# Quantitative hydrodynamic limit and regularity for Langevin dynamics

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joint work with S. Armstrong

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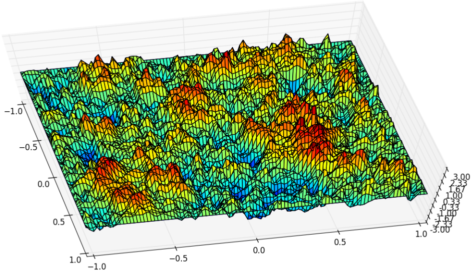
# Summary

# Introduction: Random surfaces

**Interface model:** Model the interface separating two phases.

- **An interface (or a surface)** is represented by a function $\phi : \mathbb{Z}^d \to \mathbb{R}$ ;



# Random surfaces

- *Base space:* Consider a finite set $\Lambda \subseteq \mathbb{Z}^d$, typically **a box**

$$\Lambda_L := \{-L, \dots, L\}^d \, ;$$

- *Configuration space:* Set of **functions**

$$\mathbb{R}^{\Lambda_L} := \{\phi : \Lambda_L \to \mathbb{R} \, : \, \phi = 0 \text{ on boundary}\} \, .$$

## Definition: Random surfaces

We define the **finite-volume Gibbs measure**

$$\mu_{\Lambda_L}(d\phi) := \frac{1}{Z} \exp\left( - \sum_{x \sim y} V(\phi(y) - \phi(x)) \right) \prod_{x \in \Lambda_L} d\phi(x),$$

where $V : \mathbb{R} \to \mathbb{R}$ is a convex, symmetric, interaction potential

$$0 < \lambda \leq V'' \leq \Lambda < \infty.$$

# Random surfaces and Langevin dynamics

## Definition: Langevin dynamics

The Gibbs measure is associated with **the Langevin dynamics:**

$$d\phi_t(x) = \sum_{y \sim x} V'(\phi_t(y) - \phi_t(x))dt + \sqrt{2}dB_t(x),$$

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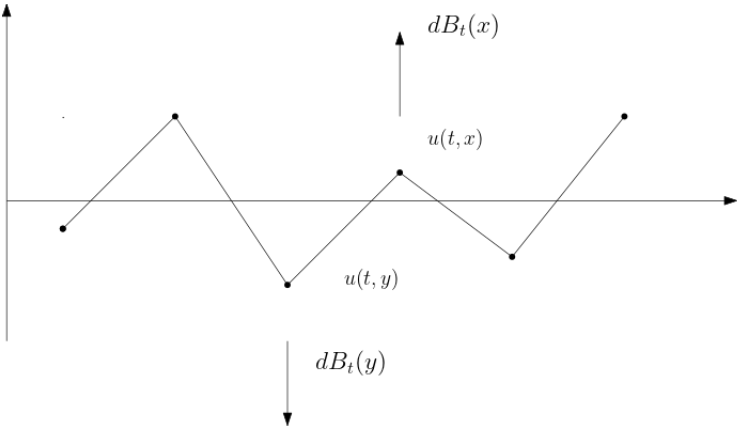

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**Objective:** Study the **large-scale behavior** of the model:

- Localization/Delocalization;
- Hydrodynamic limit (LLN);
- Scaling limit (CLT), etc.

# Literature

Random surfaces:

- **Brascamp-Lieb-Lebowitz (1975)** Localization, Delocalization;
- **Funaki-Spohn (1997)** Hydrodynamic Limit/LLN;
- **Naddaf-Spencer (1998)** Scaling Limit/CLT;
- **Deuschel-Giacomin-Ioffe (2000)** Large deviations;
- **Giacomin-Olla-Spohn (2001)** Scaling Limit/CLT;
- Lecture notes of **Funaki** (Saint Flour 2003);

**Cotar, Biskup, Bolthausen, Brydges, Fröhlich, Helffer, Kotecký, Müller, Nishikawa, Sheffield, Sjöstrand, Velenik, Yau etc.**

# Langevin dynamics as a parabolic equation

From *a PDE perspective*, one can rewrite the Langevin dynamics as

$$\underbrace{d\phi_t(x) - \nabla \cdot V'(\nabla\phi_t)dt}_{\text{Parabolic operator}} = \underbrace{\sqrt{2}dB_t(x).}_{\text{random noise}}$$

**Question:** Study **the large-scale behavior** of the Langevin dynamics:

- **Probability:** Law of Large Numbers, Central Limit Theorem, etc.
- **Analysis:** What is the regularity of the Langevin dynamics?

