we obtain that

$$\limsup_{t \to +\infty} t^{-\frac{d}{d+2}} \ln \int_{2\varepsilon}^{+\infty} \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\widehat{\mathbf{W}} \subset C(0) | L = s\} d\omega_d s^{d-1} e^{-\omega_d s^d} ds$$

$$\leq -2^{\frac{d}{d+2}} (c^d \omega_d)^{\frac{2}{d+2}} \left(\frac{d+2}{2} \right) \left(\frac{j_{(d-2)/2}^2}{d} \right)^{\frac{d}{d+2}}. \tag{30}$$

Taking the limit in (30) when $c \to 2^-$, we get

$$\limsup_{t \to +\infty} t^{-\frac{d}{d+2}} \ln \int_{2\varepsilon}^{+\infty} \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\widehat{\mathbf{W}} \subset C(0) | L = s\} d\omega_d s^{d-1} e^{-\omega_d s^d} ds$$

$$\leq -2^{\frac{3d}{d+2}} \omega_d^{\frac{2}{d+2}} \left(\frac{d+2}{2} \right) \left(\frac{j_{(d-2)/2}^2}{d} \right)^{\frac{d}{d+2}}. \tag{31}$$

Besides, let us notice that we have the following inequality for every $s < s' \in \mathbb{R}_+$:

$$\mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\mathbf{W} \subset C(0) | L = s\} \leq \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\mathbf{W} \subset C(0) | L = s'\}. \tag{32}$$

Indeed, considering a uniformly distributed point X_0 on the sphere $\mathbb{S}^{d-1}$, the equality in law

$$(\Phi | L = s) \overset{law}{=} (\Phi \cap B(0, s)^c) \cup \{s \cdot X_0\},$$

is clearly satisfied. We then deduce from (32) that

$$\int_0^{2\varepsilon} \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\mathbf{W} \subset C(0) | L = s\} d\omega_d s^{d-1} e^{-\omega_d s^d} ds$$

$$\leq \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\mathbf{W} \subset C(0) | L = 2\varepsilon\}$$

$$\leq \int_{2\varepsilon}^{+\infty} \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\mathbf{W} \subset C(0) | L = s\} d\omega_d s^{d-1} e^{-\omega_d s^d} ds.$$

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Using (31), we then obtain that

$$\limsup_{t\to+\infty} t^{-\frac{d}{d+2}} \ln \int_0^{2\varepsilon} \mathbf{E}\overline{\mathbf{P}}\{\sqrt{2t}\mathbf{W} \subset C(0)|L=s\}d\omega_d s^{d-1}e^{-\omega_d s^d}ds$$

$$\leq -2^{\frac{3d}{d+2}}\omega_d^{\frac{2}{d+2}}\left(\frac{d+2}{2}\right)\left(\frac{j_{(d-2)/2}^2}{d}\right)^{\frac{d}{d+2}}. \tag{33}$$

Combining the equality (28) with (30) and (33), we get the estimation (26), which completes the proof of the convergence (3).

Let us now focus on the proof of Lemma 1.

Proof of Lemma 1. Let us fix $x \in \mathbb{R}^d$. Using the identity

$$\overline{\mathbf{E}}e^{-(2t)^{\frac{d}{2}}V_d(\widehat{\mathbf{W}},x)} = (2t)^{\frac{d}{2}}\int_0^\infty \overline{\mathbf{P}}\left(V_d(\widehat{\mathbf{W}},x) \leq u\right)e^{-(2t)^{\frac{d}{2}}u}du, \quad t > 0,$$

we obtain that it suffices to show the following inequality.

$$\overline{\mathbf{P}}\left\{V_d(\widehat{\mathbf{W}},x) \leq u\right\} \leq \overline{\mathbf{P}}\left\{V_d(\widehat{\mathbf{W}},0) \leq u\right\}, \quad u \geq 0. \tag{34}$$

Let us notice that (34) is not a direct consequence of the inclusion of a set into an other one.

The following lemma provides a useful deterministic result of set theory.

Lemma 2 *For every fixed $N \geq 1$ and $u \geq 0$, the set*

$$A_u = \left\{z = (z_1,\cdots,z_N) \in (\mathbb{R}^d)^N; V_d\left(\bigcup_{i=1}^N B(z_i,||z_i||)\right) \leq u\right\}, \quad d \geq 2,$$

is convex and symmetric.

