Large deviation probabilities for the number of vertices of random polytopes in the ball *

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Abstract

In this paper, we establish large-deviation results on the number of extreme points of a homogeneous Poisson point process in the unit ball of $\mathbb{R}^d$. In particular, we deduce an almost-sure law of large numbers in any dimension. As an auxiliary result we prove strong localization of the extreme points in an annulus near the boundary of the ball.

Introduction and main results

Let us denote by $U_1, \cdots, U_n$, $n \in \mathbb{N}^*$, n independent and uniformly distributed variables in the unit ball $\mathbb{B}^d$ of the Euclidean space $\mathbb{R}^d$, $d \geq 2$, and let X_t, $t > 0$, be a homogeneous Poisson point process in $\mathbb{B}^d$ of intensity measure $(t/\omega_d)\mathbf{1}_{\mathbb{B}^d}(x)dx$ where ω_d is the Lebesgue measure of the unit ball. We consider the convex hull C_n (resp. C_t) of the set $\{U_1, \cdots, U_n\}$ (resp. of X_t) and N_n (resp. $\widetilde{N}_t$) its number of vertices. The asymptotic behaviour of N_n when $n \to +\infty$ has been widely investigated in the literature. For $d = 2$ Rényi and Sulanke [10] obtained in 1963 the convergence of means of N_n. Their work has been followed by Efron [5], Buchta & Müller [2], Groeneboom [6] who obtained a central limit theorem and Massé [8] who proved a law of large numbers in probability. More recently, estimating precisely the variance of N_n for all dimensions $d \geq 2$, Reitzner [9] deduced an almost-sure convergence for the number of vertices of random polyhedra in any convex set of $\mathbb{R}^d$, $d \geq 4$, with a $\mathcal{C}^2$ boundary and positive Gaussian curvature. For all $d \geq 2$ the asymptotic behaviour of $\mathbb{E}N_n$ is known to be

$$\mathbb{E}N_n \sim c_d n^{(d-1)/(d+1)}. \tag{1}$$

where the dimension-dependent constant c_d is known explicitly, see Wieacker [13] (for $d = 3$), Bárány [1], Schütt [12] as well as (7) in Reitzner [9] and the references therein. Note that $\alpha_s \sim \beta_s$ stands for $\lim_{s\to\infty} \alpha_s/\beta_s = 1$. Throughout the paper we make use of 'O, Ω, Θ' notation. Recall that $O(X)$ stands for quantities bounded above by X multiplied by a constant, $\Omega(X)$ for quantities bounded below by X multiplied by a constant, while $\Theta(X) = O(X) \cap \Omega(X)$.

Let us remark that (1) implies the same type of asymptotics for $\widetilde{N}_t$ when $t \to +\infty$, i.e.

$$\mathbb{E}\widetilde{N}_t \sim c_d t^{(d-1)/(d+1)}. \tag{2}$$

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Indeed, using that a homogeneous Poisson process in $\mathbb{B}^d$ of intensity t coincides in distribution with $\{U_1, \cdots, U_S\}$ where S is a Poisson variable with mean t, we obtain

$$|E\widetilde{N}_t - \sum_{|k-t|\leq t^{2/3}} \mathbb{E}N_k \mathbb{P}\{S=k\}| \leq \sum_{|k-t|\leq t^{2/3}} k\mathbb{P}\{S=k\} = t\mathbb{P}\{S \notin [t-1-t^{2/3}, t-1+t^{2/3}]\}.$$

It remains to use standard moderate-deviation results on the Poisson distribution to prove that the RHS goes to zero when $t \to +\infty$ and that the sum in the LHS tends to $c_d t^{(d-1)/(d+1)}$.

The purpose of this paper is to establish the following large-deviation results for N_n (resp. $\widetilde{N}_t$):

Theorem 1 *For each $\varepsilon > 0$,*

$$\liminf_{n\to\infty} \frac{1}{\log n} \log\left(-\log \mathbb{P}\left\{\left|\frac{N_n}{\mathbb{E}N_n} - 1\right| > \varepsilon\right\}\right) \geq \frac{d-1}{3d+5}. \tag{3}$$

Theorem 2 *For each $\varepsilon > 0$,*

$$\liminf_{t\to\infty} \frac{1}{\log t} \log\left(-\log \mathbb{P}\left\{\left|\frac{\widetilde{N}_t}{\mathbb{E}\widetilde{N}_t} - 1\right| > \varepsilon\right\}\right) \geq \frac{d-1}{3d+5}. \tag{4}$$

Let us remark that these results are of the same type as the concentration results for volumes of unions of random closed sets obtained in [11]. We believe that the concentration rate $1 - \frac{2d+6}{3d+5}$ on the RHS of (1) and (2) is not optimal and we conjecture that the optimal value should be $\frac{d-1}{d+1}$, coinciding with the exponent determining the asymptotics of the expected number of vertices. However, we were not able to verify this conjecture with our current methods.

In particular, we deduce the almost sure law of large numbers for N_n (resp. $\widetilde{N}_t$) in any dimension $d \geq 2$.

Corollary 1 *We have*

$$\lim_{n\to\infty} N_n/\mathbb{E}N_n = 1 \ \ a.s.$$

and

$$\lim_{t\to\infty} \widetilde{N}_t/\mathbb{E}\widetilde{N}_t = 1 \ \ a.s..$$

Corollary 1 is a direct consequence of Theorems 1 and 2 and the Borel-Cantelli lemma.

Our technique of proof strongly relies on the localization of extreme points in a small annulus near the boundary of the unit ball, which allows us to use a standard concentration of measure result due to Ledoux [7]. The following proposition shows that with an overwhelming probability, going exponentially fast to one, the vertices of the convex hull of the points inside the ball are located in an annulus centered at the origin of thickness of order $n^{-2/(d+1)}$ (resp. $t^{-2/(d+1)}$).

