Free energy in non-convex mean-field spin glass models

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Rencontres de Probabilités 2024 in Rouen
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Outline

- The Sherrington—Kirkpatrick model
- Convex vs non-convex
- A PDE approach

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Spin configuration:

$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N) \in \{-1, +1\}^N$

Sherrington—Kirkpatrick model ('75)

$N = 6$

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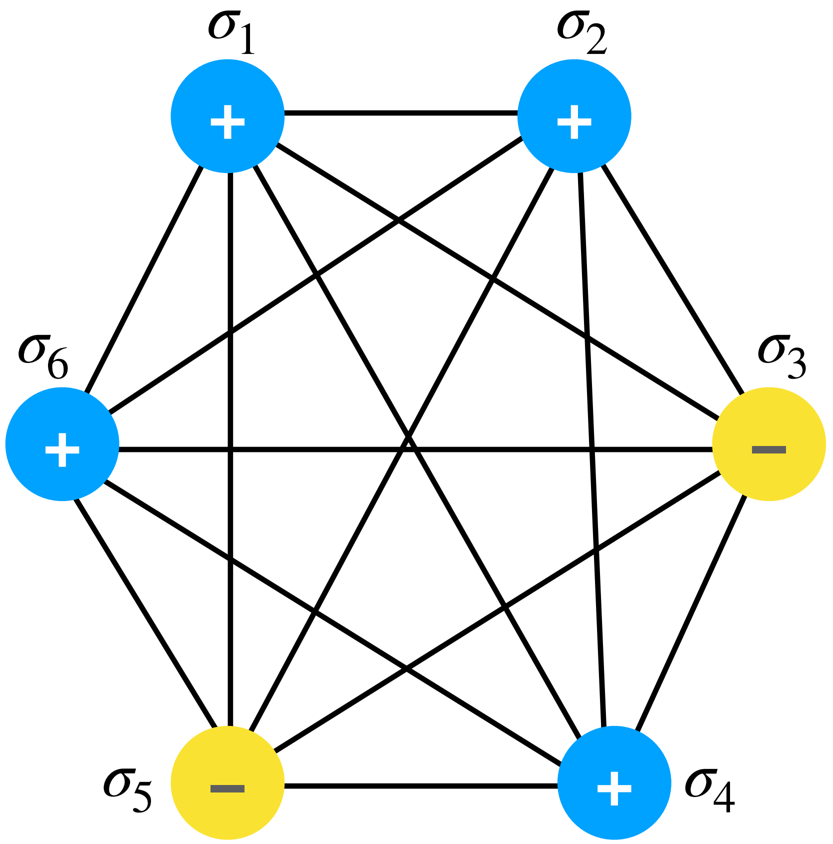

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Energy/Hamiltonian

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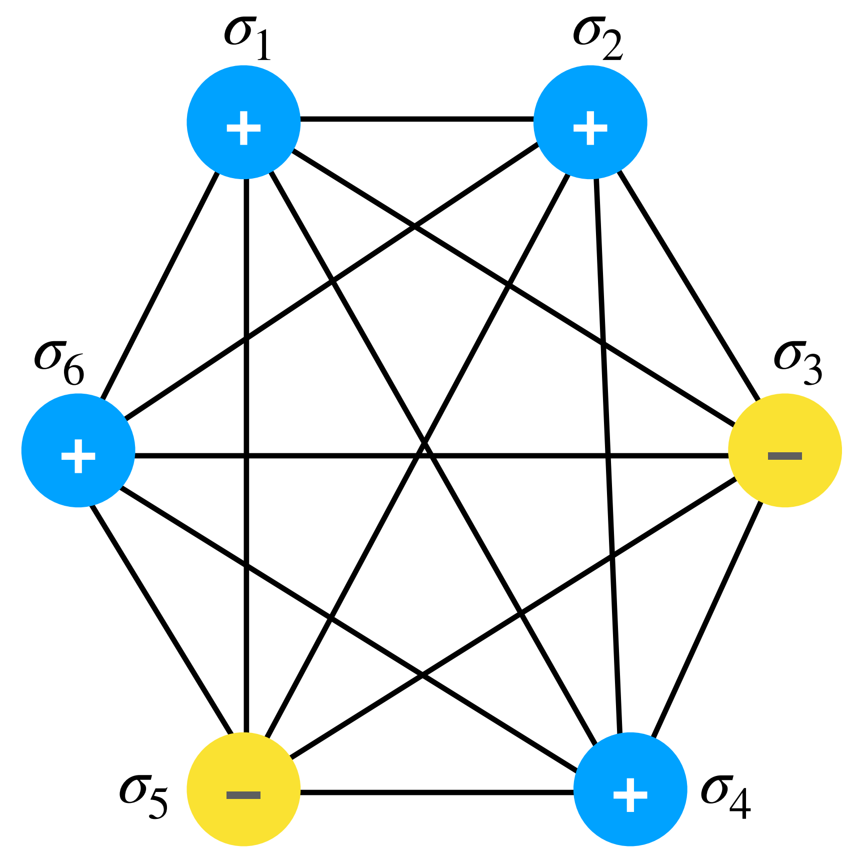

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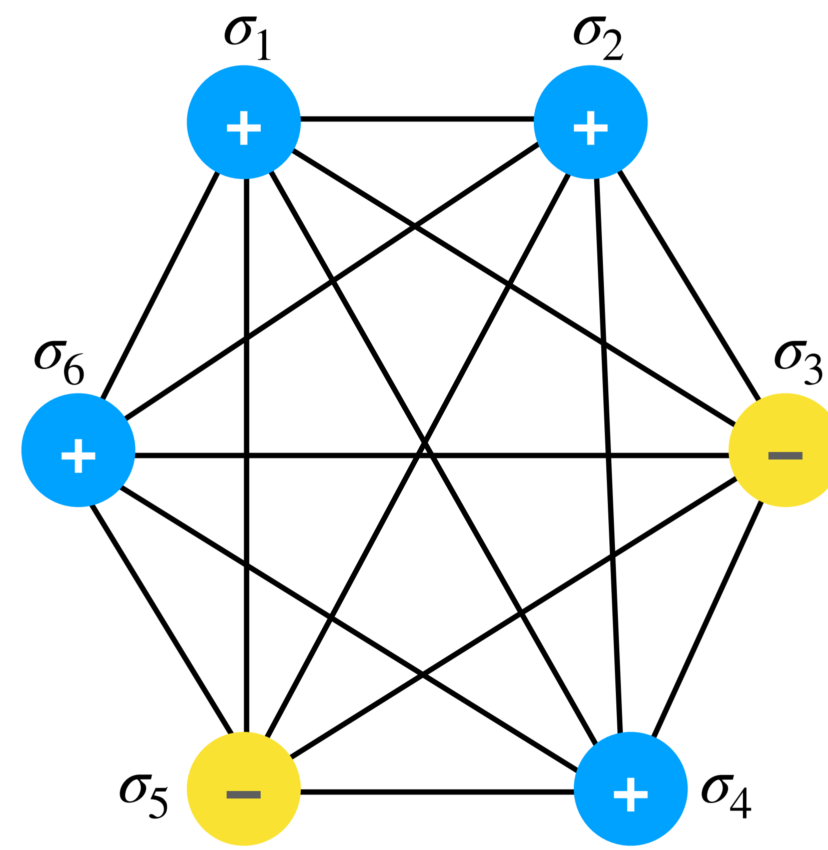

