Duality and symmetry in interacting particle systems

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Lecture 1: General

1. Setting the stage
2. Duality
3. Symmetries

Lecture 2: Applications

4. Symmetric simple exclusion process
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1. Setting the stage

Q1: How can we describe the behavior of a VERY large number of interacting particles?

Q2: Emergence of large scale behavior from microscopic interactions?

1.1 Interacting particle systems [Spitzer, 1970], ...

- Many particles hopping randomly on a lattice (Markovian)

– Lattice gases: Particle numbers of all species conserved

– Reaction-diffusion systems: some particle numbers not conserved

- Approach: Statistical Physics and Probability Theory

Three types of problems:

- Microsopic properties: Description on lattice scale

– Invariant measures

– Correlations

– Large deviations

– ...

- Macrosopic properties: Large-scale behavior

– Hydrodynamic limit

– Fluctuations

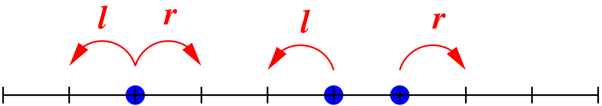
– Large deviations

– ...

- Universality: Are there specific macroscopic properties that do not depend on microscopic details of the interaction?

Paradigmatic example: 1-dim (A)symmetric simple exclusion process

- (A)symmetric nearest neighbor jumps on integer lattice
- Exclusion principle: at most one particle per site



- Finite or (semi-)infinite integer lattice Λ
- Local state space $\mathbb{S}_{loc} = \{0, 1\}$
- Local occupation variables $\eta_k \in \mathbb{S}_{loc}$ for $k \in \Lambda$
- (Global) state space $\mathbb{S} = \mathbb{S}_{loc}^{\Lambda}$
- Configuration $\boldsymbol{\eta} = \{\eta(1), \ldots, \eta(L)\} \in \mathbb{S}$

1.2 Continuous time Markov chains

[Liggett, Continuous Time Markov Processes: An Introduction, 2010]

- Conventions:

$\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}_0^+ = [0, \infty)$

- Ingredients in definition of CTMC:

− *Countable* state space $\mathbb{S}$, time $t \in \mathbb{R}_0^+$

− *Path space* Ω: Set of right continuous functions $\omega : \mathbb{R}_0^+ \to \mathbb{S}$ with finitely many jumps in any finite time interval

− Time shift $\theta_s : \Omega \to \Omega$, $s \in \mathbb{R}_0^+$ defined by $(\theta_s \omega)(t) = \omega(t + s)$

− Stochastic process $X(t)$ where $X(t, \omega) = \omega(t)$

− $\mathcal{F}$: Smallest σ-algebra s.t. the mapping $\omega \to \omega(t)$ is measurable for each $t \geq 0$

Definition 1.1 *Markov chain on $\mathbb{S}$:*

(i) Collection of probability measures $\{P^x, x \in \mathbb{S}\}$ on Ω

(ii) Right continuous filtration $\mathcal{F}_t, t \geq 0$ s.t. $X(t)$ is $\mathcal{F}_t$ measurable for each $t \geq 0$ and $P^x(X(0) = x) = 1$

(iii) Markov property $E^x(Y \circ \theta_s | \mathcal{F}_s) = E^{X(s)} Y$ for all $x \in \Omega$ and all bounded measurable Y on Π

• Specific chain defined by *generator* $\mathcal{L}$: Linear operator on $C(\mathbb{S})$ s.t. for all $x, y \in \mathbb{S}$

$$(\mathcal{L}f)(x) = \sum_{y \in \mathbb{S} \setminus x} w(x \to y)[f(y) - f(x)]$$

with transition rates $w(x \to y)$ from configuration x to configuration y

• Equivalent definition by *intensity matrix* Q:

$$(\mathcal{L}f)(x) = \sum_{y \in \mathbb{S}} Q_{xy} f(y)$$

with $Q_{xy} = w(x \to y)$ for $x \neq y$ and $Q_{xx} = -\sum_{y \in \mathbb{S} \setminus x} w(x \to y)$ (conservation of probability)

Definition 1.2 *Probability semigroup:*

Family of continuous linear operators S_t, $t \in \mathbb{R}_0^+$ *on* $C(\mathbb{S})$ *satisfying*

(i) $S_0 f = \lim_{t \searrow 0} S_t f = f$ *for all* $f \in C(\mathbb{S})$

(ii) $S_{s+t} f = S_s S_t f$ *for all* $f \in C(\mathbb{S})$

(iii) $S_t f \geq 0$ *for all nonnegative* $f \in C(\mathbb{S})$

(iv) Compact $\mathbb{S}$*:* $S_t 1 = 1$ *for all* $t \geq 0$, *noncompact* $\mathbb{S}$*: ...*

- Semigroup and generator: $S_t = \mathrm{e}^{\mathcal{L}t} := \lim_{n \to \infty} (I - \frac{t}{n}\mathcal{L})^{-n}$

- Semigroup and intensity matrix for finite state space: $S_t = \mathrm{e}^{Qt}$

- Notation for action of semigroup: $f_t := S_t f$ for measurable functions $f : \mathbb{S} \to \mathbb{R}$, and $\mu_t := \mu S_t$ for probability measure μ on state space $\mathbb{S}$

- Transition probability $p_t(x,y) := P^x(X(t)=y)$ satisfying

– $\frac{\mathrm{d}}{\mathrm{d}t} p_t(x,y)|_{t=0} = Q_{xy}$

– Chapman-Kolmogorov equation $p_{s+t}(x,y) = \sum_{z\in\mathbb{S}} p_s(x,z)p_t(z,y)$

– Kolmogorov forward equation $\frac{\mathrm{d}}{\mathrm{d}t} p_t(x,y) = \sum_{z\in\mathbb{S}} Q_{xz}p_t(z,y)$

– Kolmogorov backward equation $\frac{\mathrm{d}}{\mathrm{d}t} p_t(x,y) = \sum_{z\in\mathbb{S}} p_t(x,z)Q_{zy}$

– $p_t(x,y) = (\mathrm{e}^{Qt})_{xy}$ (finite $\mathbb{S}$, mild conditions for countable $\mathbb{S}$)

- Notation for expectations:

– $E^x(A)$ for $A \in \mathcal{F}$ w.r.t. probability measure P^x on path space Ω

– $\langle f \rangle_\mu = \sum_{x\in\mathbb{S}} f(x)\mu(x)$ w.r.t. probability measure μ on state space $\mathbb{S}$

– $\langle f_t \rangle_\mu = \langle f \rangle_{\mu_t} = \sum_{x\in\mathbb{S}} f(x)\mu_t(x) = \sum_{x\in\mathbb{S}} \mu(x)E^x f(X(t))$

Definition 1.3 *Stationary and reversible measure: A measure π on the state space $\mathbb{S}$ is said to be stationary if for all $x, y \in \mathbb{S}$, $t > 0$*

$$\pi(y) = \sum_{x \in \mathbb{S}} \pi(x) p_t(x, y)$$

and reversible if

$$\pi(y) p_t(y, x) = \pi(x) p_t(x, y).$$

- Every reversible measure is stationary
- Stationary measure: $\pi S_t = \pi$ and $\sum_{x \in \mathbb{S}} \pi(x) q_{xy} = 0$
- Ergodic process: $\pi(x) > 0$ for all $x \in \mathbb{S}$