Here $B(r)$, $r > 0$, denotes the ball centered at the origin and of radius r. In the sequel we agree to use c to denote a positive constant, possibly varying between different occurrences of c.

Proposition 1 *(i) There exist constants $c > 0$, $K > 0$ such that for every $0 < \alpha < 2/(d+1)$, we have*

$$\mathbb{P}\{B(1-Kn^{-\alpha}) \not\subseteq C_n\} = O(\exp(-cn^{1-\alpha(d+1)/2})). \tag{5}$$

(ii) In the same way, we have

$$\mathbb{P}\{B(1-Kt^{-\alpha}) \not\subseteq \widetilde{C}_t\} = O(\exp(-ct^{1-\alpha(d+1)/2})). \tag{6}$$

Our main motivation is the extension to higher dimensions $d \geq 2$ of our previous results in the Euclidean plane on the number of sides [4] and the radius of the circumscribed ball [3] of the typical Poisson-Voronoi cell. Indeed, we established a connection between the sides of the typical cell and the extreme points of an inhomogeneous Poisson point process in the unit ball via an action of the classical inversion. In dimension $d \geq 3$, the same argument provides a relation between the number of hyperfaces (resp. the radius of the circumscribed ball) of the typical cell and the number of vertices (resp. the inradius) of the convex hull of the Poisson process inside the ball. We will deduce from Theorem 2, Corollary 1 and Proposition 1 some new results on the geometry of the typical Poisson-Voronoi cell that will be developed in a future work.

The paper is structured as follows. We first obtain an auxiliary proposition stating the localization of the extreme points near the boundary of the ball. Using concentration of measure arguments due to Ledoux, we then prove the main large deviation result for the number N_n (Theorem 1). We deduce Theorem 2 from Theorem 1 and a large-deviation property of the Poisson distribution. Finally, concluding remarks are listed about the extensions of these results.

1 Proof of Proposition 1.

(i). For a fixed $u_0 \in \mathbb{S}^{d-1}$ ($\mathbb{S}^{d-1}$ being the unit sphere of $\mathbb{R}^d$), let $S_n = \sup_{1\leq i\leq n} U_i \cdot u_0$, where $\cdot$ denotes the usual scalar product in the Euclidean space $\mathbb{R}^d$. There exists $c > 0$ such that for every $\alpha \in (0, 2/(d+1))$, we have

$$\mathbb{P}\{S_n \leq 1 - n^{-\alpha}\} = O(\exp(-cn^{1-(d+1)\alpha/2})). \tag{7}$$

Indeed, for a fixed $\alpha \in (0, 2/(d+1))$, we have that

$$\mathbb{P}\{S_n \leq 1 - n^{-\alpha}\} = \left(1 - \frac{V_d(\{x \in \mathbb{B}^d; x \cdot u_0 > 1 - n^{-\alpha}\})}{\omega_d}\right)^n, \tag{8}$$

where V_d and ω_d are respectively the Lebesgue measure in $\mathbb{R}^d$ and the volume of the unit ball $\mathbb{B}^d$. By an elementary computation, we obtain

$$\begin{aligned} V_d(\{x \in \mathbb{B}^d; x \cdot u_0 > 1 - n^{-\alpha}\}) &= \omega_{d-1}\int_0^{\arccos(1-n^{-\alpha})} \sin^d(\theta)d\theta \\ &\underset{n\to+\infty}{\sim} \frac{\omega_{d-1}}{d+1}2^{(d+1)/2}n^{-\alpha(d+1)/2}. \end{aligned} \tag{9}$$

Combining (8) with (9), we deduce the result (7).

We consider then for a fixed $\alpha \in (0, 2/(d+1))$ a deterministic covering of the sphere $\mathbb{S}^{d-1}$ by spherical caps of height $n^{-\alpha}$ (i.e. of angular radius equal to $\arccos(1 - n^{-\alpha})$) such that the total number of caps is of order $\Theta(n^{\frac{\alpha}{2}(d-1)})$. In addition, we suppose that every cap intersects at most a fixed number ς of other caps. Let us remark that the existence of such a covering can be proved by induction over the dimension d.

Indeed, let us suppose that for every $\varepsilon > 0$, a covering of $\mathbb{S}^{d-1}$ by $N_d^{\text{cap}}(\varepsilon)$ spherical caps of angular radius $\varepsilon > 0$ exists. Then we can construct a covering of the cylinder $\mathcal{C} = \mathbb{S}^{d-1} \times [-\pi/2, \pi/2] \subseteq \mathbb{R}^{d+1}$ as follows : for every integer k with $|k| \leq \lfloor\frac{\pi}{\varepsilon}\rfloor$ we use the induction hypothesis to choose $N_d^{\text{cap}}(\varepsilon/2)$ caps of radius $\varepsilon/2$ on the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1} \times \{k\varepsilon/2\}$, fully covering this sphere and centered at $a_1^k, \cdots, a^k_{N_d^{\text{cap}}(\varepsilon/2)}$. A covering by $(2\lfloor\pi/\varepsilon\rfloor + 1)N_d^{\text{cap}}(\varepsilon/2)$ balls of the set $\mathcal{C}$ is then obtained by considering the balls of radius ε and centered at the points a_i^j, $1 \leq i \leq N_d^{\text{cap}}(\varepsilon/2)$, $-\lfloor \frac{\pi}{\varepsilon} \rfloor \leq j \leq \lceil \frac{\pi}{\varepsilon} \rceil$. To proceed, note that the mapping $(u, \theta) \longrightarrow (u \sin\theta + \mathbf{e}_{d+1} \cos\theta)$, where $\mathbf{e}_{d+1} = (0, \cdots, 0, 1) \in \mathbb{R}^{d+1}$, is a surjection from $\mathcal{C}$ onto the unit sphere $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$ satisfying the Lipschitz condition with constant 1. This observation allows us to transform the above covering of $\mathcal{C}$ into a covering of $\mathbb{S}^d$ with $(2\lfloor \pi/\varepsilon \rfloor + 1) N_d^{\text{cap}}(\varepsilon/2)$ caps of radius ε. By induction, it means that $\mathbb{S}^{d-1}$ can be covered with $\Theta(\varepsilon^{d-1})$ spherical caps of radius ε. Moreover, the proof above also shows that the covering thus constructed satisfies the requirement that every cap intersects at most a fixed number ς of other caps.