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$(g_{ij})_{ij}$ **i.i.d. standard Gaussian**

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Interested in $\lim_{N\to\infty} F_N(\beta)$

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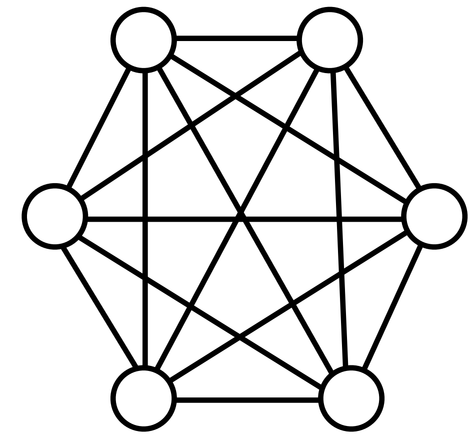
Panchenko ('13+) gave a more insightful proof

SK is convex



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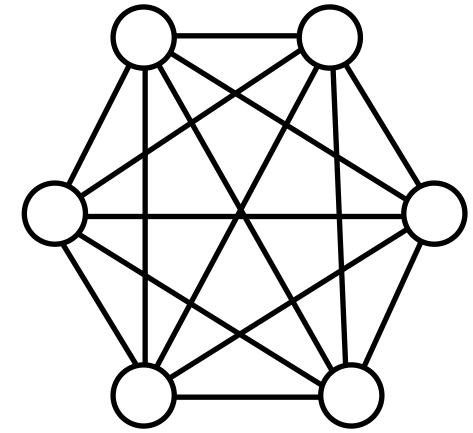

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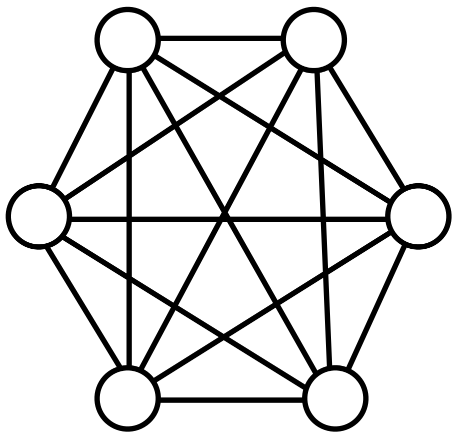

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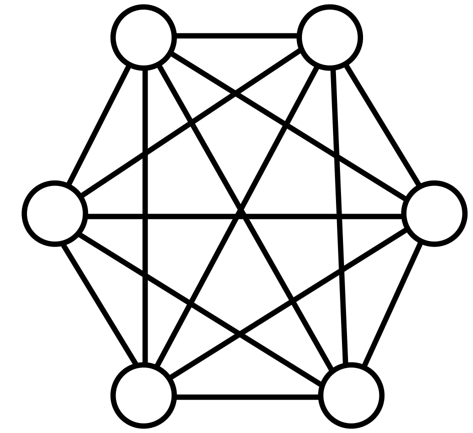

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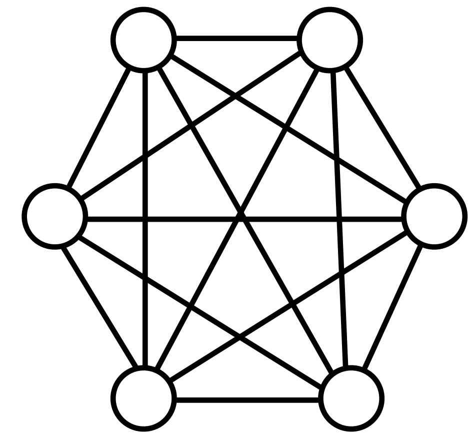

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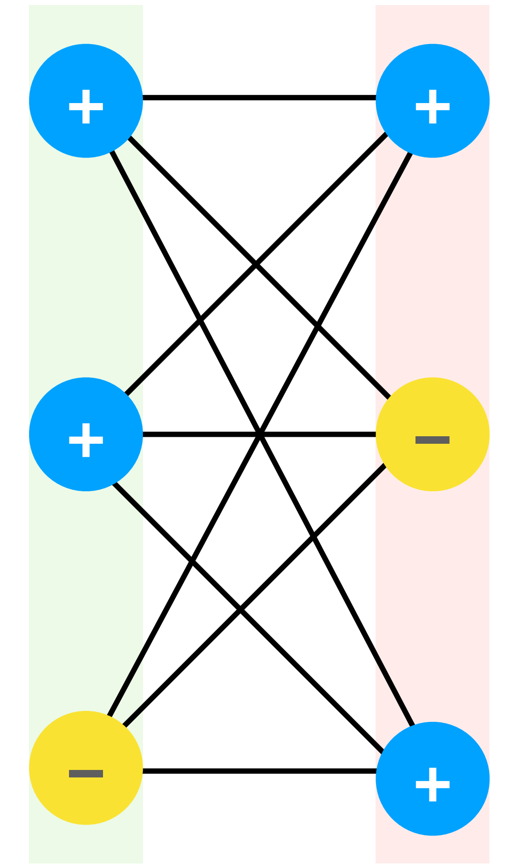
Overlap $\sim$ **# spins with the same sign**

Nonconvex: bipartite model



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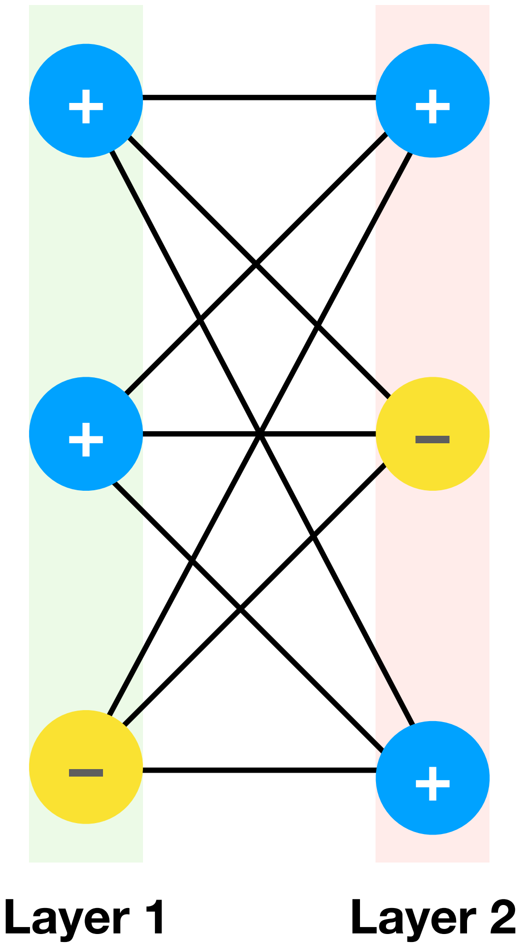