Definition 1.4 *(Time-)reversed process:*
Process with transition rates $q^{rev}_{x,y} = \frac{\pi(y)}{\pi(x)} q_{y,x}$
Reversible process: $Q^{rev} = Q$

1.3 Some linear algebra

> Focus now on finite state space, cardinality $|\mathbb{S}| = d_S$

• Bra-ket notation: $\{\langle\, n\,|, n \in \{0, 1, \ldots, d_s - 1\} =$ canonical basis vectors $\mathfrak{e}_n$ of $\mathbb{C}^{d_S}$, represented as row vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with component 1 at position n and 0 else

• Consider vector $\vec{f} \in \mathbb{C}^{d_S}$ with components $f(n) \in \mathbb{C}$ with complex conjugate $f^*(n)$, then:

– Bra vector: $\langle\, f\,| = \sum_{n=0}^{d_S-1} f^*(n)\langle\, n\,|$ (row vector)

– Ket vector: $|\, f\,\rangle = \sum_{n=0}^{d_S-1} f(n)|\, n\,\rangle^T$ (column vector)

• Matrix product: Consider $d_{S_1} \times d_{S_2}$ matrix A and $d_{S_2} \times d_{S_3}$ matrix B. Then $(A \cdot B)_{mn} \equiv (AB)_{mn} = \sum_{k=0}^{d_{S_2}} A_{mk} B_{kn}$

• Scalar product: $\langle \vec{f}, \vec{g} \rangle := \sum_{n=0}^{d_S-1} f^*(n)g(n) = \langle f \mid g \rangle$ with *matrix multiplication* $\langle f \mid g \rangle := \langle f \mid \cdot \mid g \rangle$

Notice: $\langle f \mid A \mid g \rangle = \langle \vec{f}, (A\vec{g}) \rangle = \langle (A^{*T}\vec{f}), \vec{g} \rangle$

• Kronecker product: Consider $d_{S_1} \times d_{S_2}$ matrix A and $d_{S_3} \times d_{S_4}$ matrix B. Then $(A \otimes B)_{pd_{S_3}+m, qd_{S_4}+n} = A_{pq}B_{mn}$

$$A \otimes B = \begin{pmatrix} A_{00}B & A_{01}B & A_{02}B & \dots \\ A_{10}B & A_{11}B & A_{12}B & \dots \\ A_{20}B & A_{21}B & A_{22}B & \dots \\ A_{56}B & A_{56}B & A_{32}B & \dots \\ A_{40}B & A_{41}B & A_{42}B & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

• Dyadic product: $\mid f \rangle\langle g \mid := \mid f \rangle \otimes \langle g \mid$ with matrix elements $(\mid f \rangle\langle g \mid)_{mn} = f(m)g^*(n)$

What does all of this have to do with probability theory?

• Bijective enumeration function $\imath : \mathbb{S} \to Z$ (label all configurations by some integer) $\Longrightarrow$ bijective mapping $\mathbb{S} \to \mathbb{C}^{d_S}$, $x \mapsto \langle\, \imath(x) \,| \equiv \langle\, x \,|$

• Special vectors and matrices:

– Probability vector $\langle\, \mu \,| = \sum_{x\in\mathbb{S}} \mu(x) \langle\, x \,|$

– Summation vector $|\, s \,\rangle = \sum_{x\in\mathbb{S}} |\, x \,\rangle$ (all components 1)

– Identity matrix $\mathbf{1} = \sum_{x\in\mathbb{S}} |\, x \,\rangle\langle\, x \,|$

– Probability matrix $\hat{\mu} = \sum_{x\in\mathbb{S}} \mu(x) |\, x \,\rangle\langle\, x \,|$ (diagonal)

– Function matrix $\hat{f} = \sum_{x\in\mathbb{S}} f(x) |\, x \,\rangle\langle\, x \,|$ (diagonal)

– Intensity matrix $Q = \sum_{x\in\mathbb{S}} \sum_{y\in\mathbb{S}} q_{xy} |\, x \,\rangle\langle\, y \,|$

- Some probabilistic concepts:

– Normalization of probability measure: $\langle \mu \,|\, s \,\rangle = 1$

– Conservation of probability: $Q|\, s \,\rangle = 0$

– Stationarity : $\langle \pi \,|\mathrm{e}^{Qt} = \langle \pi \,|$ and $\langle \pi \,|Q = 0$

– Generator: $(\mathcal{L}f)(x) = \langle x \,|Q|\, f \,\rangle$

– Measure on state space $\mathbb{S}$ at time t, starting from μ: $\langle \mu_t \,| = \langle \mu \,|\mathrm{e}^{Qt}$

– Reversed process: $Q^{rev} = \hat{\pi}^{-1} Q^T \hat{\pi}$

– Reversible measure: $\mathrm{e}^{Q^T t} \hat{\mu} = \hat{\mu} \mathrm{e}^{Qt}$ (no normalization needed)

- Transition probability $p_t(x,y) = \langle\, x\,|\mathrm{e}^{Qt}|\, y\,\rangle$

- Chapman-Kolmogorov equation $p_{s+t}(x,y) = \langle\, x\,|\mathrm{e}^{Q(s+t)}|\, y\,\rangle$

$$= \langle\, x\,|\mathrm{e}^{Q(s)}\mathbf{1}\mathrm{e}^{Q(t)}|\, y\,\rangle = \sum_{z\in\mathbb{S}} \langle\, x\,|\mathrm{e}^{Qs}|\, z\,\rangle\langle\, z\,|\mathrm{e}^{Qt}|\, y\,\rangle = \sum_{z\in\mathbb{S}} p_s(x,z)p_t(z,y)$$

- Kolmogorov forward equation $\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,y) = (\langle\, x\,|Q)\mathrm{e}^{Qt}|\, y\,\rangle$

- Kolmogorov backward equation $\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,y) = \langle\, x\,|\mathrm{e}^{Qt}(Q|\, y\,\rangle)$

- Expectation w.r.t measure μ: $\langle\, f\,\rangle = \langle\, \mu\,|\, f\,\rangle = \langle\, \mu\,|\hat{f}|\, s\,\rangle$

- Expectation at time t with initial measure μ:

$$\langle\, f_t\,\rangle_\mu \equiv \sum_{x\in\mathbb{S}} \mu(x)E^x f(X(t)) = \langle\, \mu\,|\mathrm{e}^{Qt}|\, f\,\rangle = \langle\, \mu\,|\mathrm{e}^{Qt}\hat{f}|\, s\,\rangle$$

Symmetry:

Definition 1.5 *Let* $\Sigma : \mathbb{S} \times \mathbb{S} \to \mathbb{C}$ *be a function and* S *be a matrix with elements* $S_{xy} = \Sigma(x, y)$. S *is called a symmetry of a process if the intensity matrix* Q *and* S *satisfy the commutation relation*

$$[Q, S] = 0.$$

except if $S = \mathbf{1}$. *If* S *is diagonal then it is called a diagonal symmetry.*