Let $\mathcal{D}_n$ be the event that the set $\{U_1, \cdots, U_n\}$ intersects the interiors of all the caps of the covering. Since the number of caps is polynomial in $n^{\alpha/2}$ and the probability that $\{U_1, \cdots, U_n\}$ does not intersect one cap is bounded subexponentially by the estimate (7), we obtain that

$$\mathbb{P}(\mathcal{D}_n^c) = O(\exp(-cn^{1-(d+1)\alpha/2})),$$

where c is a positive constant. In order to get (5), it remains to notice that there exists a positive constant K such that the hyperplanes spanned by the facets of a polyhedron with a vertex in each cap are at least at distance $(1 - Kn^{-\alpha})$ from the origin, which means that

$$\mathcal{D}_n \subset \{C_n \supset B(1 - Kn^{-\alpha})\}.$$

Let us remark that the constant K can be taken equal to 4. Indeed, the interior of any circular cap of height $4n^{-\alpha}$ contains at least one cap of height $n^{-\alpha}$ of the initial covering (since the angular radius of the larger cap, $\arccos(1 - 4n^{-\alpha})$, is greater than the angular diameter of the smaller cap, i.e. $2\arccos(1 - n^{-\alpha})$). Being in $\mathcal{D}_n$ implies then that any cap of height $4n^{-\alpha}$ contains a point of $\{U_1, \cdots, U_n\}$ in its interior and, consequently, that any facet of the convex hull of $\{U_1, \cdots, U_n\}$ is at least at a distance $(1-4n^{-\alpha})$ from the origin. This completes the argument.

(ii). Replacing (1) by the equality

$$\mathbb{P}\{\widetilde{S}_t \leq 1 - t^{-\alpha}\} = \exp(-t \cdot V_d(\{x \in \mathbb{B}^d; x \cdot u_0 > 1 - t^{-\alpha}\})),$$

where $\widetilde{S}_t = \sup_{x \in X_t} x \cdot u_0$, the proof of Proposition 1 for the Poisson point process is very similar to what we did for the binomial point process.

□

2 Proof of Theorem 1

An important obstacle in the study of the number of vertices of convex hulls of large samples is that adding a new vertex may discard an arbitrarily large number of other vertices. The idea underlying this proof is to circumvent this difficulty by providing an appropriate *artificial* modification of the number of vertices functional N_n which, while being very close to N_n, is better behaved as satisfying a Lipschitz-type condition so that appropriate measure concentration tools can be applied.

To proceed with this construction, we choose $0 < \alpha < 2/(d+1)$ and $\beta \in (1 - \alpha(d+1)/2, 1)$ and we construct the functional $N_n^{\alpha,\beta}$ in the following way. Using the same type of covering as in the proof of Proposition 1, we can cover the shell $\mathbb{B}^d \setminus B(1 - Kn^{-\alpha})$ (with K given by Proposition 1) with a number of order $\Theta(n^{\alpha((d+1)/2-1)})$ of equal-sized spherical caps of volume $\Theta(n^{-\alpha(d+1)/2})$ each and such that from each point of $\mathbb{B}^d \setminus B(1 - Kn^{-\alpha})$ only a fixed number ς of caps are seen within $\mathbb{B}^d \setminus B(1 - Kn^{-\alpha})$. For a sample $\mathcal{X}$ in $\mathbb{B}^d$, within each of the spherical caps Ξ constructed above we observe the subsample $\Xi \cap \mathcal{X}$ and, in case $\#(\Xi \cap \mathcal{X}) > n^\beta$ (we say that Ξ is overfull in such case), we order the points of $\Xi \cap \mathcal{X}$ in a certain deterministic way (e.g. by decreasing distance to the origin) and we reject those with their numbers exceeding n^β. We shall refer to this procedure as to the overfull-rejection. We also reject all the sample points falling outside $\mathbb{B}^d \setminus B(1 - Kn^{-\alpha})$. Let us notice that a sample point of $\mathcal{X}$ can be rejected more than once, if belonging to several different caps. Writing $\widehat{\mathcal{X}}$ for the so reduced sample we define the functional $\Phi_n^{\alpha,\beta}(\mathcal{X})$ to be the number of vertices of the convex hull of $\widehat{\mathcal{X}}$. A crucial observation is that adding or removing (consequently, also moving) a single point in the sample $\mathcal{X}$ may change the value of $\Phi_n^{\alpha,\beta}(\mathcal{X})$ by at most $\Theta(n^\beta)$. To see it note first that when adding a new point x we encounter the following four possibilities:

- $x \in \mathrm{conv}(\hat{\mathcal{X}})$ and x does not fall into an overfull region; in this case the value of $\Phi_n^{\alpha,\beta}$ remains unchanged,
- $x \in \mathrm{conv}(\hat{\mathcal{X}})$ but x falls into an overfull region. If x is rejected, $\Phi_n^{\alpha,\beta}$ does not change, otherwise x causes overfull-rejection of another point and may possibly become itself a new vertex of the convex hull of the reduced sample, possibly discarding some vertices of $\mathrm{conv}(\hat{\mathcal{X}})$. Both these changes result in $\Phi_n^{\alpha,\beta}$ changing by at most $\Theta(n^\beta)$ because at most $\Theta(n^\beta)$ points of the reduced sample can be seen from any given point of $\mathbb{B}^d \setminus B(1 - Kn^{-\alpha})$,
- $x \notin \mathrm{conv}(\hat{\mathcal{X}})$ and x does not fall into an overfull region; in this case x becomes a new vertex, discarding at most $\Theta(n^\beta)$ vertices of $\hat{\mathcal{X}}$,
- $x \notin \mathrm{conv}(\hat{\mathcal{X}})$ and x does fall into an overfull region. If x is itself rejected, nothing changes, otherwise x becomes a new vertex, possibly causing overfull-rejection of another vertex of $\mathrm{conv}(\hat{\mathcal{X}})$ and possibly discarding some vertices of $\mathrm{conv}(\hat{\mathcal{X}})$. As above, both these changes result in $\Phi_n^{\alpha,\beta}$ changing by at most $\Theta(n^\beta)$ since at most $\Theta(n^\beta)$ points of the reduced sample can be seen from any given point of $\mathbb{B}^d \setminus B(1 - Kn^{-\alpha})$.

