Metastable Hierarchy in Abstract Low-Temperature Lattice Models

Seonwoo Kim

KIAS HCMC
June E Huh Center for Mathematical Challenges

Rencontres de Probabilités 2024

Université de Rouen Normandie, September 26th, 2024

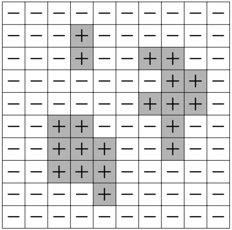
Basic Setup

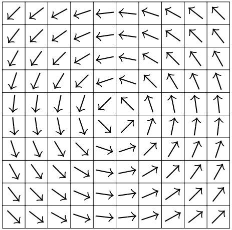
- **Lattice Model**: connected space $\Omega = S^{\Lambda}$, Hamiltonian $\mathbb{H} : \Omega \to \mathbb{R}$

Ising Model $\Omega = \{-1, +1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\frac{1}{2} \sum_{x \sim y} \eta_x \eta_y - \frac{h}{2} \sum_{x \in \Lambda} \eta_x$ (h small)

Potts Model $\Omega = \{1, 2, \ldots, q\}^{\Lambda}$, $\mathbb{H}(\eta) = -\sum_{x \sim y} \mathbf{1}\{\eta_x = \eta_y\}$

XY Model $\Omega = (\mathbb{R}/\mathbb{Z})^{\Lambda}$, $\mathbb{H}(\eta) = -\sum_{x \sim y} J_{xy} \cos(\eta_x - \eta_y) - \sum_{x \in \Lambda} h_x \cos \eta_x$





Basic Setup

- **Lattice Model**: connected space $\Omega = S^{\Lambda}$, Hamiltonian $\mathbb{H} : \Omega \to \mathbb{R}$

  Ising Model $\Omega = \{-1, +1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y} \eta_x \eta_y - \frac{h}{2}\sum_{x\in\Lambda} \eta_x$ (h small)

  Potts Model $\Omega = \{1, 2, \dots, q\}^{\Lambda}$, $\mathbb{H}(\eta) = -\sum_{x\sim y} \mathbf{1}\{\eta_x = \eta_y\}$

  XY Model $\Omega = (\mathbb{R}/\mathbb{Z})^{\Lambda}$, $\mathbb{H}(\eta) = -\sum_{x\sim y} J_{xy} \cos(\eta_x - \eta_y) - \sum_{x\in\Lambda} h_x \cos \eta_x$

- **Gibbs Measure**: $\mu_\beta(\eta) = Z_\beta^{-1} e^{-\beta \mathbb{H}(\eta)}$, β: inverse temperature $\to \infty$

Basic Setup

- **Lattice Model**: connected space $\Omega = S^{\Lambda}$, Hamiltonian $\mathbb{H} : \Omega \to \mathbb{R}$

  Ising Model $\Omega = \{-1,\, +1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x$ (h small)

  Potts Model $\Omega = \{1,\, 2,\, \dots,\, q\}^{\Lambda}$, $\mathbb{H}(\eta) = -\sum_{x\sim y}\mathbf{1}\{\eta_x = \eta_y\}$

  XY Model $\Omega = (\mathbb{R}/\mathbb{Z})^{\Lambda}$, $\mathbb{H}(\eta) = -\sum_{x\sim y}J_{xy}\cos(\eta_x - \eta_y) - \sum_{x\in\Lambda}h_x\cos\eta_x$

- **Gibbs Measure**: $\mu_\beta(\eta) = Z_\beta^{-1}e^{-\beta\mathbb{H}(\eta)}$, β: inverse temperature $\to\infty$

- **Metropolis Dynamics**: continuous-time MC $\{\eta_\beta(t)\}_{t\ge 0}$ in Ω with jump rate

$$r_\beta(\eta,\, \xi) = \begin{cases} e^{-\beta\max\{\mathbb{H}(\xi)-\mathbb{H}(\eta),\, 0\}} & \text{if} \quad \eta\sim\xi, \\ 0 & \text{otherwise.}\end{cases}$$

  *Reversible w.r.t. the Gibbs measure μ_β

$$\mathbb{H}(\xi)\le\mathbb{H}(\eta): \quad \textit{fast jump}, \quad \mathbb{H}(\xi)>\mathbb{H}(\eta): \quad \textit{slow jump}$$

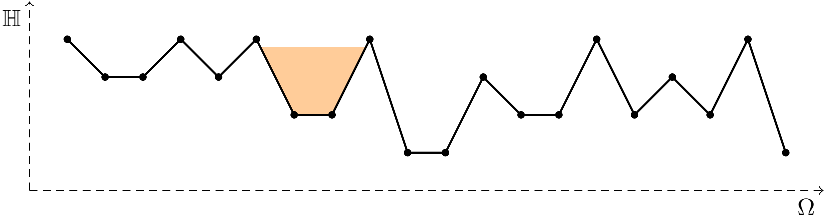
Metastable Behavior as $\beta \to \infty$

- **Metropolis Dynamics**: continuous-time MC $\{\eta_\beta(t)\}_{t\geq 0}$ in Ω with jump rate

$$r_\beta(\eta,\,\xi) = \begin{cases} e^{-\beta \max\{\mathbb{H}(\xi)-\mathbb{H}(\eta),\,0\}} & \text{if} \quad \eta \sim \xi, \\ 0 & \text{otherwise.} \end{cases}$$

*Reversible w.r.t. the Gibbs measure μ_β

$$\mathbb{H}(\xi) \leq \mathbb{H}(\eta):\quad \textit{fast jump},\quad \mathbb{H}(\xi) > \mathbb{H}(\eta):\quad \textit{slow jump}$$



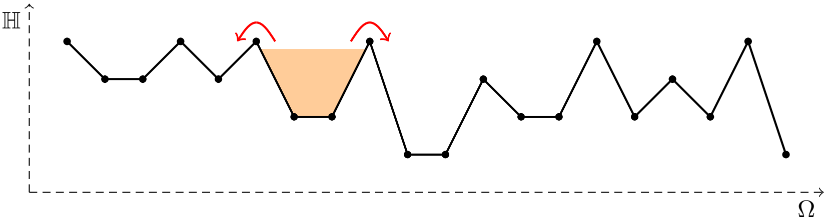
Metastable Behavior as $\beta \to \infty$