• If S is a symmetry and $\langle \pi |$ is a stationary probability vector, then also $\langle \pi_S | := \langle \pi | S$ is stationary

• Existence of a diagonal symmetry implies that the process is not ergodic. The number of ergodic subspaces is larger or equal to the number of distinct eigenvalues of S

2. Duality

Ingredients:

- Two Markov processes $x(t)$ and η_t with state spaces $\mathbb{X}$ and $\mathbb{S}$
- Duality function $D : \mathbb{X} \times \mathbb{S} \to \mathbb{R}$

2.1 Definition and basic properties

Definition 2.1 *Let $x(t)$ be a Markov process with state space $\mathbb{X}$ and $\eta(t)$ be a Markov process with state space $\mathbb{S}$. Furthermore, let $D : \mathbb{X} \times \mathbb{S} \to \mathbb{R}$ be a bounded measurable function. The processes $x(t)$ and $\eta(t)$ are said to be dual w.r.t. the duality function D if*

$$\mathbf{E}^x D(x(t), \eta) = \mathbf{E}^\eta D(x, \eta(t)).$$

Use of duality: Express properties of one process in terms of another (possibly simpler) one.

• For Markov chains with countable state spaces the matrix

$$\hat{D} := \sum_{x \in \Xi} \sum_{\eta \in \Omega} D(x, \eta) | \, x \, \rangle \langle \, \eta \, |$$

with matrix elements $D_{x,\eta} = D(x, \eta)$ is called the duality matrix.

• A duality function of the form $D(x, \eta) = \sum_x d(x) \delta_{x,\eta}$ is called diagonal.

• If the intensity matrices of the two processes are equal then the process is said to be self-dual w.r.t. D.

• A process with strictly positive stationary measure π and its reversed are dual w.r.t. the diagonal duality function $D(x, y) = \sum_x \pi^{-1}(x) \delta_{x,y}$.
$\Longrightarrow$ Dualities always exist!

Remark 2.2 *In terms of transition probabilities $p_t(\cdot|\cdot)$ for $x(t)$ and $\tilde{p}_t(\cdot|\cdot)$ for $\eta(t)$ the duality property reads*

$$\sum_{y \in \mathbb{X}} p_t(x,y) D(y,\eta) = \sum_{\zeta \in \mathbb{S}} D(x,\zeta) q_t(\zeta,\eta).$$

With intensity matrix Q for $X(t)$ and $\tilde{Q}$ for $\eta(t)$ this means

$$\mathrm{e}^{Qt} D = D \mathrm{e}^{\tilde{Q}^T t}$$

for all $t > 0$.

• Take time derivative at $t = 0$: Duality becomes [Sudbury et al. (1995), Giardinà et al. (2009)]

$$QD = D\tilde{Q}^T$$

• Slightly stronger version of Theorem 2.6 in Giardinà et al. (2009) (making no assumption on the existence of S^{-1}) [Belitsky and GMS (2015)]:

Theorem 2.3 (Belitsky and GMS (2015)) *Let Q be the intensity matrix of an ergodic Markov process $X(t)$ with countable state space and stationary measure π and Q^{rev} be the intensity matrix of the reversed process $X^{rev}(t)$. Assume that there exists an intertwiner S such that*

$$QS = SQ^{rev}.$$

Then $X(t)$ is self-dual with duality matrix

$$D = S\hat{\pi}^{-1}.$$

Proof: Chain of equalities from the hypothesis of the theorem and the definition of reversed process:

$$QS\hat{\pi}^{-1} = SQ^{rev}\hat{\pi}^{-1} = S\hat{\pi}^{-1}\hat{\pi}Q^{rev}\hat{\pi}^{-1} = S\hat{\pi}^{-1}Q^{T}$$

□

Corollary 2.4 *If $X(t)$ is reversible then the hypothesis reads $QS = SQ$, i.e. S is a symmetry of $X(t)$.*

$\Longrightarrow$ Use of duality: Exploit symmetries to express properties of one process in terms of another (possibly simpler) one. [GMS and Sandow (1994)]

• $D(x,\eta)$ can be understood as a family of measurable functions $f^x : \mathbb{S} \mapsto \mathbb{R}$ indexed by $x \in \mathbb{X}$ and defined by $f^x(\eta) := D(x,\eta)$, or, alternatively as a family of functions $g^\eta : \mathbb{X} \mapsto \mathbb{R}$ indexed by η and defined by $g^\eta(x) := D(x,\eta)$. $\Longrightarrow$ Reformulation of duality

$$\langle\, f_t^x \,\rangle_\eta = \langle\, g_t^\eta \,\rangle_x$$

with $\langle\, f_0^x \,\rangle_\eta = \langle\, g_0^\eta \,\rangle_x = D(x,\eta)$.

• Intensity matrix has nontrivial right invariant subspace if $|\mathbb{S}| < |\mathbb{X}|$ c.f. [Redig, Sau (2019)]

• Paradigmatic example: Selfdual symmetric simple exclusion process (SSEP) where hard-core particles perform lattice random walk

– Expectation of local density at time t for many-particle initial state given in terms of transition probability for just one particle

– Joint expectation for N particles at times $t_1, \ldots, t_N$ given in terms of transition probability for N particles

– Origin: $SU(2)$ symmetry of generator (apparent through relationship to quantum XXX Heisenberg spin chain [GMS and Sandow, 1994])

• Simple example:

(1): Symmetric random walk $X(t)$ on state space $\mathbb{X} = \mathbb{Z}$ with jump rate $w(x \to x \pm 1) = 1$

(2) Coin tossing $\eta(t)$ with flip rate $w(1 \to -1) = w(-1 \to 1) = 1$ (state space $\mathbb{S} = \{1, -1\}$

– RW: $Q_{xy} = \delta_{x,y-1} + \delta_{x,y+1} - 2\delta_{x,y}$ (discrete Laplacian)
$p_t(x,y) = \mathrm{e}^{-2t} I_{x-y}(2t)$ (modified Bessel function)

– Coin: $\tilde{Q}_{\eta\zeta} = \delta_{\eta,-\zeta} - \delta_{\eta,\zeta}$, $\tilde{p}_t(\eta,\zeta) = \mathrm{e}^{-t}\left(\delta_{\eta,\zeta}\cosh t + \delta_{\eta,-\zeta}\sinh t\right)$

Q1: What is the probability p_t^+ to find $X(t)$ at time t on an even site?

A1: Duality function: $D(x,\eta) = \frac{1}{2}\left[1 + (-1)^x \eta\right] \Longrightarrow QD = \tilde{Q}^T D$

$\Longrightarrow p_t^+ = \frac{1}{2}\left(1 + \mathrm{e}^{-2t}\right)$ for $p_0^+ = 1$ and $p_t^+ = \frac{1}{2}\left(1 - \mathrm{e}^{-2t}\right)$ for $p_0^+ = 0$

Proof: $\langle\, x\,|\mathrm{e}^{Qt} D|\,\eta\,\rangle = \langle\, x\,|D\mathrm{e}^{\tilde{Q}^T t}|\,\eta\,\rangle = \sum_{\zeta\in\mathbb{S}} \langle\, x\,|D|\,\zeta\,\rangle\langle\,\zeta\,|\mathrm{e}^{\tilde{Q}^T t}|\,\eta\,\rangle$

$$= \sum_{\zeta\in\mathbb{S}} \langle\, x\,|D|\,\zeta\,\rangle\langle\,\eta\,|\mathrm{e}^{\tilde{Q} t}|\,\zeta\,\rangle = \sum_{\zeta\in\mathbb{S}} D(x,\zeta)\tilde{p}_t(\eta,\zeta) = \frac{1}{2}\left[1 + (-1)^x \eta\, \mathrm{e}^{-2t}\right]$$

Q2: How did I find this duality (dual process and duality function)?