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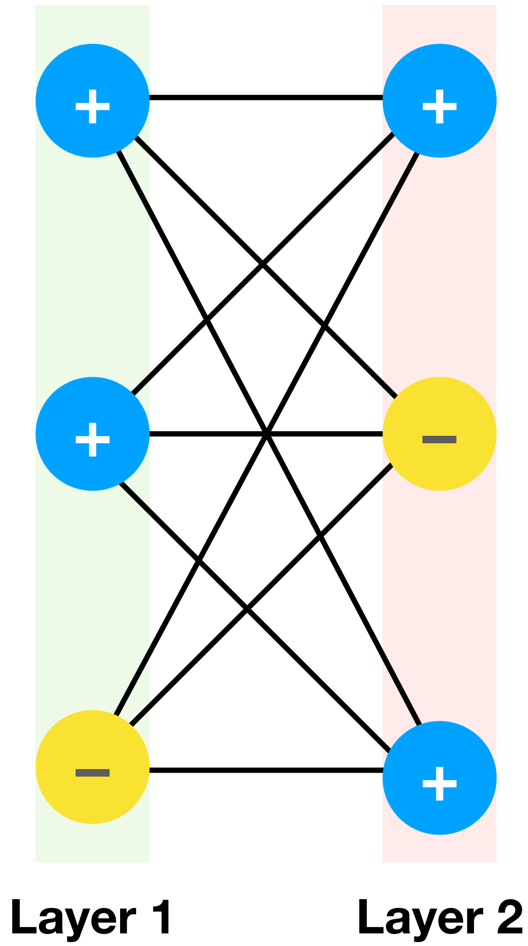

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$(x_1, x_2) \mapsto x_1 x_2$ **is not convex.**

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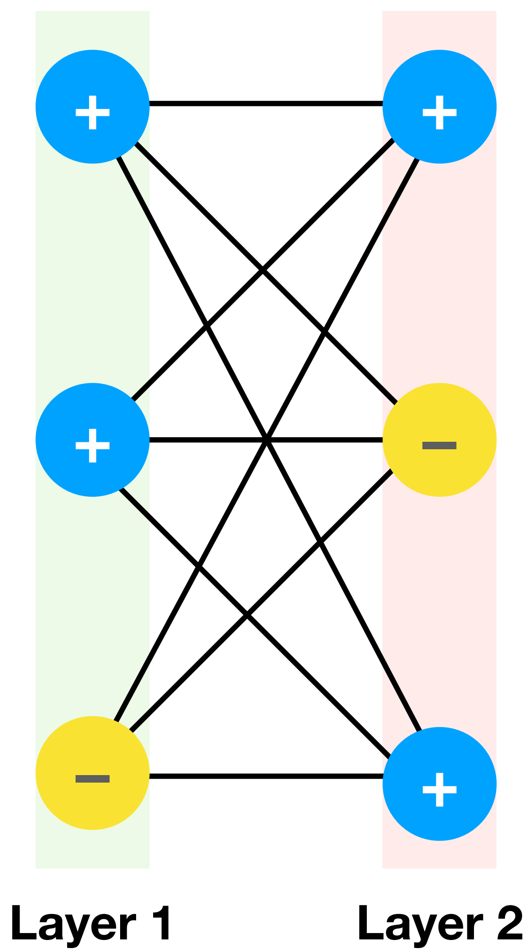

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Problem: theory based on Parisi formula is not available!

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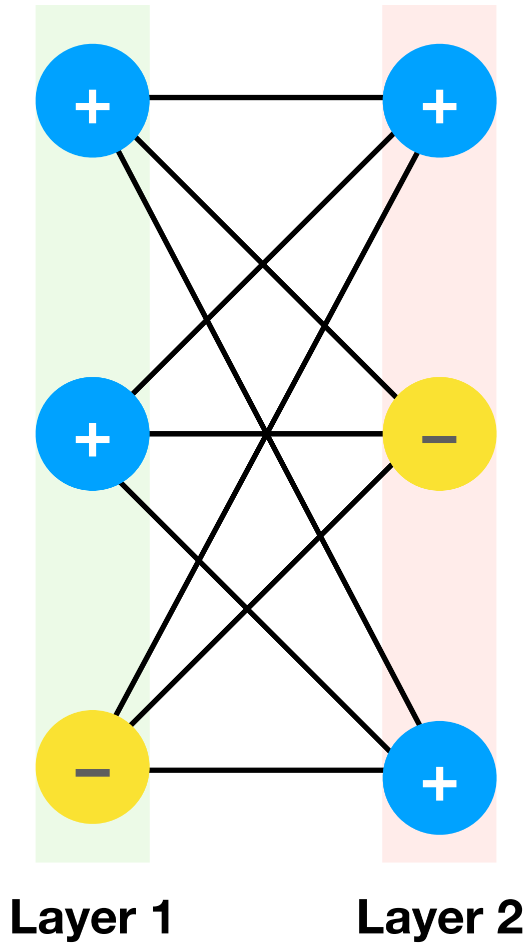

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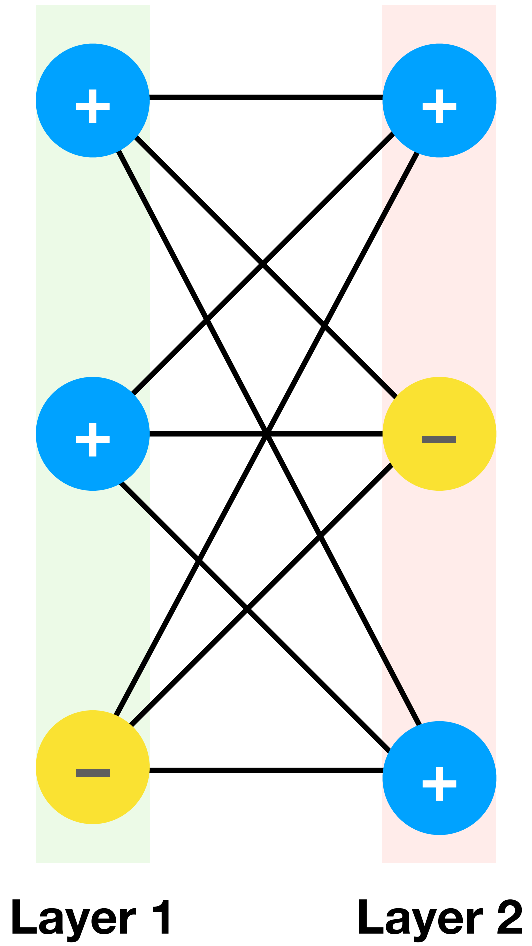

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Guerra $\lim F_N(\beta) \leq \inf \mathscr{P}_\beta(\mu)$ **Breaks down**

Talagrand/Panchenko $\lim F_N(\beta) \geq \inf \mathscr{P}_\beta(\mu)$ **Maybe not sharp**

General vector spin glass

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No prediction for the limit

A PDE perspective

Physics: Agliari, Barra, Burioni, Di Biasio, Guerra, Tantari… (2010s)

Math: Mourrat (2019+)

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$$\langle g(\sigma)\rangle = \frac{1}{Z}\sum_{\sigma} g(\sigma)e^{\sqrt{2t}H_N(\sigma)+\sqrt{h}z\cdot\sigma}$$

Where $Z = \sum_{\sigma} e^{\sqrt{2t}H_N(\sigma)+\sqrt{h}z\cdot\sigma}$

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For SK, $\xi(o) = o^2$**; r.h.s. is** $-\mathrm{Var}_{\mathbb{E}\langle\cdot\rangle}(\mathbf{O})$

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Conjecture *In general case, even when ξ is non-convex,*

$$\lim_{N\to\infty} F_N = f$$

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$$\partial_t f + \xi(-\partial_h f) = 0 \text{ (HJ)}$$

on an infinite dimensional cone (empty interior)

Digression to PDE

❌ **External field** $\sqrt{2hz} \cdot \sigma$**, for** $h \geq 0$

h **— an increasing path** $[0,1] \to \mathrm{PSD}_{D \times D}$

$\partial_t f + \xi(-\partial_h f) = 0$ **(HJ)**

on an infinite dimensional cone (empty interior)

Theorem [C.—Xia 20, 22a, 22b]
The Cauchy problem of **(HJ)** *is well-posed: existence, uniqueness, and comparison principle.*
Moreover, 1) natural finite-dimensional approximations exist; 2) variational formulas for solution exist when ξ *is convex or* $f(0,\cdot)$ *is concave.*

A PDE perspective

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Needs **Panchenko**'s ultrametricity result.