A similar argument shows that also removing a sample point results in overall change of $\Phi_n^{\alpha,\beta}$ by at most $\Theta(n^\beta)$.

From now on, let us consider

$$N_n^{\alpha,\beta} := \Phi_n^{\alpha,\beta}(\{U_1, \cdots, U_n\}).$$

The proof of Theorem 1 is divided into three steps. Using Proposition 1, we show in Lemma 1 that $N_n^{\alpha,\beta}$ is a good approximation of N_n. Then we give in Lemma 2 some concentration properties on the number of points of the sample $\{U_1, \cdots, U_n\}$ falling into the annulus $\mathbb{B}^d \setminus B(1 - Kn^{-\alpha})$. In Lemma 3, which is the key result of our proof, we deduce from Lemma 2 and a classical measure concentration result a large-deviation property for $N_n^{\alpha,\beta}$. Theorem 1 is then easily concluded from Lemmas 1 and 3.

Lemma 1 *There exists a positive constant c such that*

$$\mathbb{P}(N_n \neq N_n^{\alpha,\beta}) \leq O\left(e^{-cn^{1-\alpha\frac{d+1}{2}}}\right).$$

In particular, $|\mathbb{E}(N_n^{\alpha,\beta}) - \mathbb{E}(N_n)| \leq n\mathbb{P}(N_n \neq N_n^{\alpha,\beta}) = O\left(e^{-cn^{1-\alpha\frac{d+1}{2}}}\right)$.

Proof. Let $\mathcal{A}_n$ and $\mathcal{B}_n$ be the events such that, respectively, there is at least one extreme point of $\{U_1, \cdots, U_n\}$ in $B(1 - Kn^{-\alpha})$ and there is at least one spherical cap containing more than n^β points.

Using Proposition 1, we have

$$\mathbb{P}(\mathcal{A}_n) = O(e^{-cn^{1-\alpha(d+1)/2}}). \tag{10}$$

Moreover, denoting by $\text{Bin}(n,p)$ a binomial variable with parameters n and p, we obtain that

$$\mathbb{P}(\mathcal{B}_n) \leq \Psi_n \mathbb{P}\{\text{Bin}(n, v_n) \geq n^\beta\}, \tag{11}$$

where Ψ_n is the number of spherical caps and v_n is the Lebesgue measure of a single cap divided by ω_d.

Using the Legendre transform, we have

$$\begin{aligned} \mathbb{P}\{\text{Bin}(n, v_n) \geq n^\beta\} &\leq \inf_{t\geq 0}\left\{e^{-tn^\beta}\mathbb{E}(e^{t\text{Bin}(n,v_n)})\right\} \\ &= \inf_{t\geq 0}\left\{\exp\{-tn^\beta + n\log(e^t v_n + 1 - v_n)\}\right\} \\ &= \exp\left\{n\log\left((1-v_n)\frac{n^\beta}{n-n^\beta} + 1 - v_n\right) - n^\beta \log\left((v_n^{-1}-1)\frac{n^\beta}{n-n^\beta}\right)\right\}, \end{aligned}$$

where the last equality is obtained by taking $t = \log((v_n^{-1} - 1)n^\beta/(n - n^\beta))$.

Since $v_n = \Theta(n^{-\alpha(d+1)/2})$, it follows that

$$\mathbb{P}(\text{Bin}(n, v_n) \geq n^\beta) = O(e^{-cn^\beta}).$$

Combining (11) with the estimate $\Psi_n = O(n^{\alpha(d-1)/2})$, we deduce that

$$\mathbb{P}(\mathcal{B}_n) = O(e^{-cn^\beta}). \tag{12}$$

Besides, it comes from the definition of $N_n^{\alpha,\beta}$ that

$$\mathbb{P}\{N_n \neq N_n^{\alpha,\beta}\} \leq \mathbb{P}(\mathcal{A}_n) + \mathbb{P}(\mathcal{B}_n). \tag{13}$$

Inserting the estimations (10) and (12) in (13), the proof of Lemma 1 is completed.

□

Let M_n be the number of U_i, $1 \leq i \leq n$, contained in $\mathbb{B}^d \setminus B(1-Kn^{-\alpha})$. M_n has then a binomial distribution with parameters n and w_n, where $w_n = V_d(\mathbb{B}^d \setminus B(1 - Kn^{-\alpha}))/\omega_d$.

The following lemma collects some technical estimates for M_n needed for the proof of Lemma 3 below.