Proof. First, A_u is clearly symmetric. In order to prove that A_u is convex, let us fix $\overline{z} = (z_1,\cdots,z_N)$, $\overline{y} = (y_1,\cdots,y_N) \in A_u$, $0 < \gamma < 1$, and define

$$\overline{v} = (v_1,\cdots,v_N) = (1-\gamma)z + \gamma y.$$

We will also use the notation $z_0 = y_0 = v_0 = 0$.

From the equality (12), we obtain

$$V_d\left(\bigcup_{i=1}^N B(v_i,||v_i||)\right)$$

$$= \frac{2^d}{d}\int_{\mathbb{S}^{d-1}}\left[\sup_{i=0,\cdots,N}(v_i \cdot u)\right]^d d\nu_d(u)$$

$$= \frac{2^d}{d}\int_{\mathbb{S}^{d-1}}\left[\sup_{i=0,\cdots,N}(\gamma z_i \cdot u + (1-\gamma)y_i \cdot u)\right]^d d\nu_d(u)$$

$$\leq \frac{2^d}{d}\left[\gamma\int\left(\sup_{i=0,\cdots,N}(z_i \cdot u)\right)^d d\nu_d(u) + (1-\gamma)\int\left(\sup_{i=0,\cdots,N}(y_i \cdot u)\right)^d d\nu_d(u)\right]$$

$$= \gamma V_d\left(\bigcup_{i=1}^N B(z_i,||z_i||)\right) + (1-\gamma)V_d\left(\bigcup_{i=1}^N B(y_i,||y_i||)\right)$$

$$\leq u$$

which proves Lemma 2.

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□

Returning now to the proof of the inequality (34), let us fix $N \geq 2$ ans select $0 \leq t_1 \leq \cdots \leq t_N \leq 1$. The law of the random vector

$$\mathcal{W} = (W(t_1), \cdots, W(t_N)) \in (\mathbb{R}^d)^N$$

is a centered Gaussian measure. Let us observe that

$$\overline{\mathbf{P}}\left\{V_d\left(\bigcup_{i=1}^N B(W(t_i), ||W(t_i) - x||)\right) \leq u\right\} = \overline{\mathbf{P}}\left\{\mathcal{W} \in A_u + \overline{x}\right\} \tag{35}$$

with $\overline{x} = (x, \cdots, x) \in (\mathbb{R}^d)^N$.

since the set A_u, $u > 0$, is convex and symmetric, we may apply Anderson's lemma [1] which gives

$$\overline{\mathbf{P}}\left\{\mathcal{W} \in A_u + \overline{x}\right\} \leq \overline{\mathbf{P}}\left\{\mathcal{W} \in A_u\right\}. \tag{36}$$

Let us now take a countable set $(t_i)_{i \geq 1}$ dense in $[0, 1]$. By the continuity of the Brownian bridge sample paths, the set $D = (W(t_i))_{i \geq 1}$ is a dense subset of W. Then by (9), we have

$$V_d(D, x) = V_d(\mathbf{W}, x) = V_d\left(\overline{\cup_{y \in D} B(y, ||y - x||)}\right), \quad x \in \mathbb{R}^d.$$

Thus the increasing sequence $V_d\left(\cup_{i=1}^N B(y_i, ||y_i - x||)\right)$, $N \geq 1$, converges to $V_d(\mathbf{W}, x)$ for all $x \in \mathbb{R}^d$, which implies

$$\overline{\mathbf{P}}\left\{V_d\left(\widehat{\mathbf{W}}, x\right) \leq u\right\} = \lim_{N \to +\infty} \overline{\mathbf{P}}\left\{V_d\left(\bigcup_{i=1}^N B(W(t_i), ||W(t_i) - x||)\right) \leq u\right\}, \quad u \geq 0.$$

Combining this with (36), we obtain the inequality (34) and the proof of Lemma 1 is completed.

□

Proof of (4). Let us first notice that the distribution of the first eigenvalue $\mu_1 = j_{(d-2)/2}^2 / R_m^2$ of the ball $B(0, R_m)$ may be explicited by using the equality (22).