Recall **Guerra**'s upper bound fails when non-convex

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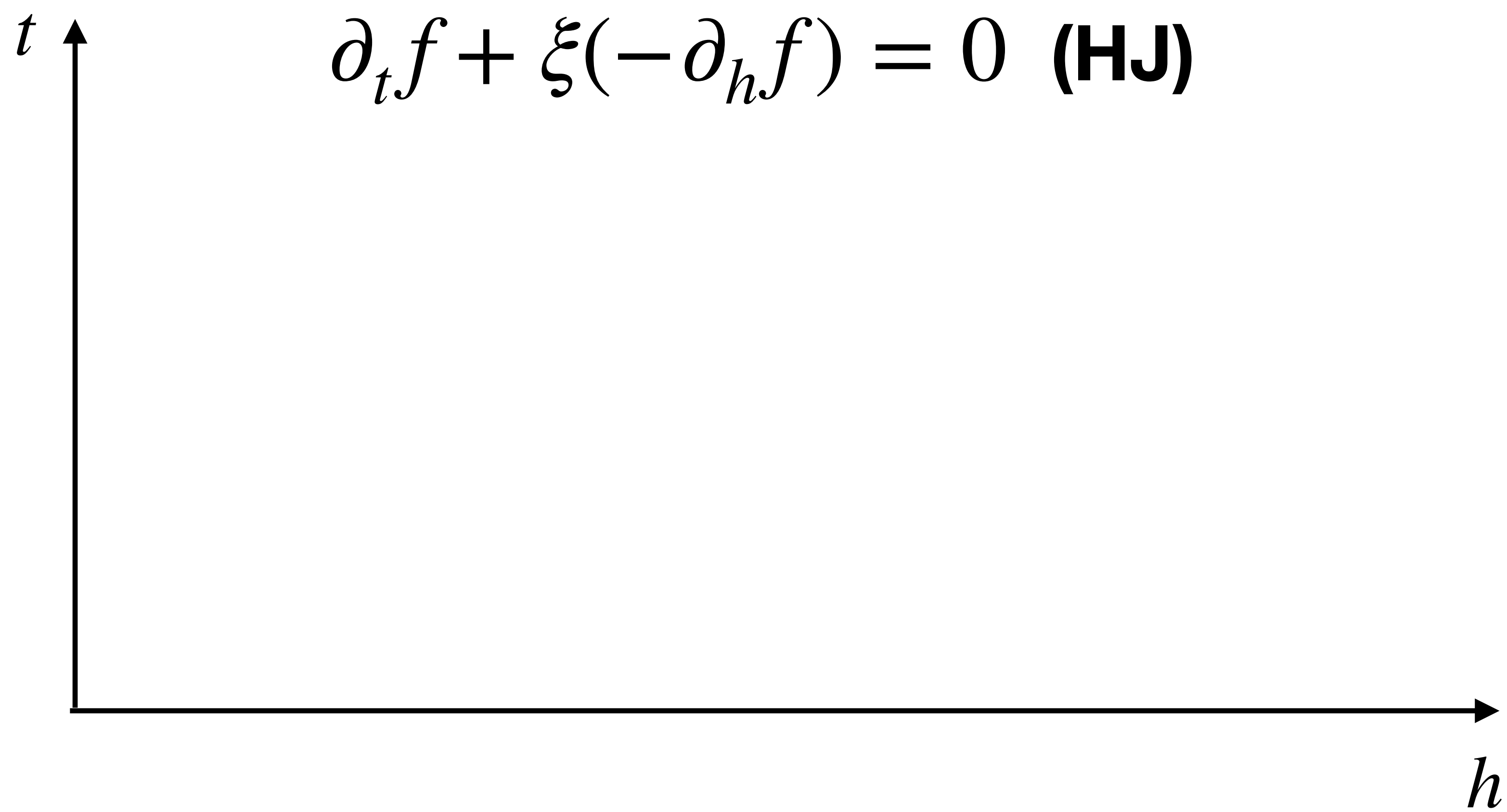

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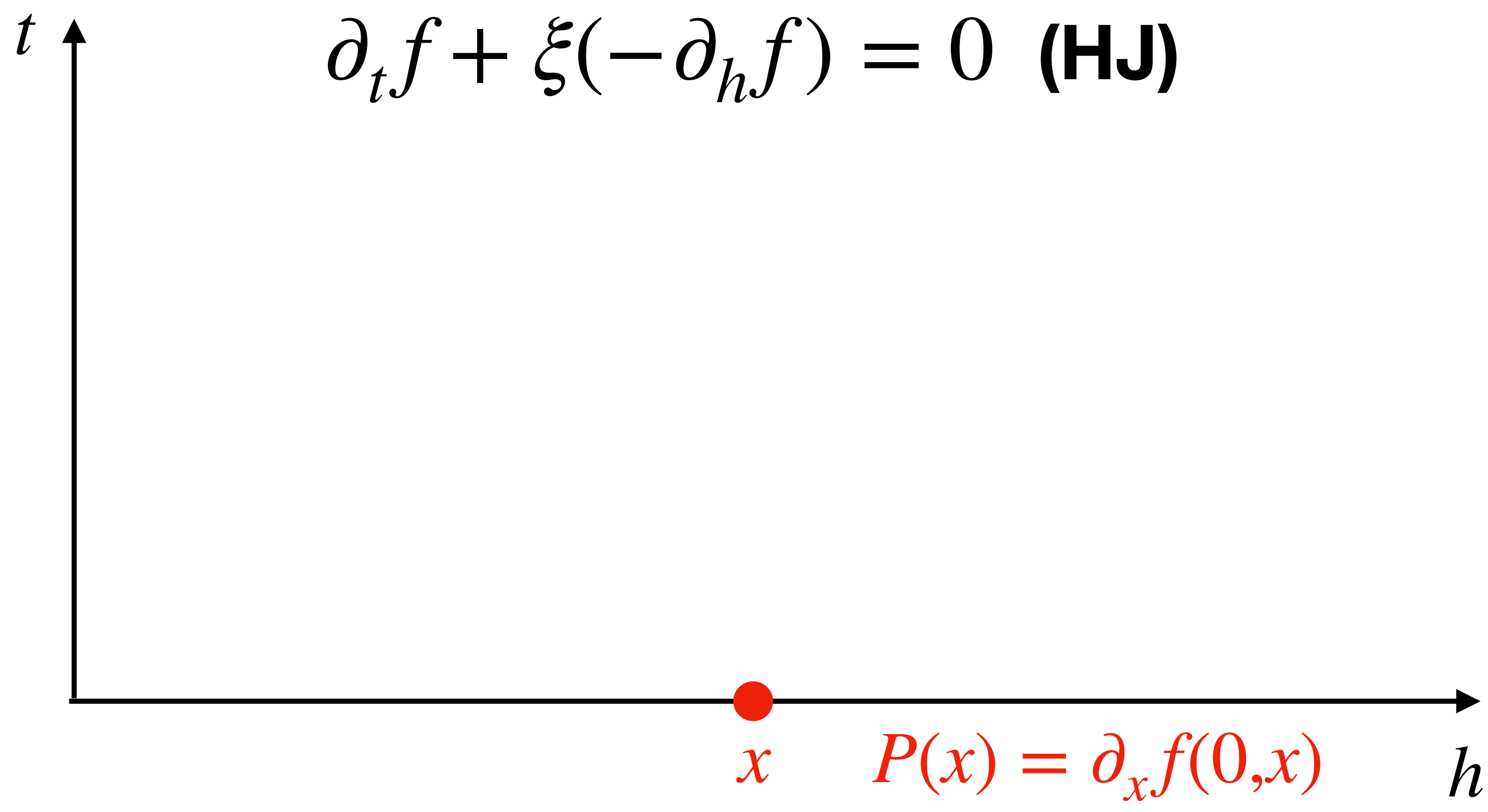