- **Metropolis Dynamics**: continuous-time MC $\{\eta_\beta(t)\}_{t\geq 0}$ in Ω with jump rate

$$r_\beta(\eta,\,\xi) = \begin{cases} e^{-\beta \max\{\mathbb{H}(\xi)-\mathbb{H}(\eta),\,0\}} & \text{if} \quad \eta \sim \xi, \\ 0 & \text{otherwise.} \end{cases}$$

*Reversible w.r.t. the Gibbs measure μ_β

$$\mathbb{H}(\xi) \leq \mathbb{H}(\eta): \quad \textit{fast jump}, \quad \mathbb{H}(\xi) > \mathbb{H}(\eta): \quad \textit{slow jump}$$



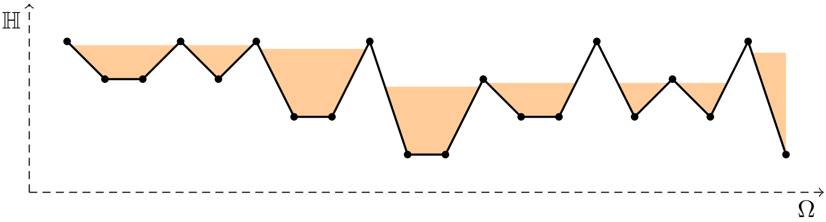
Metastable Behavior as $\beta \to \infty$

- **Metropolis Dynamics**: continuous-time MC $\{\eta_\beta(t)\}_{t\geq 0}$ in Ω with jump rate

$$r_\beta(\eta,\,\xi) = \begin{cases} e^{-\beta\max\{\mathbb{H}(\xi)-\mathbb{H}(\eta),\,0\}} & \text{if} \quad \eta\sim\xi, \\ 0 & \text{otherwise.} \end{cases}$$

*Reversible w.r.t. the Gibbs measure μ_β

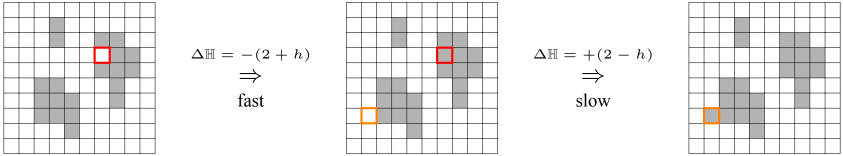
$$\mathbb{H}(\xi)\leq\mathbb{H}(\eta): \quad \textit{fast jump}, \quad \mathbb{H}(\xi)>\mathbb{H}(\eta): \quad \textit{slow jump}$$



Two Examples: Glauber & Kawasaki Dynamics

- **Glauber Dynamics**: $\Omega = \{-1, +1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x$

$\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single spin flip



Two Examples: Glauber & Kawasaki Dynamics

- **Glauber Dynamics**: $\Omega = \{-1, +1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single spin flip

  *Whole graph Ω is connected w.r.t. the Glauber dynamics

Two Examples: Glauber & Kawasaki Dynamics

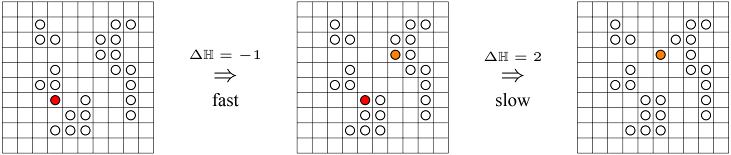
- **Glauber Dynamics**: $\Omega = \{-1, +1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single spin flip

  *Whole graph Ω is connected w.r.t. the Glauber dynamics

- **Kawasaki Dynamics**: $\Omega = \{0, 1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\sum_{x\sim y}\eta_x\eta_y$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single particle jump



Two Examples: Glauber & Kawasaki Dynamics

- **Glauber Dynamics**: $\Omega = \{-1, +1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single spin flip

  *Whole graph Ω is connected w.r.t. the Glauber dynamics

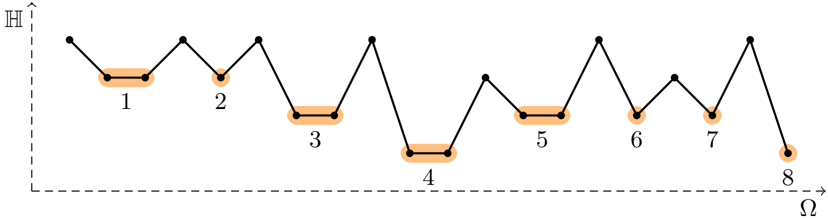
- **Kawasaki Dynamics**: $\Omega = \{0, 1\}^{\Lambda}$, $\mathbb{H}(\eta) = -\sum_{x\sim y}\eta_x\eta_y$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single particle jump

  *Subgraph $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda}\eta_x = \mathcal{N}\}$ is connected

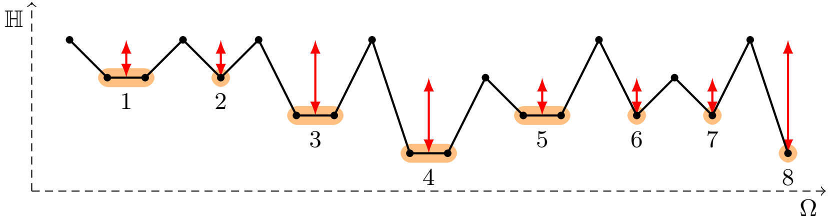
Stable Plateau and Initial Depth

- **Stable Plateau** $\mathcal{P}$: nonempty connected subset s.t.
  - $\mathbb{H}(\eta) = \mathbb{H}(\mathcal{P})$ for all $\eta \in \mathcal{P}$
  - $\mathbb{H}(\zeta) > \mathbb{H}(\mathcal{P})$ for all $\zeta \in \partial\mathcal{P}$