A2: (i) Define $|+\rangle := \sum_{x\in\mathbb{Z}} |2x\rangle, |-\rangle := \sum_{x\in\mathbb{Z}} |2x-1\rangle, \langle y_t| := \langle y|\mathrm{e}^{Qt}$

(ii) Form two-dimensional vector $(p_t^+, p_t^-) = \langle y_t|(|+\rangle, |-\rangle)$

(iii) Kolmogorov forward equation for $p_t^{\pm} = \langle y|\mathrm{e}^{Qt}|\pm\rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_t^+, p_t^-) = \langle y_t|(Q|+\rangle, Q|-\rangle) = \langle y_t| \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} (|+\rangle, |-\rangle)$$

$\Longrightarrow \langle +|$ and $\langle -|$ span two-dimensional left-invariant subspace of Q

– Dual intensity matrix pops up: $\frac{\mathrm{d}}{\mathrm{d}t}(p_t^+, p_t^-) = \tilde{Q}^T(p_t^+, p_t^-)$

– Correspondence between family of expectations and duality yields duality matrix D

$\Longrightarrow$ Computation of $p_t^{\pm}$ without use of random walk transition probability

2.2 Three different concepts of duality

• Consider two Markov processes $\boldsymbol{\eta}(t)$ and $\mathbf{x}(t)$ with generally different countable state spaces and intensity matrices $Q_{\boldsymbol{\eta}\boldsymbol{\eta}'} = w(\boldsymbol{\eta} \to \boldsymbol{\eta}')$, $\tilde{Q}_{\mathbf{x}\mathbf{x}'} = w(\mathbf{x} \to \mathbf{x}')$

• Quantum Hamiltonian formalism: $H = -Q^T$, $G = -\tilde{Q}^T$

– Invariant measures $\mu(\boldsymbol{\eta})$, $\pi(\mathbf{x})$

– Probability vectors $|\,\mu\,\rangle$, $|\,\pi\,\rangle$

– Stationarity: $H|\,\mu^*\,\rangle = 0$, $G|\,\pi^*\,\rangle = 0$

– Reverse processes for strictly positive invariant measures:

$$H_{rev} = \hat{\mu} H^T \hat{\mu}^{-1}, \quad G_{rev} = \hat{\pi} G^T \hat{\pi}^{-1}$$

• (Conventional) Duality: Relationship between two processes that yields time-dependent expectations of one process in terms of the dual in terms of a duality function $D(\mathbf{x}, \boldsymbol{\eta})$

• Duality at the level of generators: $DH = G^T D$

• Useful information about expectations if dual process has simple properties:

For family of functions $f^{\mathbf{x}}(\boldsymbol{\eta}) := D(\mathbf{x}, \boldsymbol{\eta})$:

$$\langle f^{\mathbf{x}}(t) \rangle_{\mu} = \sum_{\mathbf{y}} P(\mathbf{x}, t|\mathbf{y}, 0) \langle f^{\mathbf{y}}(0) \rangle_{\mu}$$

with transition probability $P(\mathbf{x}, t|\mathbf{y}, 0)$ of dual process

• Reverse duality: $HR = RG^T$ [GMS, 2023]

with reverse duality matrix R and duality function $R_{\boldsymbol{\eta}\mathbf{x}} = R(\boldsymbol{\eta}, \mathbf{x})$

• Useful information about measures if reverse dual process has simple properties

For family of measures $\mu_{\boldsymbol{\eta}}^{\mathbf{x}}(t) := R(\boldsymbol{\eta}, \mathbf{x})$:

$$\mu_{\boldsymbol{\eta}}^{\mathbf{x}}(t) = \sum_{\mathbf{y}} P(\mathbf{x}, t|\mathbf{y}, 0)\mu_{\boldsymbol{\eta}}^{\mathbf{y}}(0)$$

• Duality function can take negative values (corresponding to signed measures)

• Reversible process $H = G^T$: Reverse duality = Symmetry

• Intertwining duality: $BH = GB$

with intertwining duality matrix B and duality function $B_{\boldsymbol{\eta}\mathbf{x}} = B(\boldsymbol{\eta}, \mathbf{x})$

$$\langle f^{\mathbf{x}}(t) \rangle_{\mu} = \sum_{\mathbf{y}} P(\mathbf{y}, t|\mathbf{x}, 0) \langle f^{\mathbf{y}}(0) \rangle_{\mu}$$

$$\mu_{\boldsymbol{\eta}}^{\mathbf{x}}(t) = \sum_{\mathbf{y}} P(\mathbf{y}, t|\mathbf{x}, 0) \mu_{\boldsymbol{\eta}}^{\mathbf{y}}(0)$$

- Link with conventional duality: $B = \hat{\pi}^* D$
- Invertible B: Similarity of processes H and $G = BHB^{-1}$
- Selfduality $G = H$: Intertwining duality = Symmetry

3 Symmetries

3.1 Lie algebras

Definition 3.1 *A Lie algebra is vector space* $\mathfrak{g}$ *over a field* F *and binary map* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *(Lie bracket), satisfying the following axioms:*

(i) Bilinearity:

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], [Z, aX + bY] = a[Z, X] + b[Z, Y]$$

for all $a, b \in F$ *and all* $X, Y, Z \in \mathfrak{g}$*.*

(ii) Alternating property: $[X, X] = 0$ *for all* $X \in \mathfrak{g}$*.*

(iii) Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$
for all $X, Y, Z \in \mathfrak{g}$*.*

• Structure constants: Specific Lie algebra with n generators x_i, $i \in \{1, \ldots, n\}$ is defined by the structure constants in Lie bracket

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k$$

• Representation of Lie algebra: Set of matrices M_i that satisfy the Lie bracket relations with commutator $[M_i, M_j] := M_i M_j - M_j M_i$

• Example: Lie algebra $\mathfrak{sl}(2, \mathbb{C})$: Three generators $X^{\pm}, X^z$, Lie brackets

$$[X^+, X^-] = 2X^z, [X^z, X^{\pm}] = \pm X^{\pm}$$

• Representations: $X^{\alpha} \mapsto s^{\alpha}$

$$s^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; s^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; s^z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• Universal enveloping algebra: Unital associative algebra generated by elements $X_1, \ldots X_n$ subject to the relations $X_i X_j - X_j X_i = \sum_{k=1}^{n} c_{ijk} X_k$.

• A Casimir element C_i is an element of the center of the universal enveloping algebra of a Lie algebra.