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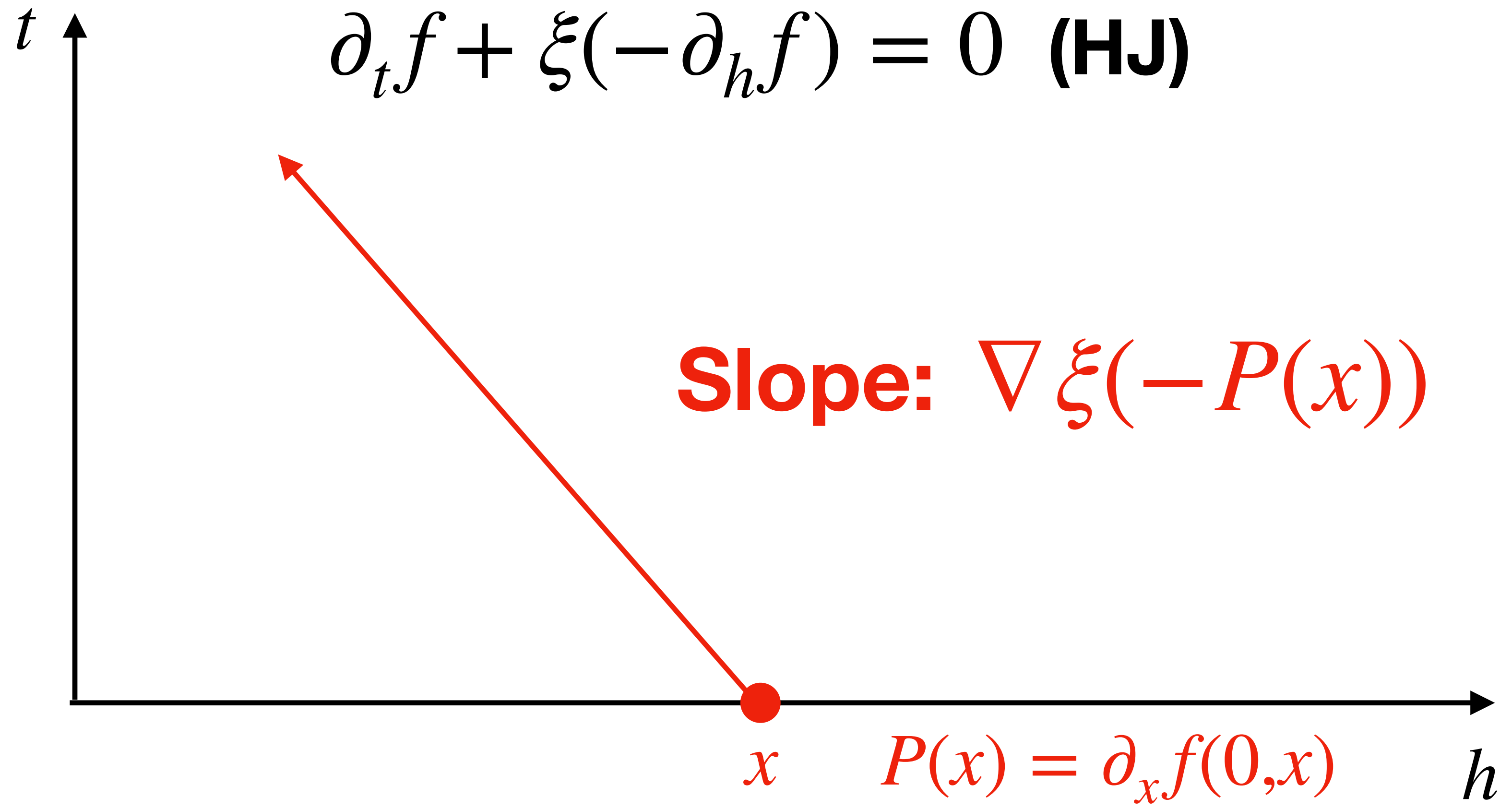