Lemma 2 *(i) With* $k_n^- = \lfloor nw_n - n^{1-\frac{2}{d+1}} \rfloor$ *and* $k_n^+ = \lceil nw_n + n^{1-\frac{2}{d+1}} \rceil$, *we have*

$$\liminf_{n\to+\infty} \frac{1}{\log(n)} \log(-\log \mathbb{P}\{M_n \notin [k_n^-, k_n^+]\}) \geq 1 + \alpha - 4/(d+1). \tag{14}$$

(ii) Besides,

$$\lim_{n\to+\infty} \sup_{k\in[k_n^-,k_n^+]} \left| \frac{\mathbb{E}(N_n^{\alpha,\beta}|M_n = k)}{\mathbb{E}(N_n^{\alpha,\beta})} - 1 \right| = 0. \tag{15}$$

Proof. (i) Using Tchebychev's inequality, we get

$$\begin{aligned}\mathbb{P}\{M_n \notin [k_n^-, k_n^+]\} &\le \inf_{u\ge 1} u^{-k_n^+}(w_n u + 1 - w_n)^n + \inf_{0<v<1} v^{-k_n^-}(w_n v + 1 - w_n)^n \\ &\le \left(1+\frac{n^{-\frac{2}{d+1}}}{w_n}\right)^{-k_n^+}\left(1+n^{-\frac{2}{d+1}}\right)^n + \left(1-\frac{n^{-\frac{2}{d+1}}}{w_n}\right)^{-k_n^-}\left(1-n^{-\frac{2}{d+1}}\right)^n,\end{aligned}$$

where we set $u := (1+n^{-\frac{2}{d+1}}/w_n)$ and $v := (1-n^{-\frac{2}{d+1}}/w_n)$. Taking logarithms of both sides we come to

$$-\log \mathbb{P}\{M_n \notin [k_n^-, k_n^+]\} \ge -\log 2$$

$$-\max\left(n\log\left(1+n^{-\frac{2}{d+1}}\right) - k_n^+\log\left(1+\frac{n^{-\frac{2}{d+1}}}{w_n}\right), n\log\left(1-n^{-\frac{2}{d+1}}\right) - k_n^-\log\left(1-\frac{n^{-\frac{2}{d+1}}}{w_n}\right)\right).$$

Thus, applying the second-order Taylor expansion $\log(1+x) = x - x^2/2 + o(x^2)$ yields

$$-\log \mathbb{P}\{M_n \notin [k_n^-, k_n^+]\} \ge -\log 2$$

$$-\max\left\{n\cdot\left(n^{-\frac{2}{d+1}} + O(n^{-\frac{4}{d+1}})\right) - [nw_n + n^{1-\frac{2}{d+1}}]\left(\frac{n^{\frac{-2}{d+1}}}{w_n} - \frac{n^{\frac{-4}{d+1}}}{2w_n^2} + o\left(\frac{n^{\frac{-4}{d+1}}}{w_n^2}\right)\right),\right.$$

$$\left. n\cdot\left(-n^{-\frac{2}{d+1}} + O(n^{-\frac{4}{d+1}})\right) - [nw_n - n^{1-\frac{2}{d+1}}]\left(-\frac{n^{\frac{-2}{d+1}}}{w_n} - \frac{n^{\frac{-4}{d+1}}}{2w_n^2} + o\left(\frac{n^{\frac{-4}{d+1}}}{w_n^2}\right)\right)\right\}$$

$$= \frac{n^{1-\frac{4}{d+1}}}{2w_n}[1+o(1)] + O(n^{1-\frac{4}{d+1}}).$$

Using the estimate $w_n \sim dKn^{-\alpha}$ when $n \to +\infty$, we obtain the required asymptotic result (14).

(ii) For $k \in \mathbb{N}$ denote by $\widehat{C}_k^n$ (resp. $N_{n,k}$) the convex hull (resp. the number of extreme points) of k i.i.d. points uniformly distributed in $\mathbb{B}^d \setminus B(1-Kn^{-\alpha})$. Conditionally on $\{M_n = k\}$, when the convex hull $\widehat{C}_k^n$ is strictly smaller than C_n, it implies that C_n does not contain $\mathbb{B}^d \setminus B(1-Kn^{-\alpha})$. Consequently, we have

$$\begin{aligned}0 \le \mathbb{E}\{N_n | M_n = k\} - \mathbb{E}\{N_{n,k}\} &\le \mathbb{E}\{N_n \mathbf{1}_{\mathcal{A}_n} | M_n = k\} \\ &\le n\mathbb{P}(\mathcal{A}_n | M_n = k),\end{aligned} \tag{16}$$

where $\mathcal{A}_n = \{C_n \not\supset B(1-Kn^{-\alpha})\}$ is the event already defined at the beginning of the proof of Lemma 1.

As in the proof of Proposition 1 (i) (i.e. $\lim_{n\to+\infty}\mathbb{P}\{\mathcal{A}_n\} = 0$), the covering with Ψ_n spherical caps of $\mathbb{B}^d \setminus B(1-Kn^{-\alpha})$ can be exploited to deduce that

$$\mathbb{P}(\mathcal{A}_n | M_n = k) \le \Psi_n\left(1 - \frac{V_d(x \in \mathbb{B}^d \setminus B(1-Kn^{-\alpha}); x\cdot u_0 > 1 - n^{-\alpha}\})}{V_d(\mathbb{B}^d \setminus B(1-Kn^{-\alpha}))}\right)^{k_n^-}, \; u_0 \in \mathbb{S}^{d-1}.$$

Still following the proof of Proposition 1 (i), we combine this last inequality with (16) to obtain

$$\lim_{n\to+\infty} \sup_{k\in[k_n^-, k_n^+]} \left(\mathbb{E}\{N_n | M_n = k\} - \mathbb{E}\{N_{n,k}\}\right) = \lim_{n\to+\infty} n \sup_{k\in[k_n^-, k_n^+]} \mathbb{P}(\mathcal{A}_n | M_n = k) = 0. \tag{17}$$

Besides, using Efron's equality (3.7) in [5] for C_{n-1} and $\widehat{C}^n_{k-1}$, we get

$$\frac{\mathbb{E}(N_n)}{\mathbb{E}(N_{n,k})} = \frac{n(1-\mathbb{E}(V_d(C_{n-1}))/\omega_d)}{k\left(1-\frac{\mathbb{E}(V_d(\widehat{C}^n_{k-1}\setminus B(1-Kn^{-\alpha})))}{V_d(\mathbb{B}^d\setminus B(1-Kn^{-\alpha}))}\right)}. \tag{18}$$