• Example: Basis elements of $U[\mathfrak{sl}(2,\mathbb{C})]$:
$\mathbf{1}, X^+, X^-, X^z, (X^+)^2, X^+X^-, X^+X^z, (X^-)^2, X^-X^z, (X^z)^2, (X^+)^3, \ldots$
but NOT $X^-X^+, X^zX^+, X^zX^-, \ldots,$

Casimirs: $C_0 = \mathbf{1}, C_1 = X^+X^- + X^-X^+ + (X^z)^2/2$

• Algebra homomorphism: Mapping that preserves defining relations of an algebra

• Coproduct $\Delta : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ where $\Delta(X) = I \otimes X + X \otimes I$

3.2 Particle systems on arbitrary lattices

The explicit form of the intensity matrix for IPS, i.e., for a suitable choice of tensor basis of the intensity matrix, often makes explicit non-abelian symmetries that allow for the derivation of non-trivial dualities.

• Let $\Gamma = (\Lambda, \Upsilon)$ be a finite graph with vertices $k \in \Lambda$ and undirected edges $\langle k, l \rangle \in \Upsilon$

• Take coproduct $\Delta(C)$ of a Casimir operator on vertices of edges $\langle k, l \rangle$ $\Longrightarrow \Delta(C) = Q_{kl} =$ Kronecker product of unit matrices for each vertex (except vertices k, l) with matrix w which is determined by C

• If Q_{kl} is the intensity matrix of a stochastic process then this process has the Lie algebra for which C is a Casimir as a symmetry $\Longrightarrow$ Selfduality with duality matrices given by arbitrary product of the generators of the Lie algebra

4 The symmetric simple exclusion process

4.1 Definition

• SSEP on an arbitrary graph Λ: Configuration $\boldsymbol{\eta} := \{\eta_k : k \in \Lambda\}$ with occupation numbers $\eta_k \in \{0,1\} \Longrightarrow \mathbb{S} = \{0,1\}^\Lambda$

• Each edge carries a "clock" that rings after an exponentially distributed random time with parameter $w_{kl} \equiv w_{lk}$. When the clock rings the occupation numbers η_k and η_l are interchanged, corresponding to a particle jump across bond $\langle k,l \rangle$ if one of the two sites is occupied and the other is empty $\Longrightarrow$ configuration $\boldsymbol{\eta}^{kl}$ with interchanged occupation numbers $\eta_j^{kl} = \eta_j + (\eta_k - \eta_l)\left(\delta_{j,l} - \delta_{k,l}\right)$

• $w(\boldsymbol{\eta} \to \boldsymbol{\eta}') = \displaystyle\sum_{\langle k,l \rangle \in \Upsilon} w_{kl}\left(\eta_k(1-\eta_l) + (1-\eta_k)\eta_l\right)\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}^{kl}}$

• Generator $\mathcal{L}f(\boldsymbol{\eta}) = \displaystyle\sum_{\langle k,l \rangle \in \Upsilon} w_{kl}$

• To compute quantum Hamiltonian take enumeration function $\imath(\boldsymbol{\eta}) = \sum_{k=1}^{L} \eta_k 2^{L-k}$ to fix the canonical basis vectors $\langle\, \boldsymbol{\eta} \,| = \langle\, e_{\imath(\boldsymbol{\eta})} \,|$ (decimal value of the binary number $\eta_1 \eta_2 \ldots \eta_L$)

$\Longrightarrow$ tensor basis $\langle\, \boldsymbol{\eta} \,| \equiv \langle\, \eta_1, \ldots, \eta_L \,| = \langle\, \eta_1 \,| \otimes \cdots \otimes \langle\, \eta_L \,|$ with the one-site basis vectors $\langle\, \eta_k \,| = (1 - \eta_k, \eta_k)$.

• Summation vector $\langle\, s \,| = (1, 1)^{\otimes L}$

• Two-dimensional unit matrix $\mathbb{1}$ and spin-lowering and raising operator

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• projectors on a particle and vacancy vector respectively:

$$\hat{n} = \frac{1}{2}(\mathbb{1} + \sigma^z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{v} = \frac{1}{2}(\mathbb{1} - \sigma^z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Longrightarrow |\, \boldsymbol{\eta}^{kl} \,\rangle = (\mathbf{1} + \sigma_k^+ \sigma_l^- + \sigma_k^- \sigma_l^+ - \hat{n}_k \hat{v}_l - \hat{v}_k \hat{n}_l)|\, \boldsymbol{\eta} \,\rangle$$

$\Longrightarrow$ Hamiltonian of the spin-1/2 Heisenberg ferromagnet

$$H = \sum_{\langle k,l \rangle} w_{kl} h_{kl}$$

with the hopping matrices

$$h_{kl} = -\left(\sigma^{+} \otimes \sigma^{-} + \sigma^{-} \otimes \sigma^{+} - \hat{n} \otimes \hat{v} - \hat{v} \otimes \hat{n}\right)_{kl}$$

- Invariant measure for fixed N: uniform
- Grand canonical Bernoulli product measure

$$|\, \pi^{*}_{L,\phi} \rangle = \begin{pmatrix} 1-\rho \\ \rho \end{pmatrix}^{\otimes L}$$

with parameter ρ (particle density)

4.2 $\mathfrak{sl}(2,\mathbb{C})$ symmetry

- $\sigma^{\pm}, \sigma^{z}$ are representation of $\mathfrak{sl}(2,\mathbb{C})$

- Coproduct of Casimirs: $\Delta(C_0) = \mathbb{1} \otimes \mathbb{1}$, $\Delta(C_1) = \sigma^{+} \otimes \sigma^{-} + \sigma^{-} \otimes \sigma^{+} + \sigma^{z} \otimes \sigma^{z}/2$

- $h = \Delta(C_1) - \Delta(C_0) \Longrightarrow [H, S^{\pm}] = [H, S^{z}] = 0$ with the representation matrices

$$S^{\pm} = \sum_{k \in \Lambda} \sigma_k^{\pm}, \quad S^{z} = \frac{1}{2} \sum_{k \in \Lambda} \sigma_k^{z}$$

which satisfy the $\mathfrak{sl}(2,\mathbb{C})$ commutation relations

$$[S^{+}, S^{-}] = 2S^{z}, \quad [S^{z}, S^{\pm}] = \pm S^{\pm}.$$

- Generalizes to higher-dimensional representations

4.3 Duality

Theorem 4.1 (GMS and Sandow (1994)) *The SSEP on a lattice* Λ *is selfdual w.r.t. the factorized duality function*

$$D(\boldsymbol{\zeta},\boldsymbol{\eta}) = \prod_{k\in\Lambda} (\alpha + \beta\eta_k)^{\gamma+\delta\zeta_k}$$

for configurations $\boldsymbol{\eta},\boldsymbol{\zeta} \in \{0,1\}^\Lambda$ *and* $\alpha,\beta,\gamma,\delta \in \mathbb{R}$*. provided that* $N(\boldsymbol{\eta}) < \infty$ *if* $\gamma \neq 0$ *and* $N(\boldsymbol{\zeta}) < \infty$ *if* $\delta \neq 0$*.*