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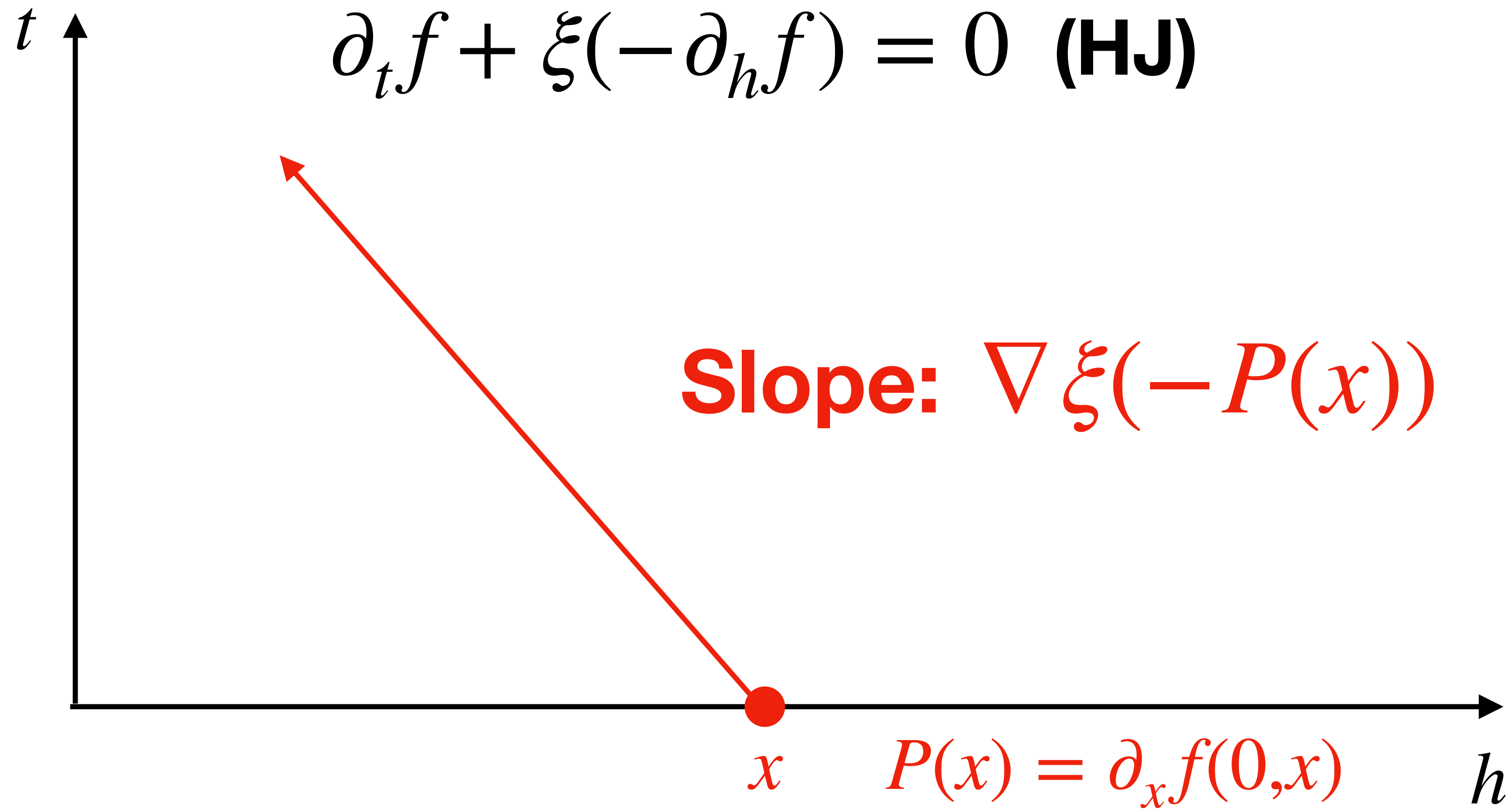

Characteristic line (starting at x): $X(s,x) = x + s\,\nabla \xi(-P(x)), \quad s \geq 0$

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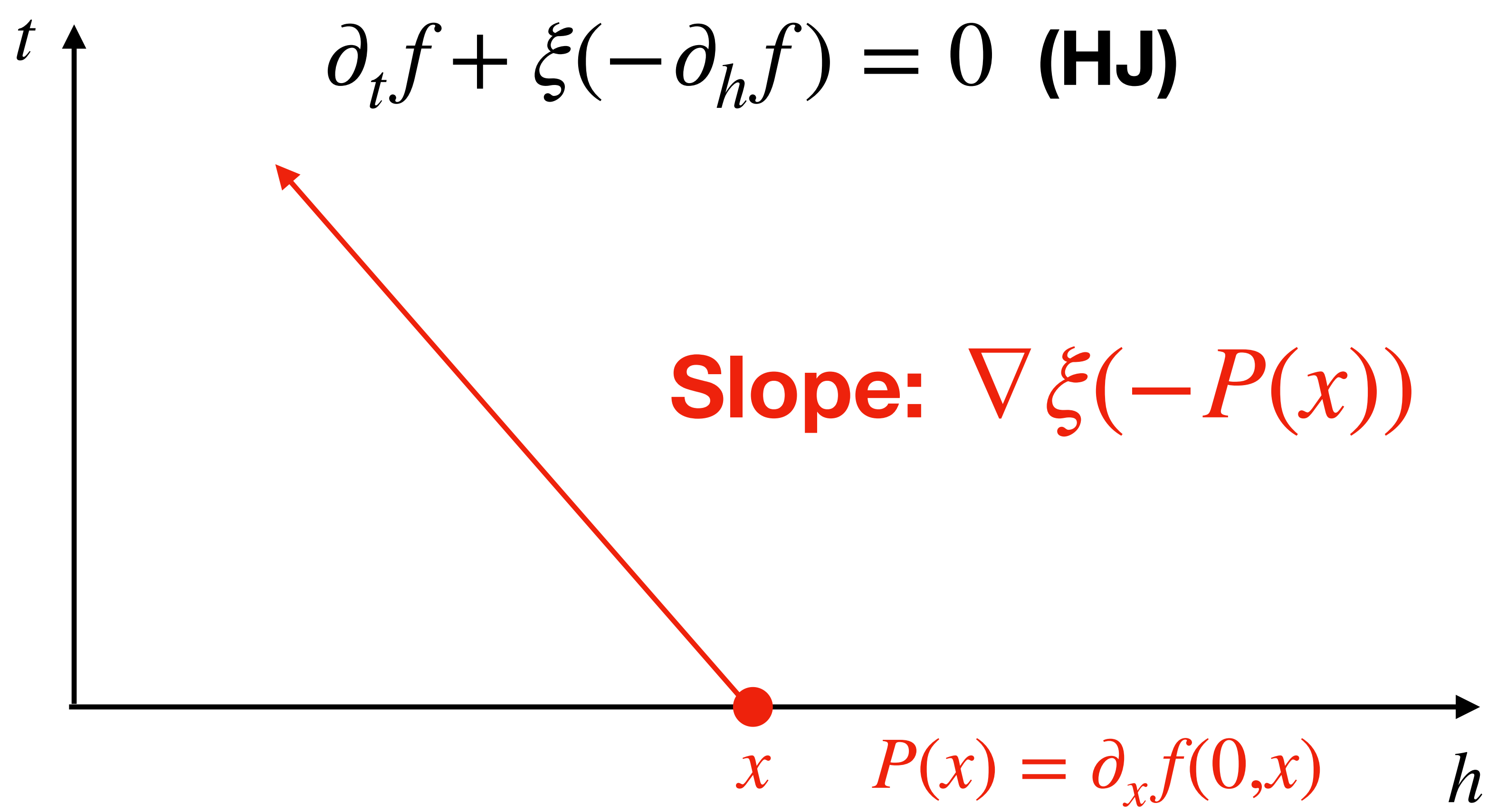

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Value along char.: $U(s,x) = f(0,x) + sP(x)\cdot\nabla\xi(-P(x)) + s\xi(-P(x))$

