Fully-connected bond percolation on $\mathbb{Z}^d$

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1. Model and motivations

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$\mathcal{E} = \{\text{edges of } \mathbb{Z}^d\}, \ \ \mathcal{E}_\Lambda = \{\text{edges of } \Lambda\}, \Lambda \subset \mathbb{Z}^d.$

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$N_{cc}^{bc(\Lambda)}$ = number of open connected components of edges in Λ with boundary condition "bc",

$$\mathcal{A}^{bc(\Lambda)} = \{N_{cc}^{bc(\Lambda)} = 1\}$$

"bc"=wired, free, periodic, left-right crossing, etc...

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Both approaches are standard in statistical physics : Gibbs measures, Ising model, FK-percoation, etc..
Issues : Existence, uniqueness, etc...

Thermodynamic approach

Definition (Thermodynamic approach)

A fully-connected bond percolation measure is any accumulation point (for the weak convergence of measures) of sequences of probability measures

$$\left(\mathbb{P}_p^{\Lambda_n}(.|N_{cc}^{bc(\Lambda_n)} = 1)\right)_{n\geq 1},$$

for any choice of boundary conditions $bc(\Lambda_n)_{n\geq 1}$.
$\mathcal{L}(p)$ denotes the space of such accumulation points. $\mathcal{L}_s(p)$ is for elements of $\mathcal{L}(p)$ which are stationary in space.

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Since Ω is a compact set, $\mathcal{L}(p)$ is not empty. $\mathcal{L}_s(p)$ is not empty as well, using "bc=periodic".

Dobrushin-Lanford-Ruelle approach

Definition (DLR approach)

A fully-connected bond percolation measure is any probability measure P on Ω such that $P(\mathcal{A}) = P(N_{cc} = 1) = 1$ and for all bounded $\Lambda \subset \mathbb{Z}^d$ and P-a.e. ω

$$P(d\omega_\Lambda | \omega_{\mathcal{E}_\Lambda^c}) = \frac{1}{Z_\Lambda(\omega_{\mathcal{E}_\Lambda^c})} 1_{\mathcal{A}}(\omega) \mathbb{P}_p^\Lambda(d\omega_\Lambda).$$

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$\mathcal{G}(p)$ could be empty !

Motivations

- Random Cluster Model (Widom Rowlinson) with "q=0" : Formally
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- Weighted random connected graph : For $p = 1/2$, $\mathbb{P}(.|N_{cc} = 1)$ samples uniformly a connected graph in $\mathbb{Z}^d$.
- Connection with the conjecture "$\theta(p_c) = 0$".

2 Results

Topological results

Theorem

For any $P \in \mathcal{L}_s(p)$,

i) $P(\textit{there exists a bounded connected component}) = 0$

ii) $P(N_{cc} = 0 \textit{ or } 1) = 1$.

For all $P \in \mathcal{G}(p)$, by definition $P(N_{cc} = 1) = 1$.

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Ideas of proof for *i)* :

- If there exists a bounded connected component $\Rightarrow$ It is not unique.
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Ideas of proof for *ii)* : Finite energy property and Burton-Keane arguments.
The event $\{N_{cc} = 0\}$ is possible for small p (microscopic connected component).

A phase transition result

Theorem

For any $d \geq 2$, *there exists a threshold* $0 < p^*(d) < 1$ *such that*

- *if* $p < p^*(d)$, $\mathcal{G}_s(p) = \emptyset$ *and* $\mathcal{L}_s(p) = \{\delta_{0^\varepsilon}\}$.
- *if* $p > p^*(d)$, *there exists* P *in* $\in \mathcal{L}_s(p) \cap \mathcal{G}_s(p)$ *with* $P(\mathcal{A}) = 1$.

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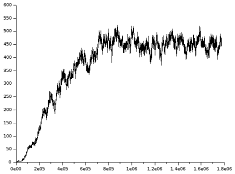
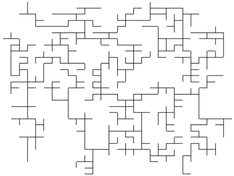
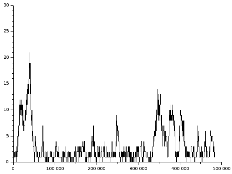
For $p > p^*(d)$, $|\mathcal{G}_s(p)| = 1$? It is mainly a conjecture. True for $d = 2$ and $p \geq 1/2$ (details in 3 slides !)

Simulation

Birth-death Metropolis Hastings algorithm with free boundary condition on a 2D grid $30 * 30$.







- At the middle, a simulation with $p = 0.2$.
- On the left, the number of open edges during the run for $p = 0.2$.
- On the right, the number of open edges during the run for $p = 0.15$.

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For $d = 2$, $p_c(2) = 1/2$ and so $0.128 < p^*(2) < 0.202$.

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$p_c(d) =$ the standard bond percolation threshold on $\mathbb{Z}^d$.

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For $d = 3$, $p_c(3) \simeq 0.25$ and so $0.075 < p^*(3) < 0.099$.