Remark 4.2 *Any Markov process whose generator is a function of the hopping matrices* $e_{k,l} = \sigma_k^x\sigma_l^x + \sigma_k^y\sigma_l^y + \sigma_k^z\sigma_l^z - \mathbf{1}$ *is* $\mathfrak{sl}(2,\mathbb{C})$ *symmetric and therefore self-dual w.r.t. the same duality functions as the SSEP.*

Remark 4.3 *Generalizes to partial exclusion with jump rate* $(\eta_k(m-\eta_l) + (m-\eta_k)\eta_l)$ *(spin-*$(m/2)$ *representation) [GMS and Sandow (1994)]*

Remark 4.4 *Let* $\mathbf{x}(\boldsymbol{\zeta}) := \{k : \zeta_k = 1\}$ *be the set of occupied sites* $x_i \in \Lambda$ *of the configuration* $\boldsymbol{\zeta}$ *and* $N(\mathbf{x}) = |\mathbf{x}|$ *be the number of particles in the configuration* $\mathbf{x}$*. For* $\gamma = 0$ *and with* $a = \alpha^\delta$*,* $b = (\alpha + \beta)^\delta - \alpha^\delta$ *the duality function becomes*

$$\tilde{D}(\mathbf{x}, \boldsymbol{\eta}) = \prod_{i=1}^{N(\mathbf{x})} (a + b\eta_{x_i})$$

for all $\mathbf{x} \in \Xi$ *and* $\boldsymbol{\eta} \in \Omega$*. For* $\alpha = 0$*,* $\beta = \delta = 1$ *corresponding to* $a = 0$ *and* $b = 1$ *one recovers the well-known duality function formulated and proved in a different way in [Liggett, 1985] and which goes back to [Spitzer, 1970].*

Proof: The $\mathfrak{sl}(2,\mathbb{C})$-symmetry implies that the L-fold Kronecker product $D = B^{\otimes L}$ is a symmetry operator for any 2×2 matrix B. Since the SSEP is reversible with uniform invariant measure this yields the duality function $D(\boldsymbol{\zeta},\boldsymbol{\eta}) = \langle\,\boldsymbol{\zeta}\,|\hat{D}|\,\boldsymbol{\eta}\,\rangle$. The factorization of the symmetry operator and also of the basis vectors yields

$$D(\boldsymbol{\zeta},\boldsymbol{\eta}) = \prod_{k\in\Lambda} \langle\,\zeta_k\,|B|\,\eta_k\,\rangle \tag{1}$$

Explicit computation of the two-dimensional bilinear form

$$\langle\,\zeta_k\,|B|\,\eta_k\,\rangle = (1-\zeta_k,\zeta_k)\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} 1-\eta_k \\ \eta_k \end{pmatrix} \tag{2}$$

yields $(\alpha+\beta\eta_k)^{\gamma+\delta\zeta_k}$ with $B_{11} = \alpha^\gamma$, $B_{12} = (\alpha+\beta)^\gamma$, $B_{12} = \alpha^{\gamma+\delta}$, $B_{22} = (\alpha+\beta)^{\gamma+\delta}$. $\square$

4.4 Density profile and dynamical structure function

- Take $\alpha = \gamma = 0$ and $\beta = \delta = 1$ in the duality function

$\Longrightarrow$ Duality matrix $D = \mathrm{e}^{S^+}$

Theorem 4.5 *For an arbitrary multi-particle initial measure μ the density profile $\rho_x(t) := \langle\, \eta_x(t) \,\rangle_\mu$ is given by*

$$\rho_x(t) = \sum_{x' \in \Lambda} \rho_{x'}(0) p_t(x'x)$$

where $p_t(x'x)$ is the transition probability of the single random walk on Λ with edge jump rates w_{kl}.

Proof: For a single site $(1,1) = (1,0)\mathrm{e}^{\sigma^+}$ and $\mathrm{e}^{\sigma^+}\hat{n}\mathrm{e}^{-\sigma^+} = \hat{n} + \sigma^+$. Therefore $\langle\, s \,| = \langle\, 0 \,|\mathrm{e}^{S^+}$. Hence

$$
\begin{aligned}
\rho_x(t) &= \langle\, s\,|\hat{n}_k \mathrm{e}^{-Ht}|\,\mu\,\rangle \\
&= \langle\, 0\,|\mathrm{e}^{S^+}\hat{n}_k \mathrm{e}^{-S^+}\mathrm{e}^{S^+}\mathrm{e}^{-Ht}|\,\mu\,\rangle \\
&= \langle\, 0\,|\mathrm{e}^{\sigma_k^+}\hat{n}_k \mathrm{e}^{-\sigma_k^+}\mathrm{e}^{S^+}\mathrm{e}^{-Ht}|\,\mu\,\rangle \\
&= \langle\, 0\,|(\hat{n}+\sigma^+)\mathrm{e}^{-Ht}\mathrm{e}^{S^+}|\,\mu\,\rangle \\
&= \langle\, k\,|\mathrm{e}^{S^+}\mathrm{e}^{-Ht}|\,\mu\,\rangle \\
&= \sum_{k'\in\Lambda}\langle\, k\,|\mathrm{e}^{-Ht}|\,k'\,\rangle\langle\, k'\,|\mathrm{e}^{S^+}|\,\mu\,\rangle \\
&= \sum_{k'\in\Lambda}\langle\, k\,|\mathrm{e}^{-Ht}|\,k'\,\rangle\langle\, s\,|\hat{n}_{k'}|\,\mu\,\rangle
\end{aligned}
$$

□

Corollary 4.6 *Take as initial state $(\hat{n}_0-\rho)|\,\rho\,\rangle$ with Bernoulli product measure $|\,\rho\,\rangle$ with density ρ. Then $S(k,t)=\langle\, s\,|\hat{n}_k\mathrm{e}^{-Ht}(\hat{n}_0-\rho)|\,\rho\,\rangle$ is the dynamical structure function with initial value $S(k,0)=\rho(1-\rho)\delta_k,0$ and $S(k,t)=\rho(1-\rho)p_t(0,k)$.*

Remark 4.7 *Theorem 4.5 generalizes to multi-time joint expectations* $\langle\, \hat{n}_{k_m}(t_m) \ldots \hat{n}_{k_m}(t_m) \,\rangle_\mu$*. For any initial measure with support on configurations with any number of particles the joint expectations of* m *occupation numbers can be expressed in terms of transition probabilities for initial states with only* m *particles.*

Remark 4.8 *On the* d*-dimensional hypercubic lattice* $\mathbb{Z}^d$ *with translation-invariant nearest-neighbour hopping the single-particle propagator satisfies a discrete diffusion equation which can be solved in explicit form in terms of modified Bessel functions*

$$I_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}p \ \mathrm{e}^{ipn - t\cos p}.$$

On $\mathbb{Z}^d$ *with hopping rates* w_i *in each direction one then has*

$$\rho_{\vec{x}}(t) = \prod_{j=1}^{d} \sum_{x'_j \in \mathbb{Z}} \rho_{x'_j}(0) \mathrm{e}^{-w_j t} I_{x_j - x'_j}(w_j t).$$

The dynamical structure function becomes

$$S_{\vec{x}}(t) = \prod_{j=1}^{d} \mathrm{e}^{-2w_j t} I_{x_j - x'_j}(2w_j t).$$

In the scaling limit $x_i(t) = r_i\sqrt{4w_i t}$ *and* $t \to \infty$ *the modified Bessel function becomes a Gaussian. Thus*

$$\prod_{j=1}^{d} \sqrt{4\pi w_j} \lim_{t\to\infty} t^{d/2} S_{\vec{x}(t)}(t) = \mathrm{e}^{-\sum_{j=1}^{d} r_j^2}.$$

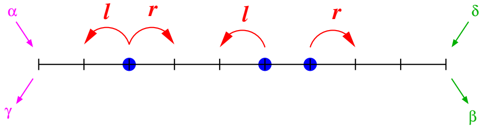
We read off the dynamical exponent $z = 2$ *and the universal Gaussian scaling function with diagonal diffusion matrix* $D_{ij} = 2w_i\delta_{ij}$.