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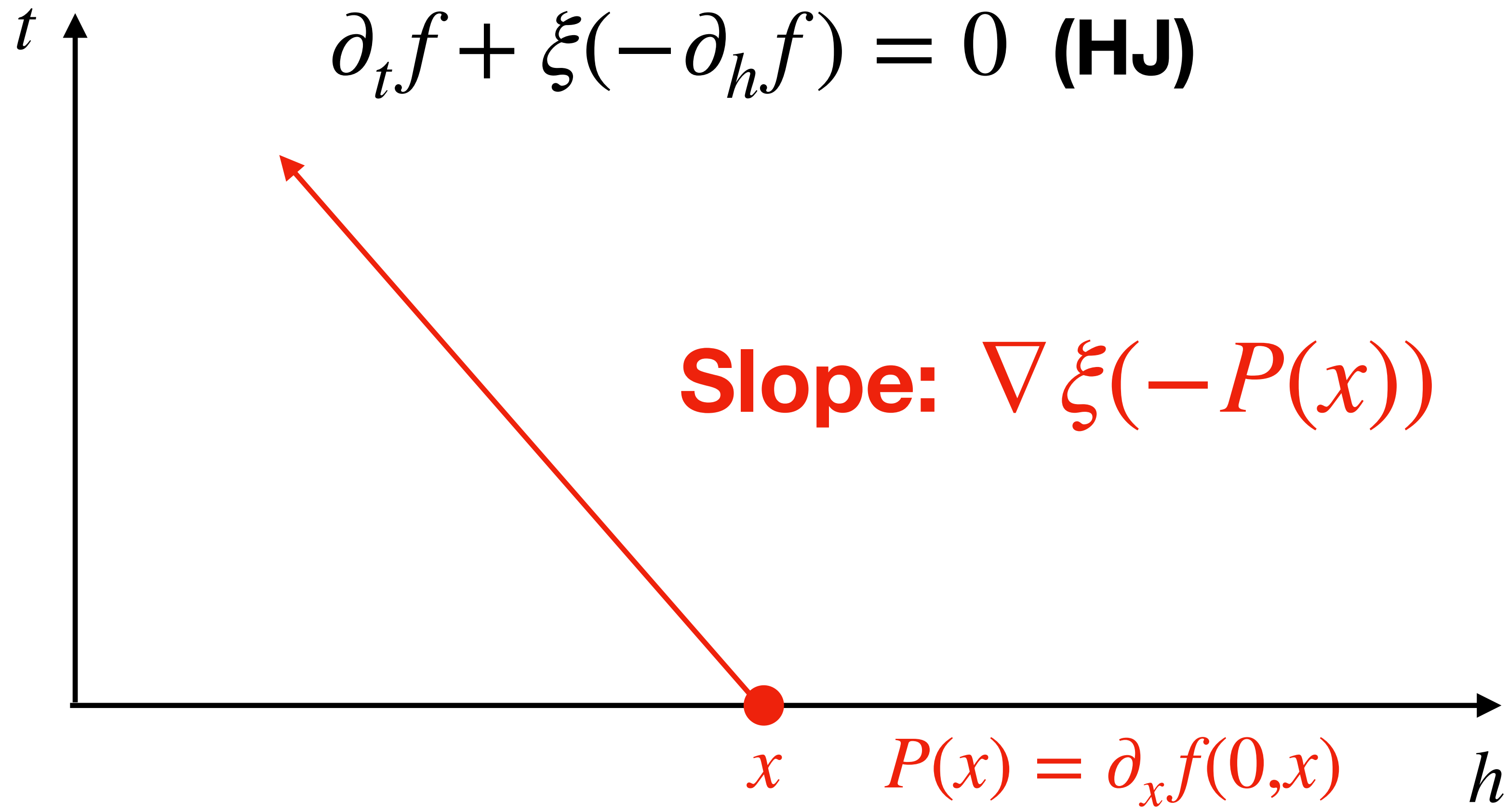

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If f is a "reasonable" solution of (HJ),
then $f \equiv U(\,\cdot\,,x)$ on $X(\,\cdot\,,x)$
before colliding with other char. lines.

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Theorem B [C.—Mourrat 23]
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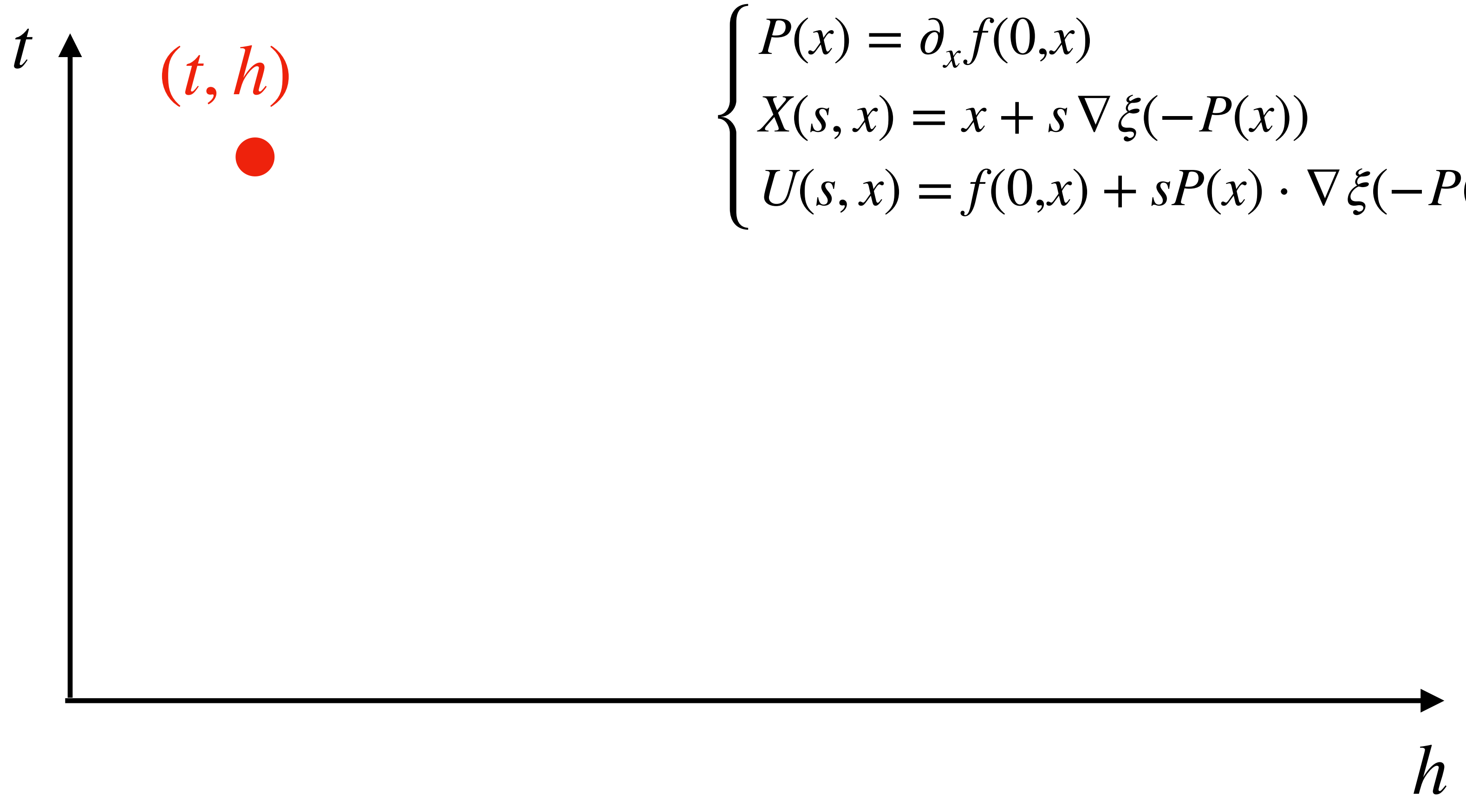