Stable Plateau and Initial Depth

- **Stable Plateau** $\mathcal{P}$: nonempty connected subset s.t.
  - $\mathbb{H}(\eta) = \mathbb{H}(\mathcal{P})$ for all $\eta \in \mathcal{P}$
  - $\mathbb{H}(\zeta) > \mathbb{H}(\mathcal{P})$ for all $\zeta \in \partial\mathcal{P}$
- **Depth** $\Gamma^{\mathcal{P}}$: minimal barrier from $\mathcal{P}$ to reach another one

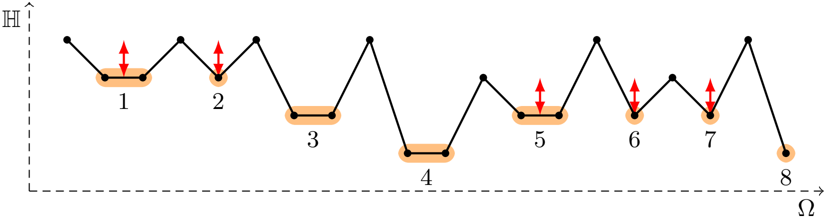


Stable Plateau and Initial Depth

- **Stable Plateau** $\mathcal{P}$: nonempty connected subset s.t.
  - $\mathbb{H}(\eta) = \mathbb{H}(\mathcal{P})$ for all $\eta \in \mathcal{P}$
  - $\mathbb{H}(\zeta) > \mathbb{H}(\mathcal{P})$ for all $\zeta \in \partial\mathcal{P}$

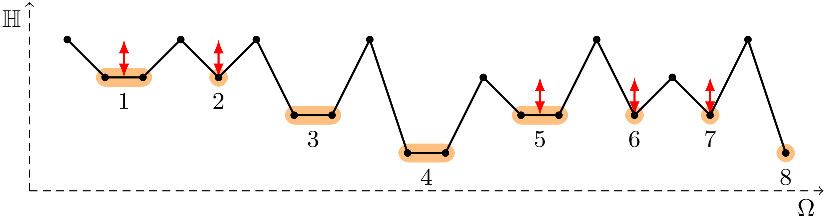
- **Depth** $\Gamma^{\mathcal{P}}$: minimal barrier from $\mathcal{P}$ to reach another one

- **Initial Depth**: $\Gamma^1 = \min\{\Gamma^{\mathcal{P}} : \mathcal{P} \text{ is a stable plateau}\}$



Metastable Transitions at Level 1

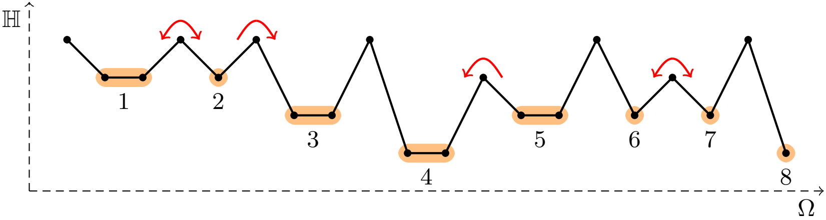
- **Initial Depth**: $\Gamma^1 = \min \{\Gamma^{\mathcal{P}} : \mathcal{P} \text{ is a stable plateau}\}$



Metastable Transitions at Level 1

- **Initial Depth**: $\Gamma^1 = \min\{\Gamma^{\mathcal{P}} : \mathcal{P} \text{ is a stable plateau}\}$

  $\Rightarrow$ accelerate the dynamics $\{\eta_\beta(t)\}_{t\geq 0}$ by the time-scale $e^{\Gamma^1 \beta} \gg 1$



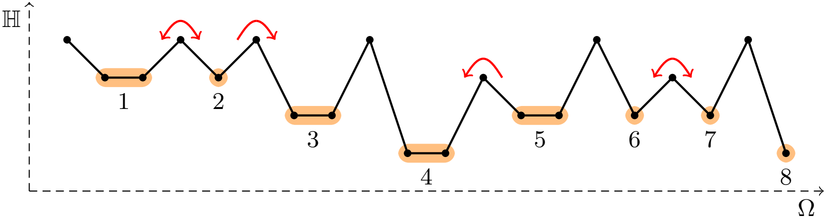
Metastable Transitions at Level 1

- **Initial Depth**: $\Gamma^1 = \min\{\Gamma^{\mathcal{P}} : \mathcal{P} \text{ is a stable plateau}\}$

  $\Rightarrow$ accelerate the dynamics $\{\eta_\beta(t)\}_{t\geq 0}$ by the time-scale $e^{\Gamma^1\beta} \gg 1$

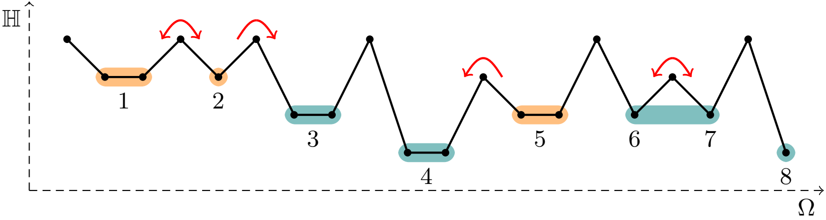
Theorem (K., *arXiv:2405.08488*)

$\{\eta_\beta(e^{\Gamma^1\beta}t)\}_{t\geq 0}$ converges to $\{\mathfrak{X}^1(t)\}_{t\geq 0}$ in the sense of marginal distributions



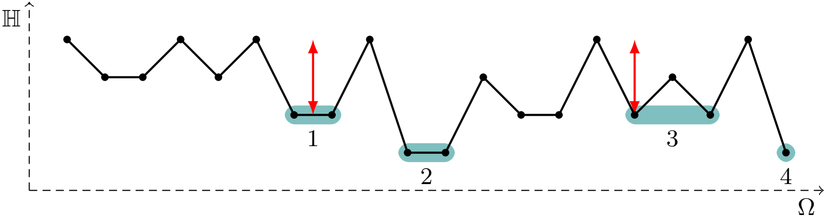
From Level 1 to Level 2

- **Next Step**: collect the *recurrent* components of $\{\mathfrak{X}^1(t)\}_{t\geq 0}$ at level 1