5. Reverse duality for the open ASEP

5.1 Open asymmetric simple exclusion process

At most one particle per site on integer lattice with L sites

Process	Transition	Rate
Jump to the right	$10 \to 01$	r
Jump to the left	$01 \to 10$	ℓ
Creation at site 1 (L)	$0 \to 1$	α (δ)
Annihilation at site 1 (L)	$1 \to 0$	γ (β)



• Quantum Hamiltonian:

$$H = -Q^T = \sum_{k=1}^{L-1} \tilde{h}_{k,k+1} + \tilde{h}^-_{L_-} + \tilde{h}^+_{L_+}$$

where

$$\begin{aligned}
\tilde{h}_{k,k+1} &= r(\hat{v}_k \hat{n}_{k+1} - \sigma_k^+ \sigma_{k+1}^-) + \ell(\hat{n}_k \hat{v}_{k+1} - \sigma_k^- \sigma_{k+1}^+) \\
\tilde{h}^-_{L_-} &= \alpha(\hat{v}_1 - \sigma_1^-) + (\gamma + r - \ell)\hat{n}_1 - \gamma \sigma_1^+ \\
\tilde{h}^+_{L_+} &= \delta(\hat{v}_L - \sigma_L^-) + (\beta - r + \ell)\hat{n}_L - \beta \sigma_L^+
\end{aligned}$$

• Invariant matrix product measure (MPM) with generally infinite-dimensional matrices [Derrida et al., 1993]

- Hopping asymmetry and time scale $q := \sqrt{\frac{r}{\ell}}, \quad w := \sqrt{r\ell}$
- Boundary densities $\rho_\pm$ and boundary jump barriers $\omega_\pm$

$$\alpha = (r+\omega_-)\rho_-, \quad \gamma = (\ell+\omega_-)(1-\rho_-)$$
$$\beta = (r+\omega_+)(1-\rho_+), \quad \delta = (\ell+\omega_+)\rho_+$$

- Fugacities: $z_\sharp \equiv z(\rho_\sharp) = \dfrac{\rho_\sharp}{1-\rho_\sharp}$
- Sandow function [Sandow, 1994]

$$\kappa_\pm(x,y) := \frac{1}{2x}(y-x+r-\ell \pm \sqrt{(y-x+r-\ell))^2+4xy})$$

$$\kappa_+(\alpha,\gamma) = z_-^{-1}, \quad \kappa_+(\beta,\delta) = z_+$$

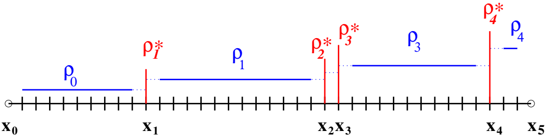
$$\kappa_-(\alpha,\gamma) = -\frac{\ell+\omega_-}{r+\omega_-}, \quad \kappa_-(\beta,\delta) = -\frac{\ell+\omega_+}{r+\omega_+}$$

5.2 Bernoulli shock measures

Definition 5.1 (Bernoulli shock measures) *With auxiliary boundary reservoir sites* $x_0 := 0$ *and* $x_{N+1} := L+1$ *the product measure* $\mu_{\boldsymbol{\eta}}^{\mathbf{x}} = \prod_{k=1}^{L} p_{\eta_k}^{\mathbf{x}}$ *with marginals*

$$p_{\eta_k}^{\mathbf{x}} = \begin{cases} (1-\rho_i^\star)(1-\eta_k) + \rho_i^\star \eta_k & k = x_i, \quad 1 \le i \le N \\ (1-\rho_i)(1-\eta_k) + \rho_i \eta_k & x_i < k < x_{i+1}, \quad 0 \le i \le N \end{cases}$$

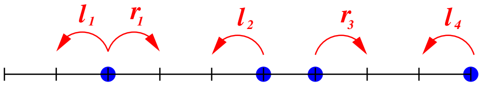
is called a Bernoulli shock measure with N *microscopic shocks at positions* $x_i \in \{1, \ldots, L\}$ *and bulk densities* ρ_i *for* $0 \le i \le N$*, and shock densities* $\rho_i^\star$ *for* $1 \le i \le N$*.*



5.3 Shock ASEP

• At most one particle per site on integer lattice with L sites, N particles, single-file jumps, closed boundaries
$\mathbf{x} = (x_1, \dots, x_N)$, $1 \le x_1 < \cdots < x_i < x_{i+1} < \cdots < x_N \le L$

Process	Transition	Rate
Jump of particle i to the right	$x_i \to x_i + 1$	r_i
Jump of particle i to the left	$x_i \to x_i - 1$	ℓ_i



$$r_i = (r - \ell)\frac{\rho_i(1-\rho_i)}{\rho_i - \rho_{i-1}}, \quad \ell_i = (r - \ell)\frac{\rho_{i-1}(1-\rho_{i-1})}{\rho_i - \rho_{i-1}}$$

with $\rho_i \in (0,1)$ and $r_i \ell_i = w^2$ for all i.

- Definition of rates implies

$$z_{i+1} = q^2 z_i \qquad \bigstar$$

for all i with free parameter z_0

- Reversible w.r.t. the unnormalized product measure

$$\pi^*_{\mathbf{x}} = \prod_{i=1}^{N} \left(\frac{r_i}{\ell_i}\right)^{x_i}$$

- Focus now on special manifolds

$$\mathcal{B}_N := \{\alpha, \beta, \gamma, \delta \in \mathbb{R}^+ : \kappa_+(\alpha,\gamma)\kappa_+(\beta,\delta) = q^{2N}\}$$

$$\mathcal{B}_N^M := \{\alpha, \beta, \gamma, \delta \in \mathcal{B}_N : \kappa_-(\alpha,\gamma)\kappa_-(\beta,\delta) = q^{-2M}\}, \quad 1 \le M \le N$$

5.4 Reverse duality

Theorem 5.2 (GMS (2023)) *Let H be the quantum Hamiltonian of the open ASEP and for parameters $\rho_0, \ldots, \rho_N$ let G be the quantum Hamiltonian of the N-particle shock exclusion process. Further, let $\mu_{\boldsymbol{\eta}}^{\mathbf{x}}$ be the BSM with left boundary density $\rho_0 = \rho_-$ and shock fugacities*

$$z_i^\star = \frac{\alpha}{\gamma} q^{2(i-1)}$$

for $1 \leq i \leq N \leq L$. The reverse-duality relation

$$HR = RG^T$$

w.r.t. the duality matrix R with matrix elements $R_{\boldsymbol{\eta}x} = \pi(\mathbf{x})\mu_{\boldsymbol{\eta}}^{\mathbf{x}}$ holds if and only if the following two conditions are satisfied:
(i) The microscopic shock stability condition $(\star)$ is satisfied for all $i \in \{1, \ldots, N\}$,
(ii) The boundary rates are on the manifold $\mathcal{B}_N^1$.