Combining (18) with (17), it follows that, uniformly in $k \in [k_n^-, k_n^+]$,

$$\frac{\mathbb{E}(N_n)}{\mathbb{E}(N_n|M_n=k)} = \frac{n(1-\mathbb{E}(V_d(C_{n-1}))/\omega_d)}{k\left(1-\frac{\mathbb{E}(V_d(\widehat{C}^n_{k-1}\setminus B(1-Kn^{-\alpha})))}{V_d(\mathbb{B}^d\setminus B(1-Kn^{-\alpha}))}\right)} + o(1). \tag{19}$$

We claim that

$$\mathbb{E}V_d(\mathbb{B}^d \setminus [\widehat{C}^n_{k-1} \cup B(1-Kn^{-\alpha})]) \sim \mathbb{E}V_d(\mathbb{B}^d \setminus C_{n-1}), \tag{20}$$

uniformly for $k \in [k_n^-, k_n^+]$. Indeed, using Proposition 1 and the relation (14) we see that

$$\mathbb{E}V_d(\mathbb{B}^d \setminus C_{n-1}) = \mathbb{E}\left[\mathbb{E}(V_d(\mathbb{B}^d \setminus [\widehat{C}^n_{M_n-1} \cup B(1-Kn^{-\alpha})])|M_n)\mathbf{1}_{\{M_n\in[k_n^-,k_n^+]\}}\right](1+o(1)).$$

Taking into account that $\mathbb{E}V_d(\mathbb{B}^d \setminus [\widehat{C}^n_{k-1} \cup B(1-Kn^{-\alpha})])$ decreases with k, we conclude that

$$\mathbb{E}V_d\left(\mathbb{B}^d \setminus [\widehat{C}^n_{k_n^+-1} \cup B(1-Kn^{-\alpha})]\right)(1+o(1)) \le \mathbb{E}V_d(\mathbb{B}^d \setminus C_{n-1}) \le$$

$$\mathbb{E}V_d\left(\mathbb{B}^d \setminus [\widehat{C}^n_{k_n^--1} \cup B(1-Kn^{-\alpha})]\right)(1+o(1)). \tag{21}$$

On the other hand, $\mathbb{E}V_d(\mathbb{B}^d\setminus[\widehat{C}^n_{k-1}\cup B(1-Kn^{-\alpha})])$ decreases with n. For $n' > n$ this can be seen by coupling the $k-1$ i.i.d. points $U_1,\dots,U_{k-1}$ uniform in $\mathbb{B}^d \setminus B(1-Kn^{-\alpha})$ with $U_1',\dots,U_{k-1}'$ given by $U_i' := \vartheta(|U_i|)\frac{U_i}{|U_i|}$, where $\vartheta(\cdot) = \vartheta_{n,n';\alpha}(\cdot)$ is a function of the form $\vartheta(r) = c_1\sqrt[d]{r^d+c_2}$ with c_1 and c_2 chosen so that $\vartheta(1) = 1$ and $\vartheta(1-Kn^{-\alpha}) = 1-Kn'^{-\alpha}$. Random points U_i' are readily verified to be i.i.d. uniformly on $\mathbb{B}^d \setminus B(1-Kn'^{-\alpha})$ and to enjoy the property that U_i' is a.s. closer to the boundary $\partial\mathbb{B}^d$ than U_i. Combining these observations with (21) and choosing $m_n^+ > m_n^-$ so that $k_n^+ = k^-_{m_n^+}$ and $k_n^- = k^+_{m_n^-}$, we see that

$$\mathbb{E}V_d(\mathbb{B}^d \setminus C_{m_n^+-1})(1+o(1)) \le \mathbb{E}V_d(\mathbb{B}^d \setminus [\widehat{C}^n_{k-1} \cup B(1-Kn^{-\alpha})]) \le \mathbb{E}V_d(\mathbb{B}^d \setminus C_{m_n^--1})(1+o(1)),$$

uniformly for $k \in [k_n^-, k_n^+]$. Since $m_n^+ \sim m_n^-$, this yields (20) as an immediate consequence. Rewrite (20) as

$$V_d(\mathbb{B}^d \setminus B(1-Kn^{-\alpha})) - \mathbb{E}(V_d(\widehat{C}^n_{k-1} \setminus B(1-Kn^{-\alpha}))) \sim \omega_d - \mathbb{E}(V_d(C_{n-1})),$$

uniformly in $k \in [k_n^-, k_n^+]$. Combining this relation with (19) leads to

$$\lim_{n\to+\infty} \sup_{k\in[k_n^-,k_n^+]} \left|\frac{\mathbb{E}(N_n|M_n=k)}{\mathbb{E}(N_n)} - 1\right| = 0.$$

In order to deduce (15), it remains to apply the same method as in Lemma 1 to get

$$\sup_{k_n^-\le k\le k_n^+} |\mathbb{E}(N_n^{\alpha,\beta}|M_n=k) - \mathbb{E}(N_n|M_n=k)| \le 2n \sup_{k_n^-\le k\le k_n^+} \mathbb{P}(\mathcal{A}_n\cup\mathcal{B}_n|M_n=k) = O(e^{-cn^{1-\alpha(d+1)/2}}).$$

$\square$

The next lemma, which is an essential step to obtain Theorem 1, shows how the Lipschitz property of the function $\Phi_n^{\alpha,\beta}$ can be used to estimate large-deviation probabilities for $N_n^{\alpha,\beta}$.

Lemma 3 *For each $\varepsilon > 0$,*

$$\liminf_{n\to+\infty} \frac{1}{\log(n)} \log\left(-\log P\left\{\left|\frac{N_n^{\alpha,\beta}}{\mathbb{E}N_n^{\alpha,\beta}} - 1\right| > \varepsilon\right\}\right) \geq 1 + \alpha - \frac{4}{d+1} - 2\beta.$$

Proof. Conditionally on $\{M_n = k\}$, $0 \leq k \leq n$, the variable $N_n^{\alpha,\beta}$ is distributed as $\Phi_n^{\alpha,\beta}(Y_1, \cdots, Y_k)$, where $Y_1, \cdots, Y_k$ are independent and uniformly distributed variables in $\mathbb{B}^d \setminus B(1 - Kn^{-\alpha})$.