From Level 1 to Level 2

- **Next Step**: collect the *recurrent* components of $\{\mathfrak{X}^1(t)\}_{t\geq 0}$ at level 1
- **2nd Depth**: Γ^2 = minimal depth of transitions between these components $> \Gamma^1$

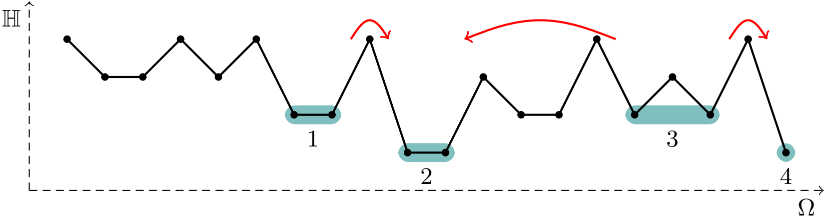


From Level 1 to Level 2

- **Next Step**: collect the *recurrent* components of $\{\mathfrak{X}^1(t)\}_{t\geq 0}$ at level 1
- **2nd Depth**: Γ^2 = minimal depth of transitions between these components $> \Gamma^1$

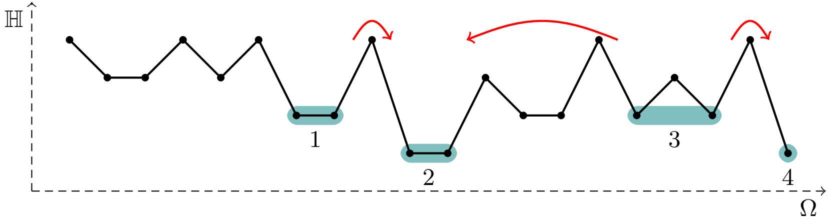
Theorem (K. *arXiv:2405.08488*)

$\{\eta_\beta(e^{\Gamma^2\beta}t)\}_{t\geq 0}$ converges to $\{\mathfrak{X}^2(t)\}_{t\geq 0}$ in the sense of marginal distributions



From Level $\ell - 1$ to Level ℓ

- **Next Step**: collect the *recurrent* components of $\{\mathfrak{X}^{\ell-1}(t)\}_{t\geq 0}$ at level $\ell - 1$
- **ℓ-th Depth**: Γ^{ℓ} = minimal depth of transitions between these components $> \Gamma^{\ell-1}$

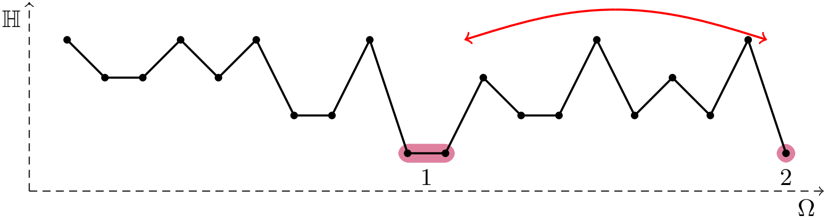


From Level $\ell - 1$ to Level ℓ

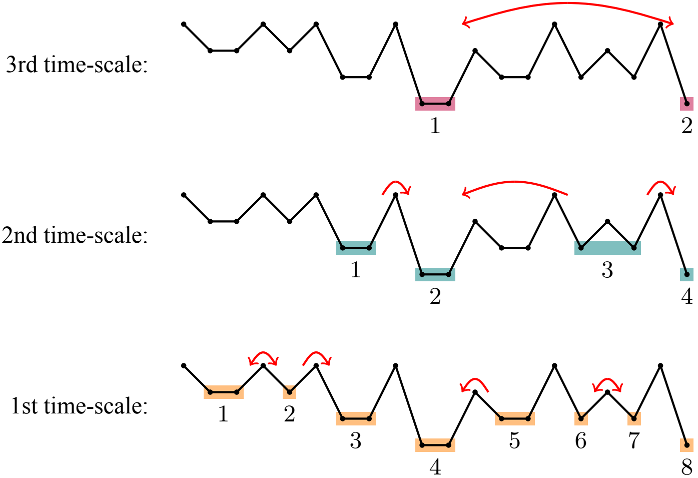
- **Next Step**: collect the *recurrent* components of $\{\mathfrak{X}^{\ell-1}(t)\}_{t\geq 0}$ at level $\ell - 1$
- **ℓ-th Depth**: Γ^ℓ = minimal depth of transitions between these components $> \Gamma^{\ell-1}$

Theorem (K. *arXiv:2405.08488*)

For each $\ell \geq 1$, $\{\eta_\beta(e^{\Gamma^\ell \beta} t)\}_{t\geq 0}$ converges to $\{\mathfrak{X}^\ell(t)\}_{t\geq 0}$ in the sense of marginal distributions



Summary: Metastable Hierarchy



Case 1: Glauber Dynamics with $h > 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^{\Lambda}, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x \sim y} \eta_x \eta_y - \frac{h}{2}\sum_{x \in \Lambda} \eta_x, \quad h > 0$

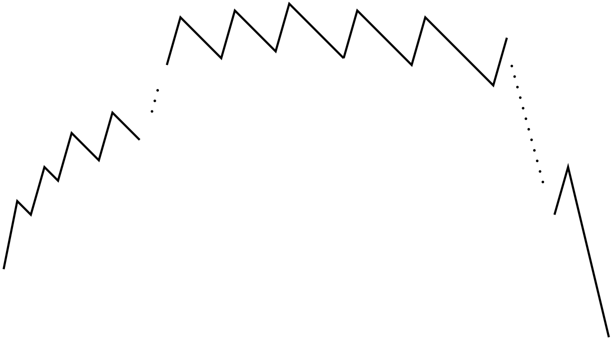
  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single spin flip