Corollary 5.3 (Shock random walk) $N = 1$: *Denote by $\mu^x_{\boldsymbol{\eta}}(t)$ the distribution at time t of the open ASEP, and let Conditions (i) - (ii) of the previous Theorem be satisfied. Then, for any $x \in \{1, \ldots, L\}$*

$$\mu^x_{\boldsymbol{\eta}}(t) = \sum_{y=1}^{L} P(y,t|x,0)\,\mu^y_{\boldsymbol{\eta}}(0)$$

where

$$P(y,t|x,0) = \frac{d_1^2 - 1}{d_1^{2L} - 1} d_1^{2(y-1)} + \frac{2}{L}\sum_{p=1}^{L-1} d_1^{y-x}\psi_p(x)\psi_p(y)\frac{w}{\epsilon_p}\mathrm{e}^{-\epsilon_p t}$$

with $\epsilon_p = w\left[d_1 + d_1^{-1} - 2\cos\left(\frac{\pi p}{L}\right)\right]$ *and* $\psi_p(y) := d_1 \sin\left(\frac{\pi p y}{L}\right) - \sin\left(\frac{\pi p(y-1)}{L}\right)$ *is the transition probability of the biased random walk starting at time $t = 0$ from x.*

Corollary 5.4 *The evolution of the open ASEP with an initial BSM with N shocks is given by the transition probabilities of the conservative N-particle shock exclusion process.*

Remark 5.5 *(1) The conservative reflective boundaries of the reverse dual are in contrast to the conventional duality for the open SSEP which is dual to the SSEP with nonconservative absorbing boundaries.* *[Spohn (1983); Carinci et al. (2013); Frassek et al. (2020)]*

(2) A reverse dual with absorbing boundaries and one shock exists on the manifolc $\mathcal{B}_1^1$. *[GMS (2023)]*

• To prove reverse duality notice:

(a) Columns of duality matrix R are the BSM probability vectors $|\, \mu^{\mathbf{x}} \,\rangle$

(b) Duality implies invariant subspace spanned by the BSM probability vectors: $H|\, \mu^{\mathbf{x}} \,\rangle \in \mathsf{span}\{|\, \mu^{\mathbf{y}} \,\rangle\}$

$\Rightarrow$ Step 1: Use local transitions to prove that $H|\, \mu^{\mathbf{x}} \,\rangle = \sum_{\mathbf{y}} G_{\mathbf{xy}}|\, \mu^{\mathbf{y}} \,\rangle\}$

$\Rightarrow$ Step 2: Prove by computation that coefficients $G_{\mathbf{xy}}$ are nonpositive for $\mathbf{x} \neq \mathbf{y}$ and conserve probability, i.e., $G_{\mathbf{xx}} = -\sum_{\mathbf{x}\neq\mathbf{y}} G_{\mathbf{xy}}$

• To prove explicit time-dependent transition probability for one shock notice that G is a tridiagonal Toeplitz matrix

• All steps involve only matrix multiplications of matrices with dimension of at most 4.

6. Factorized duality

- Search for useful dualities with factorization ansatz

- Consider reaction-diffusion systems with exclusion

- Arbitrary graph Λ with site reactions $1 \leftrightarrow 0$ and bond reactions

$\{0,0\} \to a_{21}\{0,1\} + a_{31}\{1,0\} + a_{41}\{1,1\}$ (birth/pair creation)

$\{0,1\} \to a_{12}\{0,0\} + a_{32}\{1,0\} + a_{42}\{1,1\}$ (death/diffusion/decoagulatio

$\{1,0\} \to a_{13}\{0,0\} + a_{23}\{0,1\} + a_{43}\{1,1\}$ (death/diffusion/decoagulatio

$\{1,1\} \to a_{14}\{0,0\} + a_{24}\{0,1\} + a_{34}\{1,0\}$ (pair annihilation/coagulation

- Includes SSEP, ASEP, contact process, voter model, ...

• Quantum Hamiltonian is of the form

$$H = \sum_{k \in \Lambda} g_k + \sum_{\langle k,l \rangle} h_{kl}$$

with

$$h_{kl} = - \begin{pmatrix} \cdot & a_{12} & a_{13} & a_{14} \\ a_{21} & \cdot & a_{23} & a_{24} \\ a_{31} & a_{32} & \cdot & a_{34} \\ a_{41} & a_{42} & a_{43} & \cdot \end{pmatrix}_{kl}$$

• Factorized duality: $D = B^{\otimes|\Lambda|}$ with 2×2 matrix B

$\Longrightarrow$ Duality: $Bg = \tilde{g}^T B$, $(B \otimes B)h = \tilde{h}^T (B \otimes B)$

$\Longrightarrow$ Reverse duality: $gB = B\tilde{g}^T$, $h(B \otimes B) = (B \otimes B)\tilde{h}^T$

• Finding dualities for any process with Hamiltonian H on any graph $=$ Multiplikation of 4×4 matrices [GMS (1995), Redig, Sau (2018)]

Theorem 6.1 (GMS (1995)) *On the 10-parameter manifold defined by*

$$\begin{aligned} 0 &= a_{12} + a_{32} + a_{21} + a_{41} - a_{23} - a_{43} - a_{14} - a_{34} \\ 0 &= a_{13} + a_{23} + a_{31} + a_{41} - a_{32} - a_{42} - a_{14} - a_{24} \end{aligned}$$

the Hamiltonian H with $g = 0$ has a sequence of invariant subspaces with dimensions: (1) $a_{41} \neq 0$ or $a_{41} = 0$, $a_{21}a_{31} \neq 0$ $d_k^{(1)} = \binom{|\Lambda| + k}{k}$, or (2) $a_{41} = a_{21} = a_{31} = 0$: $d_k^{(2)} = \binom{|\Lambda|}{k}$. The dual Hamiltonian is stochastic on a subset of these manifolds.

$Proof:$ Take $B = \mathbb{1} + \sigma^+$ and demand the dual $\tilde{h}$ to be tridiagonal. $\square$

Remark 6.2 $d^{(i)}$*: Cardinality of state space with k particles without exclusion (Case (1)) or with exclusion Case (2).*

Conclusions: Algebraic approach to duality

• Construction of dualities using non-Abelian symmetries

• Duality for time evolution of specific expectations for arbitrary initial measures

• Reverse duality for time evolution of specfici measures and arbitrary expectations

• Reduction of complexity through invariant subspaces

• Factorized duality for arbitrary graphs even without apparent symmetries