We then deduce from the inequality $\lambda_1 \leq \mu_1$ that

$$\liminf_{t \to 0^+} t^{\frac{d}{2}} \ln \mathbf{P}\{\lambda_1 \leq t\} \geq \lim_{t \to 0^+} t^{\frac{d}{2}} \ln \mathbf{P}\{\mu_1 \leq t\}$$
$$\geq -2^d \omega_d j_{(d-2)/2}^d.$$

It now remains to prove that

$$\limsup_{t \to 0^+} t^{\frac{d}{2}} \ln \mathbf{P}\{\lambda_1 \leq t\} \leq -2^d \omega_d j_{(d-2)/2}^d. \tag{37}$$

Let us note $c = 2^{\frac{3d}{d+2}} \omega_d^{\frac{2}{d+2}} \left(\frac{d+2}{2}\right) \left(\frac{j_{(d-2)/2}^2}{d}\right)^{\frac{d}{d+2}}$ and fix $0 < \varepsilon < 1$.

The asymptotic result (3) implies that we have for u large enough

$$\mathbf{E} e^{-u\lambda_1} \leq e^{-(1-\varepsilon)ct^{d/(d+2)}},$$

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and using Tchebychev inequality, we get

$$\mathbf{P}\{\lambda_1 \le t\} \le e^{-(1-\varepsilon)cu^{d/(d+2)}+ut}, \quad t > 0. \tag{38}$$

Taking

$$u = \left(\frac{\frac{d}{d+2}(1-\varepsilon)c}{t}\right)^{\frac{d+2}{d}}.$$

in the inequality (38), we obtain for t small enough

$$\mathbf{P}\{\lambda_1 \le t\} \le e^{-(1-\varepsilon)^{\frac{d+2}{2}}2^d\omega_d j_{(d-2)/2}^d t^{-\frac{d}{2}}},$$

which clearly provides the required result (37) and completes the proof of (4).

$\square$

4   The asymptotic behaviour of $\mathbf{E}\varphi(t)$, $t \to 0^+$.

In order to study the asymptotics of $\mathbf{E}\varphi(t)$ when $t \to 0^+$, we derive from (2) the following suitable relation.

Theorem 5 *For $k \ge 1$, the following asymptotic, when $t \to 0^+$, holds:*

$$\mathbf{E}\varphi(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\left\{\sum_{i=0}^{k-1}\frac{(-1)^i}{i!}\int \overline{\mathbf{E}}\{(V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d)^i\}e^{-\omega_d||x||^d}dx + O(t^{k/2})\right\}.$$

Proof. Let us start with the formula

$$\begin{aligned}
\mathbf{E}\varphi(t) &= \frac{1}{(4\pi t)^{\frac{d}{2}}}\int \overline{\mathbf{E}}\exp\left\{-(V_d(\sqrt{2t}\widehat{\mathbf{W}},x)\right\}dx \\
&= \frac{1}{(4\pi t)^{\frac{d}{2}}}\int \overline{\mathbf{E}}\exp\left\{-(V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d)\right\}e^{-\omega_d||x||^d}dx. \tag{39}
\end{aligned}$$

Fix $k \ge 1$. Since

$$V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d \ge 0, \quad x \in \mathbb{R}^d,$$

we have

$$|\exp\left\{-\left(V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d\right)\right\} - \sum_{i=0}^{k-1}\frac{(-1)^i}{i!}\left(V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d\right)^i|$$

$$\le \frac{1}{k!}\left(V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d\right)^k. \tag{40}$$

By Proposition 4 we have

$$\int \overline{\mathbf{E}}\left\{\left[V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d\right]^i\right\}e^{-\omega_d||x||^d}dx < +\infty, \quad \forall i \in \mathbb{N},$$

and

$$\int \overline{\mathbf{E}}\left\{\left[V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d\right]^k\right\}e^{-\omega_d||x||^d}dx \le c_k't^{k/2}, \quad 0 < t \le 1/2, \tag{41}$$

where $0 < c_k' < +\infty$ is a constant. The result follows then from (39), (40) and (41).