  Neves–Schonmann *CMP* '91, ..., Beltrán–Landim *SPA* '11

Case 1: Glauber Dynamics with $h > 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x, \quad h > 0$
  Neves–Schonmann *CMP* '91, ..., Beltrán–Landim *SPA* '11

- **Time-Scales**: $e^{h\beta} \ll e^{2h\beta} \ll \cdots \ll e^{(\mathfrak{m}-2)h\beta} \ll e^{(2-h)\beta} \ll e^{\Gamma\beta}$
  $\mathfrak{m} = \lceil \frac{2}{h} \rceil, \quad \Gamma = 4\mathfrak{m} - h(\mathfrak{m}^2 - \mathfrak{m} + 1)$



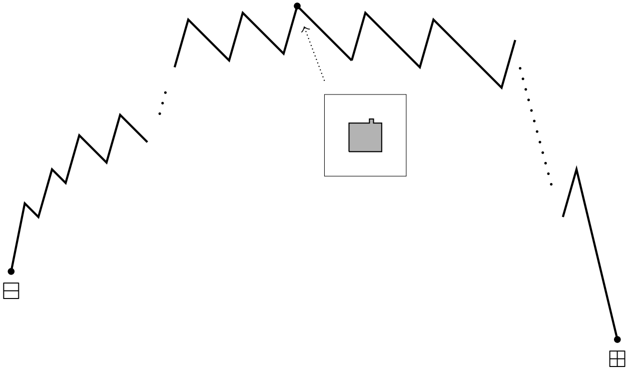
Case 1: Glauber Dynamics with $h > 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^{\Lambda}, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x, \quad h > 0$

  Neves–Schonmann *CMP* '91, ..., Beltrán–Landim *SPA* '11

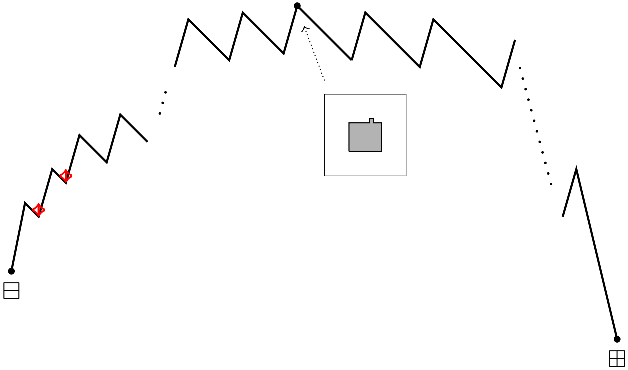
- **Time-Scales**: $e^{h\beta} \ll e^{2h\beta} \ll \cdots \ll e^{(\mathfrak{m}-2)h\beta} \ll e^{(2-h)\beta} \ll e^{\Gamma\beta}$

  $\mathfrak{m} = \lceil \frac{2}{h} \rceil, \quad \Gamma = 4\mathfrak{m} - h(\mathfrak{m}^2 - \mathfrak{m} + 1)$



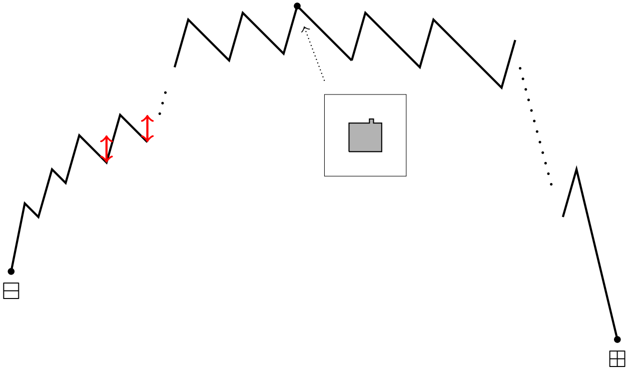
Case 1: Glauber Dynamics with $h > 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x, \quad h > 0$
  Neves–Schonmann *CMP* '91, ..., Beltrán–Landim *SPA* '11
- **Time-Scales**: $e^{h\beta} \ll e^{2h\beta} \ll \cdots \ll e^{(\mathfrak{m}-2)h\beta} \ll e^{(2-h)\beta} \ll e^{\Gamma\beta}$
  $\mathfrak{m} = \lceil \frac{2}{h} \rceil, \quad \Gamma = 4\mathfrak{m} - h(\mathfrak{m}^2 - \mathfrak{m} + 1)$



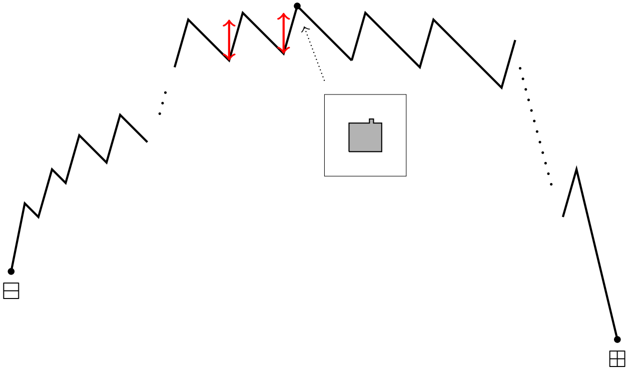
Case 1: Glauber Dynamics with $h > 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L$, $\Omega = \{\pm 1\}^\Lambda$, $\mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x$, $h > 0$
  Neves–Schonmann *CMP* '91, ..., Beltrán–Landim *SPA* '11
- **Time-Scales**: $e^{h\beta} \ll e^{2h\beta} \ll \cdots \ll e^{(\mathfrak{m}-2)h\beta} \ll e^{(2-h)\beta} \ll e^{\Gamma\beta}$
  $\mathfrak{m} = \lceil \frac{2}{h} \rceil$, $\Gamma = 4\mathfrak{m} - h(\mathfrak{m}^2 - \mathfrak{m} + 1)$