Taking into account our discussion of the properties of $\Phi_n^{\alpha,\beta}$ in the beginning of this section,

$$\Phi_n^{\alpha,\beta} : (\mathbb{B}^d \setminus B(1 - Kn^{-\alpha}))^k \longrightarrow \mathbb{N}$$

is a Lipschitz function with a Lipschitz constant equal to cn^{β} for some $c > 0$, for $(\mathbb{B}^d \setminus B(1 - Kn^{-\alpha}))^k$ endowed with the metric $\rho_k((x_1, ..., x_k), (x_1', ..., x_k')) := \mathbf{1}_{\{x_1 \neq x_1'\}} + ... + \mathbf{1}_{\{x_k \neq x_k'\}}$. Consequently, we are in a position to apply the following standard measure concentration result (see Corollary 1.17 in [7]).

Theorem 3 (Ledoux) *Let $Y_1, Y_2, ..., Y_k$ be independent random elements taking values in a metric space $(\mathcal{Y}, \rho)$ of finite diameter D. Assume that $\Phi : \mathcal{Y}^k \to \mathbb{R}$ is Lipschitz with respect to the L^1-metric $\rho_k((y_1, ..., y_k), (y_1', ..., y_k')) := \rho(y_1, y_1') + ... + \rho(y_k, y_k')$ with some Lipschitz constant L. Then, for every $\lambda \geq 0$*

$$\mathbb{P}(|\Phi(Y_1, ..., Y_k) - \mathbb{E}\Phi(Y_1, ..., Y_k)| > \lambda) \leq 2\exp\left(-\frac{\lambda^2}{2kL^2D^2}\right).$$

Applying Theorem 3 to $\lambda = \varepsilon\mathbb{E}(N_n^{\alpha,\beta}|M_n = k)$, we obtain for every $0 \leq k \leq n$,

$$\begin{aligned}
&\mathbb{P}\left\{\left|\frac{N_n^{\alpha,\beta}}{\mathbb{E}N_n^{\alpha,\beta}} - 1\right| > \varepsilon \middle| M_n = k\right\} \\
&\leq \mathbb{P}\left\{|N_n^{\alpha,\beta} - \mathbb{E}(N_n^{\alpha,\beta}|M_n = k)| > \varepsilon\mathbb{E}(N_n^{\alpha,\beta}) - |\mathbb{E}(N_n^{\alpha,\beta}|M_n = k) - \mathbb{E}(N_n^{\alpha,\beta})| \middle| M_n = k\right\} \\
&\leq 2\exp\left\{-\frac{(\varepsilon\mathbb{E}(N_n^{\alpha,\beta}) - |\mathbb{E}(N_n^{\alpha,\beta}|M_n = k) - \mathbb{E}(N_n^{\alpha,\beta})|)^2}{2c^2kn^{2\beta}}\right\}. \qquad (22)
\end{aligned}$$

Applying the relation (15), we deduce from (22) that there exists a positive constant c such that

$$\sup_{k_n^- \leq k \leq k_n^+} \mathbb{P}\left\{\left|\frac{N_n^{\alpha,\beta}}{\mathbb{E}N_n^{\alpha,\beta}} - 1\right| > \varepsilon \middle| M_n = k\right\} \leq 2\exp\left\{-\frac{c[\mathbb{E}N_n^{\alpha,\beta}]^2}{n^{1+2\beta}w_n}\right\}. \qquad (23)$$

Combining now (23) with Lemma 1, the estimate $w_n \sim cn^{-\alpha}$ and the classical result (see [10]) $\mathbb{E}N_n \sim cn^{1-\frac{2}{d+1}}$, we obtain

$$\begin{aligned}
&\liminf_{n\to+\infty} \frac{1}{\log(n)} \log\left(-\log\left(\sup_{k_n^- \leq k \leq k_n^+} \mathbb{P}\left\{\left|\frac{N_n^{\alpha,\beta}}{\mathbb{E}N_n^{\alpha,\beta}} - 1\right| > \varepsilon \middle| M_n = k\right\}\right)\right) \\
&\geq \liminf_{n\to+\infty} \frac{1}{\log(n)} \log\left(-\log\left(\exp\left\{-c\frac{[\mathbb{E}(N_n)]^2}{n^{1+2\beta-\alpha}}\right\}\right)\right) = 1 + \alpha - \frac{4}{d+1} - 2\beta. \qquad (24)
\end{aligned}$$

Besides, let us notice that

$$\mathbb{P}\left\{\left|\frac{N_n^{\alpha,\beta}}{\mathbb{E}N_n^{\alpha,\beta}}-1\right|>\varepsilon\right\} \le \sum_{k_n^- \le k \le k_n^+} \mathbb{P}\left\{\left|\frac{N_n^{\alpha,\beta}}{\mathbb{E}N_n^{\alpha,\beta}}-1\right|>\varepsilon \middle| M_n=k\right\}\mathbb{P}\{M_n=k\}+\mathbb{P}\{M_n\notin[k_n^-,k_n^+]\}. \tag{25}$$

Inserting then the results (14) and (24) in (25), we obtain Lemma 3.