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□

Let us recall that the values of the expectations of the principal geometrical characteristics of the typical cell $\mathcal{C}$ are known [20]. In particular,

$$\mathbf{E}V_d(\mathcal{C}) = 1,$$

and denoting by $V_{d-1}(\mathcal{C})$ the $(d-1)$-dimensional area of the boundary $\partial\mathcal{C}$, we have

$$\mathbf{E}V_{d-1}(\mathcal{C}) = \frac{\sqrt{\pi}d!\Gamma(2-1/d)\Gamma(d/2+1)^{-1/d}\Gamma(d/2)}{\Gamma((d+1)/2)\Gamma(d-1/2)}.$$

The following theorem provides now the explicit calculation of the three first coefficients of the preceding development for any dimension.

Theorem 6 *In dimension $d \geq 2$, we have when $t \longrightarrow 0^+$,*

$$\mathbf{E}\varphi(t) = \frac{\mathbf{E}(V_d(\mathcal{C}))}{(4\pi t)^{\frac{d}{2}}} - \frac{\mathbf{E}V_{d-1}(\mathcal{C})}{4(4\pi t)^{\frac{d-1}{2}}} + \frac{c_{d,2}}{(4\pi)^{d/2}t^{\frac{d}{2}-1}} + O\left(\frac{1}{t^{\frac{d-3}{2}}}\right),$$

where

$$c_{d,2} = \left(4^d k_d \sigma_d \frac{\Gamma(3-\frac{2}{d})}{d\omega_d^{3-\frac{2}{d}}} - I_{d,2}\sigma_d \frac{\Gamma(2-\frac{2}{d})}{d\omega_d^{2-\frac{2}{d}}}\right)$$

with

$$k_d = 4\sigma_{d-1}\sigma_{d-2}\int_{\substack{0\leq\varphi\leq\varphi'\\\varphi+\varphi'\leq\pi}}\int_{\theta=\varphi'-\varphi}^{\varphi+\varphi'}\frac{\sin^2\theta}{2}\left[\frac{\theta(2\pi-\theta)}{6(\pi-\theta)}+\frac{1}{\tan\theta}\right]$$

$$\left[\sin^2\varphi\sin^2\varphi' - (\cos\theta - \cos\varphi\cos\varphi')^2\right]^{\frac{d-4}{2}}(\cos\varphi\cos\varphi')^{d-1}\sin\varphi\sin\varphi'd\theta d\varphi d\varphi'.$$

In particular, for $d = 2$,

$$c_{2,2} = \frac{2\pi}{3}\left(\int_0^{2\pi}\frac{1-\cos t}{t}dt - 1\right). \tag{42}$$

Remark 3 By a very similar method, we can prove that Theorem 5 implies the general asymptotic development provided by (7).

Proof. Theorem 5 provides us the following expansion:

$$\mathbf{E}\varphi(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\left[1 - \int\overline{\mathbf{E}}(V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d)e^{-\omega_d||x||^d}dx\right.$$

$$\left.+\frac{1}{2}\int\overline{\mathbf{E}}\{(V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d)^2\}e^{-\omega_d||x||^d}dx + O(t\sqrt{t})\right]. \tag{43}$$

Applying Proposition 3 to $C = \sqrt{2t}\widehat{\mathbf{W}}$ we obtain

$$\overline{\mathbf{E}}(V_d(\sqrt{2t}\widehat{\mathbf{W}},x) - \omega_d||x||^d) = \sum_{j=1}^{d}I_{d,j}||x||^{d-j}(2t)^{j/2}\overline{\mathbf{E}}M_0^j + \frac{2^d}{d}\int_{\mathbb{S}^-}\overline{\mathbf{E}}[(\sqrt{2t}M_u - \rho(x,u))\vee 0]^d d\nu_d(u).$$