Case 1: Glauber Dynamics with $h > 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x, \quad h > 0$
  Neves–Schonmann *CMP* '91, ..., Beltrán–Landim *SPA* '11
- **Time-Scales**: $e^{h\beta} \ll e^{2h\beta} \ll \cdots \ll e^{(\mathfrak{m}-2)h\beta} \ll e^{(2-h)\beta} \ll e^{\Gamma\beta}$
  $\mathfrak{m} = \lceil \frac{2}{h} \rceil, \quad \Gamma = 4\mathfrak{m} - h(\mathfrak{m}^2 - \mathfrak{m} + 1)$



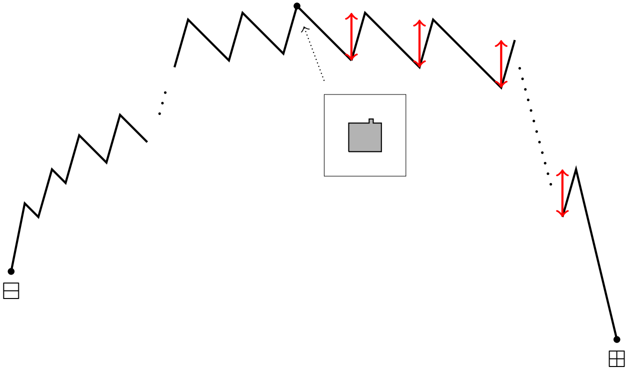
Case 1: Glauber Dynamics with $h > 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x, \quad h > 0$

  Neves–Schonmann *CMP* '91, ..., Beltrán–Landim *SPA* '11

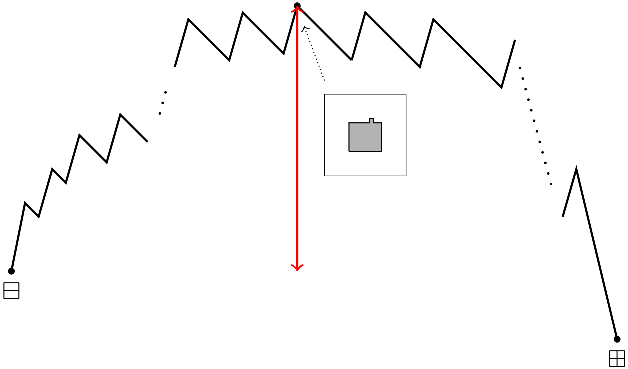
- **Time-Scales**: $e^{h\beta} \ll e^{2h\beta} \ll \cdots \ll e^{(\mathfrak{m}-2)h\beta} \ll e^{(2-h)\beta} \ll e^{\Gamma\beta}$

  $\mathfrak{m} = \lceil \frac{2}{h} \rceil, \quad \Gamma = 4\mathfrak{m} - h(\mathfrak{m}^2 - \mathfrak{m} + 1)$



Case 1: Glauber Dynamics with $h > 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y - \frac{h}{2}\sum_{x\in\Lambda}\eta_x, \quad h > 0$
  Neves–Schonmann *CMP* '91, ..., Beltrán–Landim *SPA* '11
- **Time-Scales**: $e^{h\beta} \ll e^{2h\beta} \ll \cdots \ll e^{(\mathfrak{m}-2)h\beta} \ll e^{(2-h)\beta} \ll e^{\Gamma\beta}$
  $\mathfrak{m} = \lceil \frac{2}{h} \rceil, \quad \Gamma = 4\mathfrak{m} - h(\mathfrak{m}^2 - \mathfrak{m} + 1)$



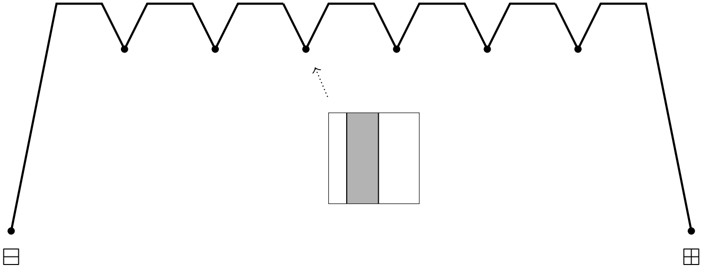
Case 2: Glauber Dynamics with $h = 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y} \eta_x \eta_y$

  Nardi–Zocca *SPA* '19, Bet–Gallo–Nardi *JSP* '21, K.–Seo *AoP* '24+

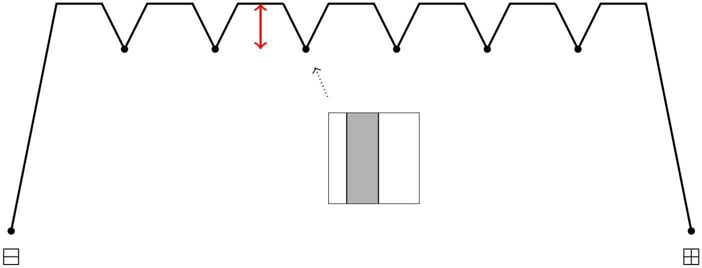
Case 2: Glauber Dynamics with $h = 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2} \sum_{x \sim y} \eta_x \eta_y$
  Nardi–Zocca *SPA* '19, Bet–Gallo–Nardi *JSP* '21, K.–Seo *AoP* '24+
- **Time-Scales**: $e^{2\beta} \ll e^{(2L+2)\beta}$



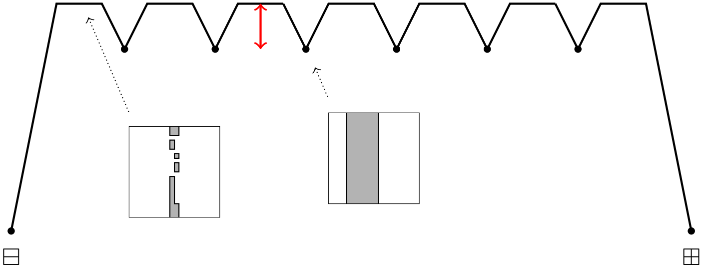
Case 2: Glauber Dynamics with $h = 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y} \eta_x \eta_y$
  Nardi–Zocca *SPA* '19, Bet–Gallo–Nardi *JSP* '21, K.–Seo *AoP* '24+
- **Time-Scales**: $e^{2\beta} \ll e^{(2L+2)\beta}$