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For $d = 3$, $p_c(3) \simeq 0.25$ and so $0.075 < p^*(3) < 0.099$.
When $d \to \infty$, $p_c(d) \sim 1/(2d)$ and so $p^*(d) \sim e^{-1}p_c(d)$.

An uniqueness result

Theorem

For $d = 2$ and $p \geq 1/2$ there exists a stationary probability measure P such that

$$\mathcal{G}(p) = \mathcal{G}_s(p) = \{P\}.$$

Moreover for any $Q \in \mathcal{L}(p)$, there exists $\alpha \in [0,1]$ such that $Q = \alpha P + (1-\alpha)\delta_{0^{\mathcal{E}}}$.

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Lemma

Let $P \in \mathcal{G}(p) \cup \mathcal{L}(p)$. Let $E \subset \mathcal{E}$ be a finite subset of edges and $\tilde{\omega} \in \mathcal{A}$ an allowed configuration. Let e be an edge in $\mathcal{E} \backslash E$. We assume that there exists an open edge f in $\tilde{\omega}_E$ having a common vertex with e. Then

$$P(e \text{ is open } | \tilde{\omega}_E) \geq p.$$

3 Sketches of proofs

The main results

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The finite volume two-parameter model

Definition (Finite volume two-parameter model)

$(\lambda, \mu) \in \mathbb{R}^2$, $\Lambda \subset \mathbb{Z}^d$ bounded and "bc" a boundary condition,

$$Q^{bc}_{\Lambda,\lambda,\mu}(\omega_\Lambda) := \frac{1}{Z^{bc}_\Lambda(\lambda,\mu)} \mathbb{1}_{\{N^{bc(\Lambda)}_{cc}(\omega_\Lambda)=1\}} e^{\lambda N_\Lambda(\omega_\Lambda)} e^{\mu \partial N^{bc}_\Lambda(\omega_\Lambda)},$$

$Z^{bc}_\Lambda(\lambda,\mu)$ the partition function.
$N_\Lambda(\omega_\Lambda)$ = the number of open edges in ω_Λ.
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For $(\lambda,\mu)=(\log(p/(1-p)),0)$

$$\mathbb{P}_p(.|N^{bc(\Lambda)}_{cc}=1)=Q^{bc}_{\Lambda,\lambda,\mu}.$$

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For $(\lambda,\mu) = (\log(p/(1-p)), 0)$

$$\mathbb{P}_p(.|N^{bc(\Lambda)}_{cc} = 1) = Q^{bc}_{\Lambda,\lambda,\mu}.$$

For $p \in (0,1)$, $\quad \mathcal{G}(p) = \mathcal{G}^2(\log(p/(1-p)), 0)$.

Phase transition

Definition

For $(\lambda, \mu) \in \mathbb{R}^2$, $\mathcal{P}(\lambda, \mu) = \lim_{n\to\infty} \frac{\log(Z_n^{\text{wired}}(\lambda,\mu))}{\#\mathcal{E}_n}$.

$$\lambda^*(\mu) = \sup \left\{\lambda \in \mathbb{R}, \mathcal{P}(\lambda, \mu) = 0\right\}.$$

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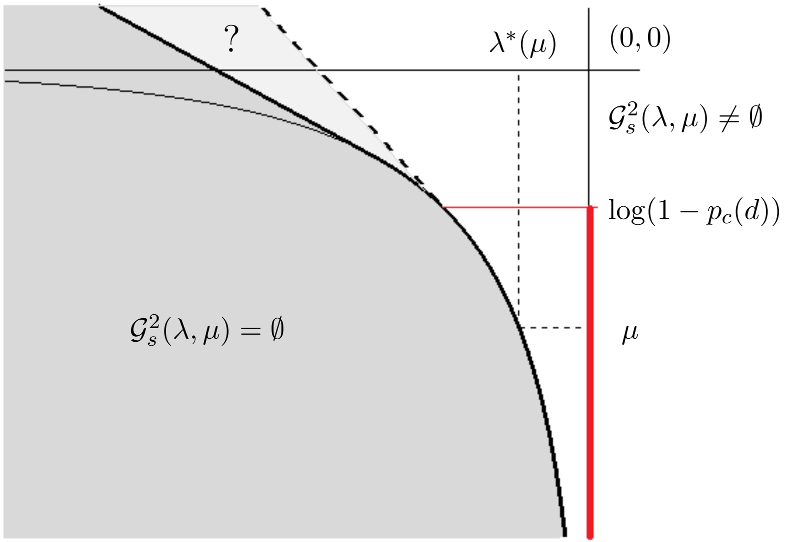
$$\frac{\partial\log(Z_n^{\text{wired}}(\lambda,\mu))}{\partial\lambda}=E_{Q_{\Lambda_n}^{\text{wired}}(\lambda,\mu)}(N_{\Lambda_n}).$$

Semi-explicit phase diagram



Details for the explicit part

Proposition

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Corollary

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Isoperimetric inequality :

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Thank you for your attention