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Suppose $d \geq 3$. Since

$$\int_{\mathbb{S}^-} \overline{\mathbf{E}}[(\sqrt{2t}M_u - \rho(x,u)) \vee 0]^d d\nu_d(u) \leq (2t)^{d/2} \sigma_d \overline{\mathbf{E}} M^d,$$

$$\overline{\mathbf{E}} M_0 = \frac{\sqrt{\pi}}{2\sqrt{2}} \quad \text{and} \quad \overline{\mathbf{E}} M_0{}^2 = \frac{1}{2},$$

then there exist $t_0 > 0$ and a constant $K_d > 0$ such that the expression above is of the form

$$\overline{\mathbf{E}}\left(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d\right) = \frac{\sqrt{\pi t}}{2}||x||^{d-1}I_{d,1} + t||x||^{d-2}I_{d,2} + t\sqrt{t}A_d(x,t), \tag{44}$$

with

$$0 \leq A_d(x,t) \leq K_d\left(1 + ||x||^{d-3}\right), \quad x \in \mathbb{R}^d, \quad 0 < t < t_0. \tag{45}$$

For $d = 2$ let us note

$$K(t, ||x||) = \frac{\sqrt{2t}M_0}{||x||}, \quad t > 0, x \in \mathbb{R}^d \setminus \{0\}.$$

We have

$$\int_{\mathbb{S}^-} \overline{\mathbf{E}}[(\sqrt{2t}M_u - \rho(x,u)) \vee 0]^2 d\nu_2(u)$$

$$= 2\mathbf{E}\left\{4tM_0{}^2 \arcsin\left(K(t, ||x||) \wedge 1\right) + ||x||^2\left[\arcsin\left(K(t, ||x||) \wedge 1\right)\right.\right.$$

$$\left.\left. -K(t, ||x||)\sqrt{(1 - K(t, ||x||)^2) \wedge 0}\right] - 4||x||\sqrt{2t}M_0\left[1 - \sqrt{(1 - K(t, ||x||)^2) \wedge 0}\right]\right\}$$

$$\leq 2\overline{\mathbf{E}}\left\{4tM_0^2 \arcsin\left(K(t, ||x||) \wedge 1\right)\right.$$

$$\left. +||x||^2\left[\arcsin\left(K(t, ||x||) \wedge 1\right) - K(t, ||x||)\sqrt{(1 - K(t, ||x||)^2) \wedge 0}\right]\right\}.$$

Considering the two cases $K(t, ||x||) \geq (1/2)||x||$ and $K(t, ||x||) < (1/2)||x||$, some elementary and somewhat lengthy calculations (using in particular the existence of a constant $\alpha > 0$ verifying

$$\arcsin(x) \leq x + \alpha x^3, \quad \forall x \in [0, \frac{1}{2}].$$

and the inequality $\sqrt{1 - x^2} \geq (1 - x^2)$, $0 \leq x \leq 1$) provide the following estimation

$$\int_{\mathbb{S}^-} \overline{\mathbf{E}}[(\sqrt{2t}M_u - \rho(x,u)) \vee 0]^2 d\nu_2(u) \leq C\frac{t\sqrt{t}}{||x||}. \tag{46}$$

where $C > 0$ is a constant.

The inequality (46) shows that the formulas (44) and (45) are valid alike for $d = 2$.

Now we may write (11) on the form

$$V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d = 2^d \sqrt{2t}\int_{\mathbb{S}^+} M_u \rho(x,u)^{d-1} d\nu_d(u) + R_d(x,t)$$

with

$$0 \leq R_d(x,t) \leq \frac{2^d}{d}\sigma_d \sum_{j=2}^d \binom{d}{j}(\sqrt{2t}M)^j ||x||^{d-j}I_{d,j}.$$

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Hence there exists $t_0 > 0$ and a constant $K'_d > 0$ such that

$$\mathbf{E}\left\{(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d)^2\right\} = 4^d k_d ||x||^{2d-2} 2t + t\sqrt{t}G_d(x,t), \quad x \in \mathbb{R}^d, \ t > 0, \qquad (47)$$

with

$$k_d = \frac{1}{||x||^{2d-2}} \iint_{(\mathbb{S}^+)^2} \overline{\mathbf{E}}(M_u M_{u'})(\rho(x,u)\rho(x,u'))^{d-1} d\nu_d(u) d\nu_d(u'), \qquad (48)$$

and

$$0 \leq G_d(x,t) \leq K'_d\left(1 + ||x||^{2d-3}\right), \quad x \in \mathbb{R}^d, \quad 0 < t < t_0.$$