# Summary

# The Hydrodynamic Limit

## Hydrodynamic Limit

Let $u$ is a solution of

$$du = \nabla \cdot V'(\nabla u)dt + dB_t \quad \text{in} \quad \mathbb{R} \times \mathbb{Z}^d.$$

Then, **over large-scales**:

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- $\bar{\nu}$ is called *the surface tension.*

# The Hydrodynamic limit

## Hydrodynamic Limit (Funaki-Spohn, 1997)

Let $u$ be a solution of

$$du = \nabla \cdot V'(\nabla u)dt + dB_t \quad \text{in} \quad \mathbb{R} \times \mathbb{Z}^d.$$

Then there exists a solution $\bar{u}$ of

$$\partial_t \bar{u} - \nabla \cdot D_p \bar{\nu}(\nabla \bar{u}) = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^d,$$

such that

$$\mathbb{E}\left[\frac{1}{L^2} \|u - \bar{u}\|^2_{\underline{L}^2(Q_L)}\right] \underset{L \to \infty}{\longrightarrow} 0$$

with $Q_L = [0, L^2] \times \Lambda_L$.

**Notation:**

$$\Lambda_L := \{-L, \ldots, L\}^d \quad \text{and} \quad \|u\|^2_{\underline{L}^2(Q_L)} = \frac{1}{L^2 |\Lambda_L|} \int_{[0,L^2]} \sum_{x \in \Lambda_L} |u(t,x)|^2 \, dt.$$

# Quantitative Hydrodynamic Limit

## Quantitative Hydrodynamic Limit (Armstrong-D. 2022)

Let $u$ be a solution of

$$du(t,\cdot) = \nabla \cdot V'(\nabla u)(t,\cdot)dt + dB_t.$$

Then, there exists a solution $\bar{u}$ of

$$\partial_t \bar{u} - \nabla \cdot D_p \bar{\nu}(\nabla \bar{u}) = 0,$$

such that

$$\frac{1}{L} \|u - \bar{u}\|_{\underline{L}^2(Q_L)} \leq \frac{(1 + \sqrt{\ln L}\mathbb{1}_{d=2})}{\sqrt{L}},$$

with $Q = [0, L^2] \times \Lambda_L$ and with **Gaussian stochastic integrability.**

**Strategy of the proof:** Two-scale expansion.

# Summary

# Two questions of regularity

**Two questions of regularity:**

1. What is the regularity of the surface tension $\bar{\nu}$?

2. What is the regularity of the Langevin dynamics

$$\begin{cases} \partial_t u = \nabla \cdot V'(\nabla u) & \implies u \in C^{1,\alpha}, \\ du = \nabla \cdot V'(\nabla u) dt + dB_t & \implies u \in \ ? \end{cases}$$

# Regularity of the surface tension

**Deuschel-Giacomin-Ioffe (2000):** If $V$ is $C^{1,1}(\mathbb{R})$ and uniformly convex then $\bar{\nu}$ is $C^{1,1}(\mathbb{R}^d)$ and uniformly convex.

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**Remarks:**

- The surface tension is **more** regular than the potential;
- The modulus of continuity of $D_p^2 \bar{\nu}$ depends on **measure theoretical** information on the map $V''$:
  - How well can the function $V''$ be approximated by continuous functions.

# Regularity of the Langevin dynamics

Consider the **parabolic equation**

$$\boxed{\partial_t u - \nabla \cdot V'(\nabla u) = 0.}$$

with the uniform ellipticity assumption

$$0 \leq \lambda \leq V''(x) \leq \Lambda < \infty$$

**Regularity:** The solution $u$ is $C^{1,\alpha}$ for a *tiny* $\alpha \in (0, 1)$.

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# Regularity for Langevin dynamics

**Large-scale regularity:**

1. If $u$ is a solution of

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**History:** Avellaneda-Lin, Armstrong-Smart, Gloria-Neukamm-Otto.

Thank you for your attention!