$\square$

Completing the proof of Theorem 1. We have that

$$\mathbb{P}\left\{\left|\frac{N_n}{\mathbb{E}N_n}-1\right|>\varepsilon\right\} \le \mathbb{P}\left\{\left|\frac{N_n^{\alpha,\beta}}{\mathbb{E}N_n^{\alpha,\beta}}-1\right|>\varepsilon\frac{\mathbb{E}N_n}{\mathbb{E}N_n^{\alpha,\beta}}-\left|\frac{\mathbb{E}N_n}{\mathbb{E}N_n^{\alpha,\beta}}-1\right|\right\}+\mathbb{P}\left\{N_n\neq N_n^{\alpha,\beta}\right\}.$$

Using Lemmas 1 and 3, we obtain that for every $\alpha\in(0,\frac{2}{d+1})$ and $\beta\in(1-\alpha\frac{d+1}{2},1)$,

$$\liminf_{n\to+\infty}\frac{1}{\log(n)}\log\left(-\log\left(\sup_{k_n^-\le k\le k_n^+}\mathbb{P}\left\{\left|\frac{N_n}{\mathbb{E}N_n}-1\right|>\varepsilon\right\}\right)\right)\ge\min(1-\alpha\frac{d+1}{2},1+\alpha-\frac{4}{d+1}-2\beta).$$

It remains to verify that

$$\sup_{\alpha\in(0,\frac{2}{d+1}),\beta\in(1-\alpha\frac{d+1}{2},1)}\min(1-\alpha\frac{d+1}{2},1+\alpha-\frac{4}{d+1}-2\beta)=1-\frac{2d+6}{3d+5}.$$

$\square$

3 Proof of Theorem 2

The method is similar to the binomial case. For some $\alpha\in(0,\frac{2}{d+1})$ and $\beta\in(1-\frac{\alpha(d+1)}{2},1)$, we consider a covering of the annulus $\mathbb{B}^d\setminus B(1-Kn^{-\alpha})$ by spherical caps (provided by the point (ii) of Proposition 1). In full analogy with the definition of $N_n^{\alpha,\beta}$ we define a modification $\widetilde{N}_t^{\alpha,\beta}$ of $\widetilde{N}_t$ by putting $\widetilde{N}_t^{\alpha,\beta}:=\Phi_t^{\alpha,\beta}(X_t)$ so that $\widetilde{N}_t^{\alpha,\beta}$ is the number of vertices of the convex hull of an appropriate subset of the intersection of the Poisson point process X_t with $\mathbb{B}^d\setminus B(1-Kt^{-\alpha})$ enjoying the property that each cap contains at most t^β points. Much along the same lines as in the proof of Lemma 1, we get

$$\mathbb{P}\{\widetilde{N}_t^{\alpha,\beta}\neq\widetilde{N}_t\}=O(e^{-ct^{1-\alpha(d+1)/2}}) \tag{26}$$

and

$$|\mathbb{E}\widetilde{N}_t^{\alpha,\beta}-\mathbb{E}\widetilde{N}_t|=O(e^{-ct^{1-\alpha(d+1)/2}}). \tag{27}$$

Moreover, let $\widetilde{M_t}$ be the number of points of $X_t\cap[B(1-Kt^{-\alpha})]^c$. $\widetilde{M_t}$ is then Poisson distributed with mean

$$\mathbb{E}\widetilde{M_t}=tV_d(\mathbb{B}^d\setminus B(1-Kt^{-\alpha}))\underset{t\to+\infty}{\sim}dK\omega_d t^{1-\alpha}.$$

As in the proof of Lemma 3, $\widetilde{M_t}$ satisfies a large-deviation inequality. Actually, denoting $k_t^-=\lfloor\mathbb{E}\widetilde{M_t}-t^{1-\frac{2}{d+1}}\rfloor$, $k_t^+=\lceil\mathbb{E}\widetilde{M_t}+t^{1-\frac{2}{d+1}}\rceil$, we get as in (14)

$$\liminf_{t\to+\infty}\frac{1}{\log(t)}\log(-\log\mathbb{P}\{\widetilde{M_t}\notin[k_t^-,k_t^+]\})\ge1+\alpha-\frac{4}{d+1}. \tag{28}$$

Further, we have

$$\mathbb{P}\left\{\left|\frac{\widetilde{N}_t^{\alpha,\beta}}{\mathbb{E}\widetilde{N}_t^{\alpha,\beta}}-1\right|>\varepsilon\right\}\le\sum_{k_t^-\le k\le k_t^+}\mathbb{P}\left\{\left|\frac{\widetilde{N}_t^{\alpha,\beta}}{\mathbb{E}\widetilde{N}_t^{\alpha,\beta}}-1\right|>\varepsilon\middle|\widetilde{M}_t=k\right\}\mathbb{P}\{\widetilde{M}_t=k\}+\mathbb{P}\{\widetilde{M}_t\not\in[k_t^-,k_t^+]\}. \tag{29}$$

The first term can be bounded in the same way as the corresponding one in (25), whereas the second term is estimated thanks to (28). Consequently, the large-deviation result is proved for $\widetilde{N}_t^{\alpha,\beta}$ and it suffices to use (26) and (27) to deduce the same for $\widetilde{N}_t$.

□

4 Concluding remarks

Remark 1 The results remain valid if we add to the random sample in the ball a fixed number of deterministic points. Indeed, the whole argument above can be repeated for such a case with only minor changes.

Remark 2 Since the asymptotic behaviour of the convex hull only depends on the geometry of the sample very close to the boundary of $\mathbb{B}^d$, the results of Proposition 1 and Theorem 2 can be extended to the class of inhomogeneous Poisson point processes Y_t with their intensity measures of the form $td\mu$, where μ is the measure (in spherical coordinates) $f(r)\mathbf{1}_{[0,1]}(r)drd\sigma_d(u)$, where $d\sigma_d$ is the area measure on the sphere $\mathbb{S}^{d-1}$. Here f is a continuous function satisfying $f(1)=1$. Indeed, in a vicinity of $\mathbb{S}^{d-1}$, the intensity measure is close to a multiple of the Lebesgue measure and once again our argument can be repeated for this case.

Remark 3 The question of the extension of these results to a general convex set is still open. The method does not apply when the ball is replaced by a polyhedron since the mean number of extreme points becomes of order $\log(n)$ which is too small for the rates obtained by measure concentration techniques to absorb the polynomial prefactors due to deterministic surface partitions as considered in our proofs.

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