The covariances $\overline{\mathbf{E}}(M_u M_{u'})$ were calculated in ([10], IV.). Precisely, if $\theta \in (0, \pi]$ is the angle spanned by the two vectors $u, u' \in \mathbb{S}^{d-1}$, then

$$\overline{\mathbf{E}}(M_u M_{u'}) = H(\theta) = \frac{\sin\theta}{2}\left[\frac{\theta(2\pi - \theta)}{6(\pi - \theta)} + \frac{1}{\tan\theta}\right]. \qquad (49)$$

Inserting (49) in the integral (48) some calculation yields:

$$k_d = 4\sigma_{d-1}\sigma_{d-2} \int_{\substack{0 \leq \varphi \leq \varphi' \\ \varphi + \varphi' \leq \pi}} \int_{\theta = \varphi' - \varphi}^{\varphi + \varphi'} H(\theta)\sin\theta\left[\sin^2\varphi\sin^2\varphi' - (\cos\theta - \cos\varphi\cos\varphi')^2\right]^{\frac{d-4}{2}} d\theta$$

$$(\cos\varphi\cos\varphi')^{d-1}\sin\varphi\sin\varphi' d\varphi d\varphi', \quad d \geq 3, \quad (50)$$

$$k_2 = \frac{\pi^2}{12}\left(1 + \frac{1}{2}\int_0^{2\pi}\frac{1 - \cos u}{u}du\right). \qquad (51)$$

In order to obtain Theorem 6 it suffices now to insert formulas (44) and (47) in (43), and to proceed to some elementary calculations thanks to (50) and (51).

$$\square$$

Remark 4 The asymptotic result of Theorem 6 and the equation (5) suggest that we may have

$$c_{2,2} = \frac{4\pi}{24}(\pi\mathbf{E}\alpha^{-1}(\mathcal{C}) - \mathbf{E}N_0(\mathcal{C}) + 2).$$

U*sing the well-known equality* $\mathbf{E}N_0(\mathcal{C}) = 6$ *(see e.g. [20]) and (51), this is equivalent to the equality*

$$\mathbf{E}\alpha^{-1}(\mathcal{C}) = \frac{4}{\pi}\int_0^{2\pi}\frac{1 - \cos t}{t}dt. \qquad (52)$$

We did not find this result in the literature so for the sake of completeness we give its proof in the Appendix.

5   Concluding remarks.

(1) A part of the above arguments works in a more general setting of Johnson-Mehl tessellation [5], [14], [19]. In that model the crystals start growing radially (in all direction at fixed speed $v > 0$) at time t_i from the nuclei x_i in such a way that

$$\Phi = \{(x_i, t_i) \in \mathbb{R}^d \times [0, +\infty)\}$$

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is a spatially-homogeneous Poisson point process. At the end of growth the whole space is covered and the construction of the Johnson-Mehl tessellation is completed. The Poisson-Voronoi tessellation corresponds to the particular case when all nuclei are born at the same time. The Johnson-Mehl crystals are star-shaped but not necessarily convex, the common boundary between two crystals, which is a part of a hyperboloid, may even be disconnected.

It can be shown that for a process Φ satisfying the canonical conditions of J. Møller [19] the expectation of the spectral function of the typical Johnson-Mehl cell can also be expressed in terms of Brownian bridge from which a two-terms expansion near the origin can be obtained (see [12]).

(2) It would be interesting to obtain the geometric significance of the coefficients $c_{d,2}$ appearing in the asymptotic of Theorem 6. In view of (6) it is likely that $c_{d,2}$ is connected with

$$\mathbf{E}\left\{ \sum_{i=1}^{N_{d-2}(\mathcal{C})} \frac{1}{3} \left(\frac{\omega_i(\mathcal{C})}{\pi} - \frac{\pi}{\omega_i(\mathcal{C})} \right) V_{d-2}(F_i(\mathcal{C})) \right\},$$

where

(i) $F_i(\mathcal{C})$, $i = 1, \cdots, N_{d-2}(\mathcal{C})$ are the $(d-2)$-dimensional faces of $\mathcal{C}$;

(ii) $\omega_i(\mathcal{C})$ is the magnitude of the dihedral angle at the face $F_i(\mathcal{C})$, $1 \leq i \leq N_{d-2}(\mathcal{C})$.