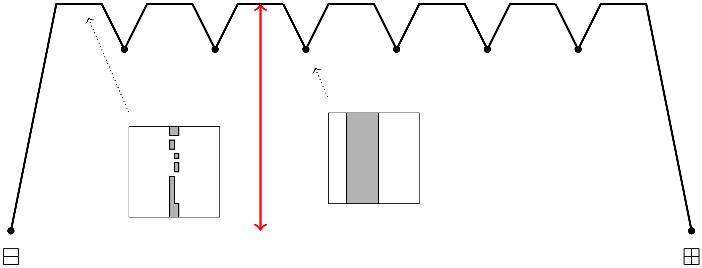
Case 2: Glauber Dynamics with $h = 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L$, $\Omega = \{\pm 1\}^\Lambda$, $\mathbb{H}(\eta) = -\frac{1}{2} \sum_{x \sim y} \eta_x \eta_y$
  Nardi–Zocca *SPA* '19, Bet–Gallo–Nardi *JSP* '21, K.–Seo *AoP* '24+
- **Time-Scales**: $e^{2\beta} \ll e^{(2L+2)\beta}$



Case 2: Glauber Dynamics with $h = 0$

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{\pm 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\frac{1}{2}\sum_{x\sim y}\eta_x\eta_y$
  Nardi–Zocca *SPA* '19, Bet–Gallo–Nardi *JSP* '21, K.–Seo *AoP* '24+
- **Time-Scales**: $e^{2\beta} \ll e^{(2L+2)\beta}$

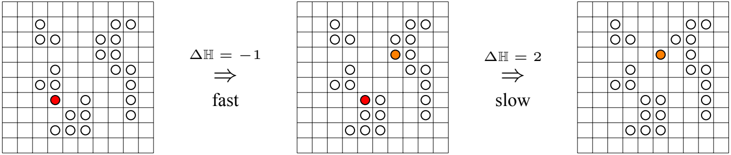


Case 3: Kawasaki Dynamics

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\, 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\sum_{x\sim y} \eta_x \eta_y$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single particle jump

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda} \eta_x = \mathcal{N}\}, \quad \mathcal{N} < \frac{L^2}{4}$



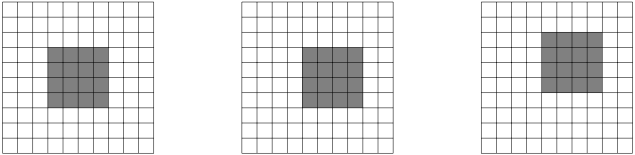
Case 3: Kawasaki Dynamics

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\, 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\sum_{x\sim y} \eta_x \eta_y$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single particle jump

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda} \eta_x = \mathcal{N}\}, \quad \mathcal{N} < \frac{L^2}{4}$

  Beltrán–Landim *AIHP* '15



Case 3: Kawasaki Dynamics

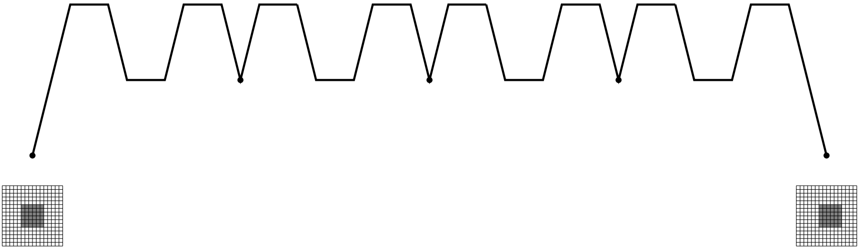
- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\, 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\sum_{x \sim y} \eta_x \eta_y$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single particle jump

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x \in \Lambda} \eta_x = \mathcal{N}\}, \quad \mathcal{N} < \frac{L^2}{4}$

  Beltrán–Landim *AIHP* '15

- **Time-Scales**: $e^\beta \ll e^{2\beta}$



Case 3: Kawasaki Dynamics

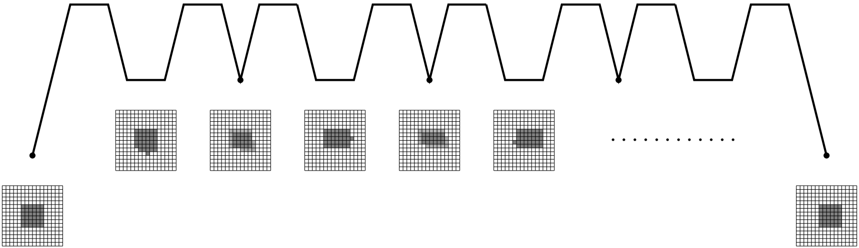
- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\ 1\}^{\Lambda}, \quad \mathbb{H}(\eta) = -\sum_{x\sim y} \eta_x \eta_y$

  $\eta \sim \xi \quad \Leftrightarrow \quad \xi$ is obtained from η by a single particle jump

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda} \eta_x = \mathcal{N}\}, \quad \mathcal{N} < \frac{L^2}{4}$

  Beltrán–Landim *AIHP* '15

- **Time-Scales**: $\quad e^{\beta} \ll e^{2\beta}$



Case 4: Kawasaki Dynamics with Macroscopic Particles

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\,1\}^\Lambda, \quad \mathbb{H}(\eta) = -\sum_{x\sim y} \eta_x \eta_y$

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda} \eta_x = \mathscr{N}\}, \quad \mathscr{N} > \frac{L^2}{4}$

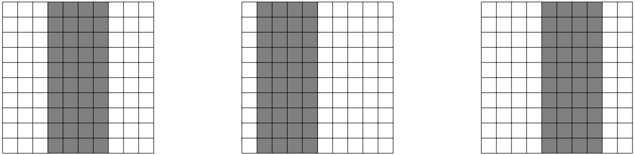
  K. *arXiv:2405.08488*

Case 4: Kawasaki Dynamics with Macroscopic Particles

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\, 1\}^{\Lambda}, \quad \mathbb{H}(\eta) = -\sum_{x\sim y} \eta_x \eta_y$