(3) To obtain the values of coefficients $c_{d,i}$, $i \geq 3$, in (7) it is necessary to calculate explicitely the covariances $\overline{\mathbf{E}} M_{u_1} \cdots M_{u_j}$, $u_1, \cdots, u_j \in \mathbb{S}^{d-1}$, $j \geq 3$, associated to the d-dimensional Brownian bridge. At our knowledge this problem is open. Note also that for a bounded convex polyhedron of $\mathbb{R}^d$ the explicit expressions of the coefficients at order $k \geq 4$ appearing in the asymptotic (near the origin) of the spectral function are unknown (see [8]).

(4) for $d = 2$, Theorem 2 can be proved without the Donsker-Varadhan theorem. Indeed, we can use the estimation provided in [11] for the distribution of the perimeter of the convex hull of the Brownian bridge $\widehat{\mathbf{W}}$.

(5) It is interesting to note the significance of Lemma 1. The inequality (46) can be rewritten under the form

$$\mathbf{E}\overline{\mathbf{P}^{\mathbf{x}}}\{T > u\} \leq \mathbf{E}\overline{\mathbf{P}^{\mathbf{0}}}\{T > u\}, \quad u \geq 0,$$

where T denotes the first exit time of the Brownian bridge from the cell $C(0)$, the notation $\overline{\mathbf{P}^{\mathbf{x}}}$ expressing the fact that the Brownian path is starting at the point $x \in \mathbb{R}^d$. If we replace the Brownian bridge by the standard Brownian motion in $\mathbb{R}^d$ in the proof of Theorem 1 we obtain the corresponding inequality

$$\mathbf{E}\overline{\mathbf{P}^{\mathbf{x}}}\{\tau > u\} \leq \mathbf{E}\overline{\mathbf{P}^{\mathbf{0}}}\{\tau > u\}, \quad u \geq 0,$$

for the first Brownian exit time τ of $C(0)$.

(6) It is well known that "the larger regions have smaller eigenvalues" and therefore the equalities (3) and (4) express that in some sense the large Voronoi cells are nearly circular. An analogous phenomenon (known under the name of D. G. Kendall conjecture) occurs for the polygons determined by a standard Poisson line process in the plane (see [11], [16]).

Appendix. Proof of (52).

Let us recall that by associating to each vertex s of the planar Poisson-Voronoi tessellation the triangle $T(s)$ whose vertices are the nuclei of the three cells containing s (and whose center of circumdisc coincides precisely with s), we obtain the dual tessellation, called the Delaunay tessellation (see [20]). The typical Delaunay cell $\mathcal{D}$ is defined (in Palm sense) by the following formula [20]:

$$\mathbf{E}h(\mathcal{D}) = \frac{1}{\lambda_0 V_2(B)} \mathbf{E} \sum_{s \in \mathcal{S} \cap B} h(T(s) - s), \tag{53}$$

for all measurable function $h : \mathcal{K} \longrightarrow \mathbb{R}_+$, and where:

(i) $B \subset \mathbb{R}^2$ is an arbitrary fixed Borel set such that $0 < V_2(B) < +\infty$;

(ii) $\mathcal{S}$ is the set of vertices of the Poisson-Voronoi tessellation.

Note that the mean number of vertices per unit area is equal to 2 (see [20]).

The distribution of the typical cell $\mathcal{D}$, which is a triangle noted $z_1 z_2 z_3$, is known explicitly by means of the distributions of the radius $\rho(\mathcal{D})$ of the circumdisc of $\mathcal{D}$ and of the three angles $\beta_1(\mathcal{D}) = \widehat{z_1 p z_2}$, $\beta_2(\mathcal{D}) = \widehat{z_2 p z_3}$, $\beta_3(\mathcal{D}) = \widehat{z_1 p z_2}$, where p is the center of the circumdisc of $\mathcal{D}$ (see [20], p. 104, for an expression of these distributions valid in any dimension and [21], p. 249, for a rewriting in dimension 2). In particular, these angles are identical in law and independent of $\rho(\mathcal{D})$. This implies that the angles $\alpha_1(\mathcal{D}), \alpha_2(\mathcal{D}), \alpha_3(\mathcal{D})$ spanned in p by any two edges of the Voronoi tessellation, are independent of $\rho(\mathcal{D})$ and of common distribution (see for example [21]):

$$\alpha_1(\mathcal{D})(P)(dt) = \frac{4}{3\pi} \sin t (\sin t - t \cos t) 1_{[0,\pi]}(t) dt. \tag{54}$$

Thus from (54) we get the following result:

Lemma 3 *We have:*

$$\mathbf{E}\frac{1}{\alpha_1(\mathcal{D})} = \frac{2}{3\pi} \int_0^{2\pi} \frac{1 - \cos t}{t} dt.$$

Let us consider now the set $\mathcal{A}$ of angles of the Poisson-Voronoi tessellation and for each $\alpha \in \mathcal{A}$, let $s(\alpha)$ be the associated vertex. Then noticing that the mean number of angles per unit area is equal to six we can define a typical angle $\overline{\alpha}$ on the following lines:

$$\mathbf{E}1_{[0,t]}(\overline{\alpha}) = \frac{1}{6V_2(B)} \mathbf{E} \sum_{\substack{\alpha \in \mathcal{A} \\ s(\alpha) \in B}} 1_{[0,t]}(\alpha), \quad 0 \leq t \leq \pi, \tag{55}$$

where B is a fixed Borel set of $\mathbb{R}^2$ such that $0 < V_2(B) < \infty$.

The typical angle $\overline{\alpha}$ can be connected, on one hand to the angles of the Voronoi typical cell $\mathcal{C}$, and on the other hand to the angles $\alpha_1(\mathcal{D}), \alpha_2(\mathcal{D}), \alpha_3(\mathcal{D})$ of the Delaunay typical cell $\mathcal{D}$. More precisely, we have:

Lemma 4 (i) *The angles $\overline{\alpha}$ et $\alpha_1(\mathcal{D})$ are identical in law.*

(ii) *for any measurable function $f : [0, \pi] \longrightarrow \mathbb{R}_+$, we have*

$$\mathbf{E}f(\overline{\alpha}) = \frac{1}{6} \mathbf{E} \sum_{i=1}^{N_0(\mathcal{C})} f(\alpha_{\mathcal{C},i}),$$

where $\alpha_{\mathcal{C},i}$, $1 \leq i \leq N_0(\mathcal{C})$, denote the inside-facing angles of the typical cell $\mathcal{C}$.

Proof. (i) Consider $0 \leq t \leq \pi$ and $B \in \mathcal{B}(\mathbb{R}^2)$, $V_2(B) = 1$.
Then by definition (53) of $\mathcal{D}$, we have

$$\frac{1}{3}\mathbf{E}\sum_{i=1}^{3} 1_{[0,t]}(\alpha_i(\mathcal{D})) = \frac{1}{6}\mathbf{E}\sum_{s \in \mathcal{S} \cap B}\left(\sum_{i=1}^{3} 1_{[0,t]}(\alpha_i(s))\right)$$

where $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$ are the three angles associated with the vertex s in the tessellation.
Moreover, by definition of the typical angle (see (55)),

$$\frac{1}{6}\mathbf{E}\sum_{s \in \mathcal{S} \cap B}\left(1_{[0,t]}(\alpha_1(s)) + 1_{[0,t]}(\alpha_2(s)) + 1_{[0,t]}(\alpha_3(s))\right) = \mathbf{E}1_{[0,t]}(\overline{\alpha}).$$

We conclude then by using the identity in law of the three angles $\alpha_1(\mathcal{D})$, $\alpha_2(\mathcal{D})$ and $\alpha_3(\mathcal{D})$.
(ii) The proof is similar to that of Prop. 3.2.2. of J. Møller [20] which connects, for $1 \leq k \leq d$,, the
k-dimensional typical face of the Voronoi tessellation to the k-dimensional faces of the typical cell.

$\square$

From Lemmas 3 and 4, we deduce that

$$\mathbf{E}\alpha^{-1}(\mathcal{C}) = 6\mathbf{E}\left(\frac{1}{\overline{\alpha}}\right) = 6\mathbf{E}\left(\frac{1}{\alpha_1(\mathcal{D})}\right) = \frac{4}{\pi}\int_0^{2\pi}\frac{1 - \cos t}{t}dt,$$

which completes the proof of (52).

$\square$

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