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda} \eta_x = \mathcal{N}\}, \quad \mathcal{N} > \frac{L^2}{4}$

  K. *arXiv:2405.08488*



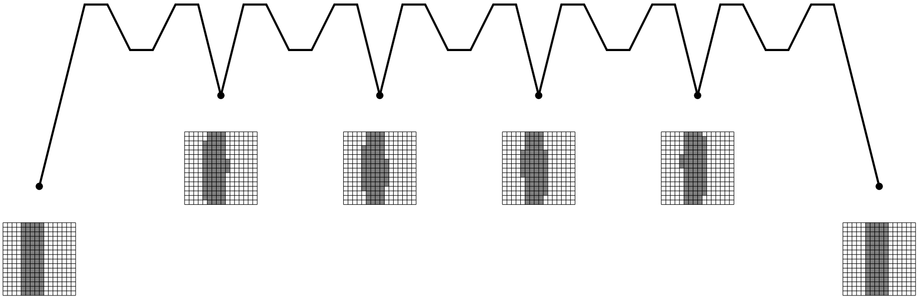
Case 4: Kawasaki Dynamics with Macroscopic Particles

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\, 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\sum_{x\sim y} \eta_x \eta_y$

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda} \eta_x = \mathcal{N}\}, \quad \mathcal{N} > \frac{L^2}{4}$

  K. *arXiv:2405.08488*

- **Time-Scales**: $\quad e^\beta \ll e^{2\beta} \ll e^{4\beta}$



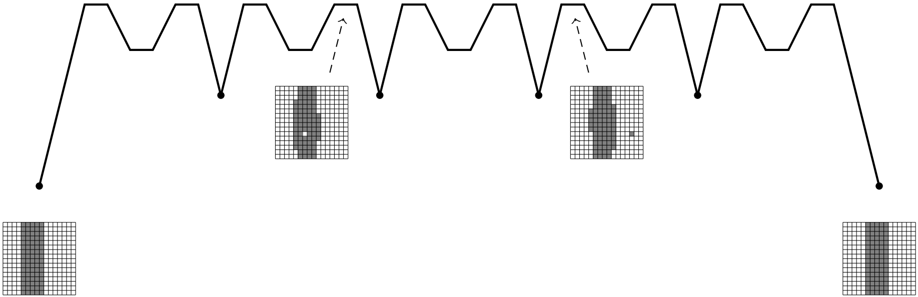
Case 4: Kawasaki Dynamics with Macroscopic Particles

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\,1\}^\Lambda, \quad \mathbb{H}(\eta) = -\sum_{x\sim y} \eta_x \eta_y$

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda} \eta_x = \mathcal{N}\}, \quad \mathcal{N} > \frac{L^2}{4}$

  K. *arXiv:2405.08488*

- **Time-Scales**: $\quad e^\beta \ll e^{2\beta} \ll e^{4\beta}$



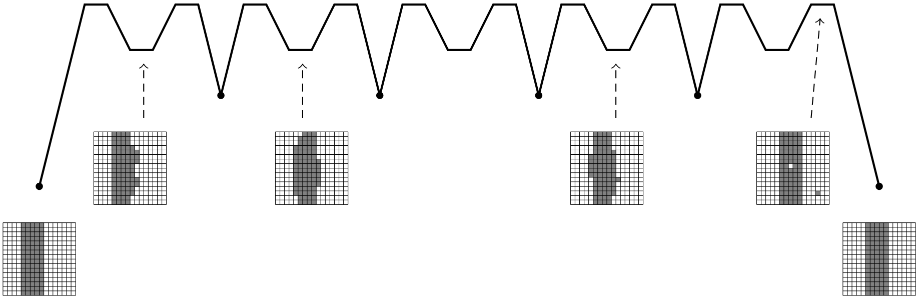
Case 4: Kawasaki Dynamics with Macroscopic Particles

- $\Lambda = \mathbb{T}_L \times \mathbb{T}_L, \quad \Omega = \{0,\, 1\}^\Lambda, \quad \mathbb{H}(\eta) = -\sum_{x\sim y} \eta_x \eta_y$

  Subset $\Omega' = \{\eta \in \Omega : \sum_{x\in\Lambda} \eta_x = \mathscr{N}\}, \quad \mathscr{N} > \frac{L^2}{4}$

  K. *arXiv:2405.08488*

- **Time-Scales**: $\quad e^{\beta} \ll e^{2\beta} \ll e^{4\beta}$



Further Directions

- **Regime of $\beta \to \infty$ & $|\Lambda| \to \infty$**: for Case 3, *Brownian motion* Gois–Landim *AoP* '15

- **Large Deviation Approach**: empirical measure

$$\mu_{\beta,T} := \frac{1}{T}\int_0^T \delta_{\eta_\beta(t)}\mathrm{d}t \xrightarrow{T\to\infty} \mu_\beta, \quad \mathbb{P}[\mu_{\beta,T}\in A] \sim \mathcal{I}_\beta(A).$$

μ_β does not explain metastability, but $\mathcal{I}_\beta$ does! Bertini–Gabrielli–Landim *AAP* '24

$$\mathcal{I}_\beta \simeq \mathcal{I}^0 + \sum_{\ell=1}^{\mathfrak{m}} \frac{1}{\theta_\beta^\ell}\mathcal{I}^\ell$$

- **2.5 LD Rate Function**: empirical flow

$$Q_{\beta,T} := \frac{1}{T}\sum_{t\in[0,T]:\,\eta_\beta(t-)\neq\eta_\beta(t)} \delta_{(\eta_\beta(t-),\,\eta_\beta(t))}.$$

LDP of $(\mu_{\beta,T},\, Q_{\beta,T})$ is called *level 2.5* LDP Bertini–Faggionato–Gabrielli *AIHP* '15

Thank you! Merci beaucoup!