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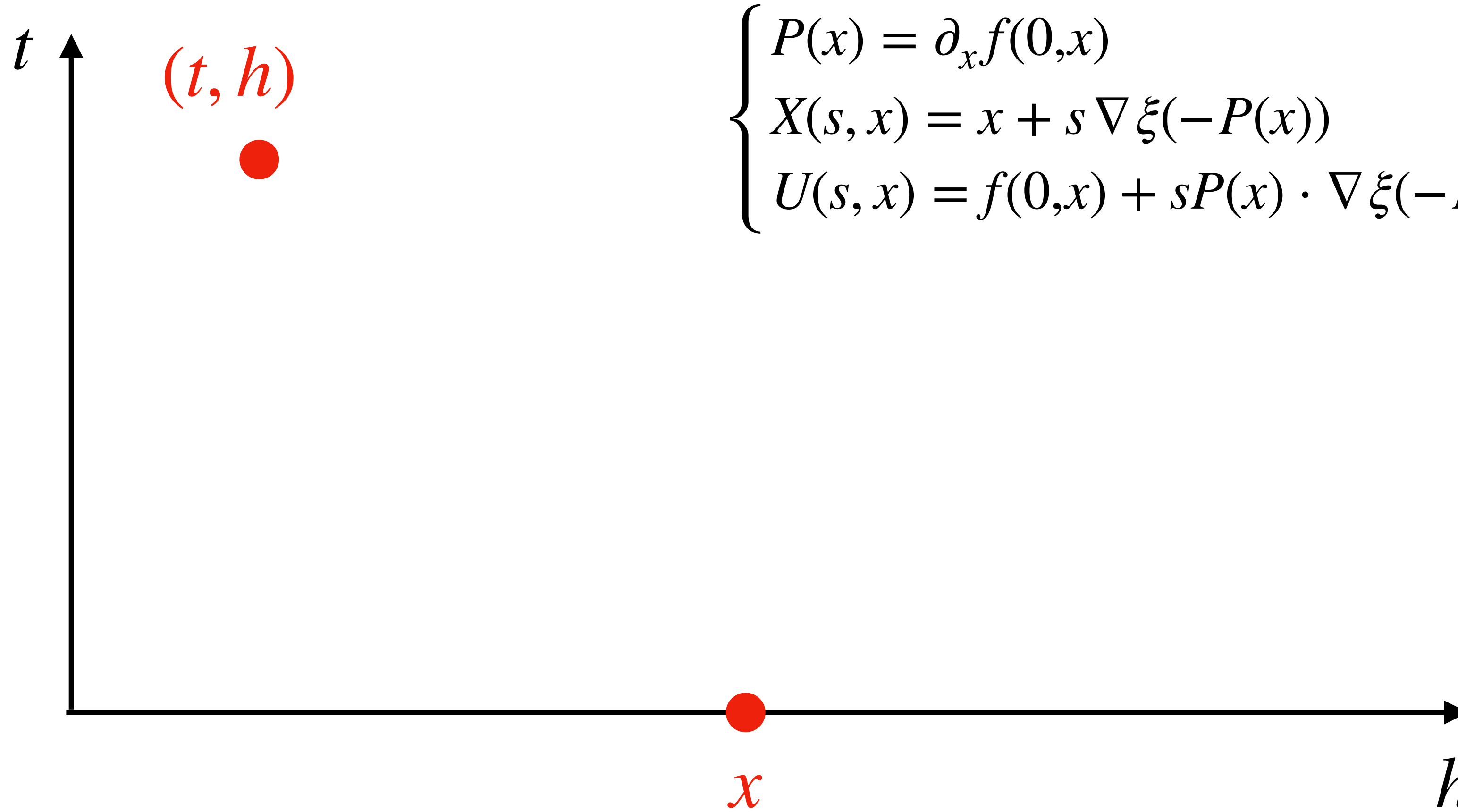

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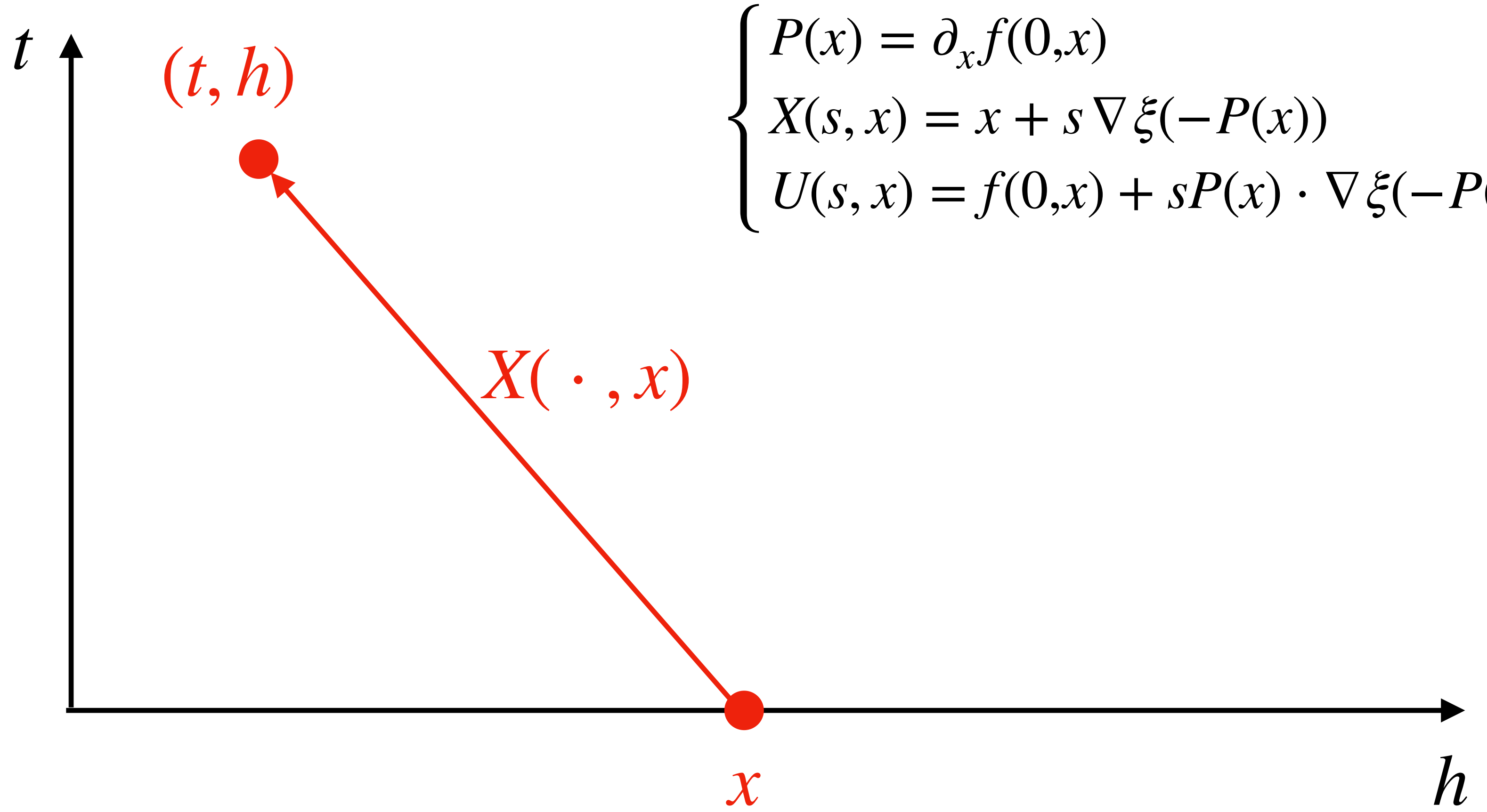

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f is determined by some char. regardless of colliding

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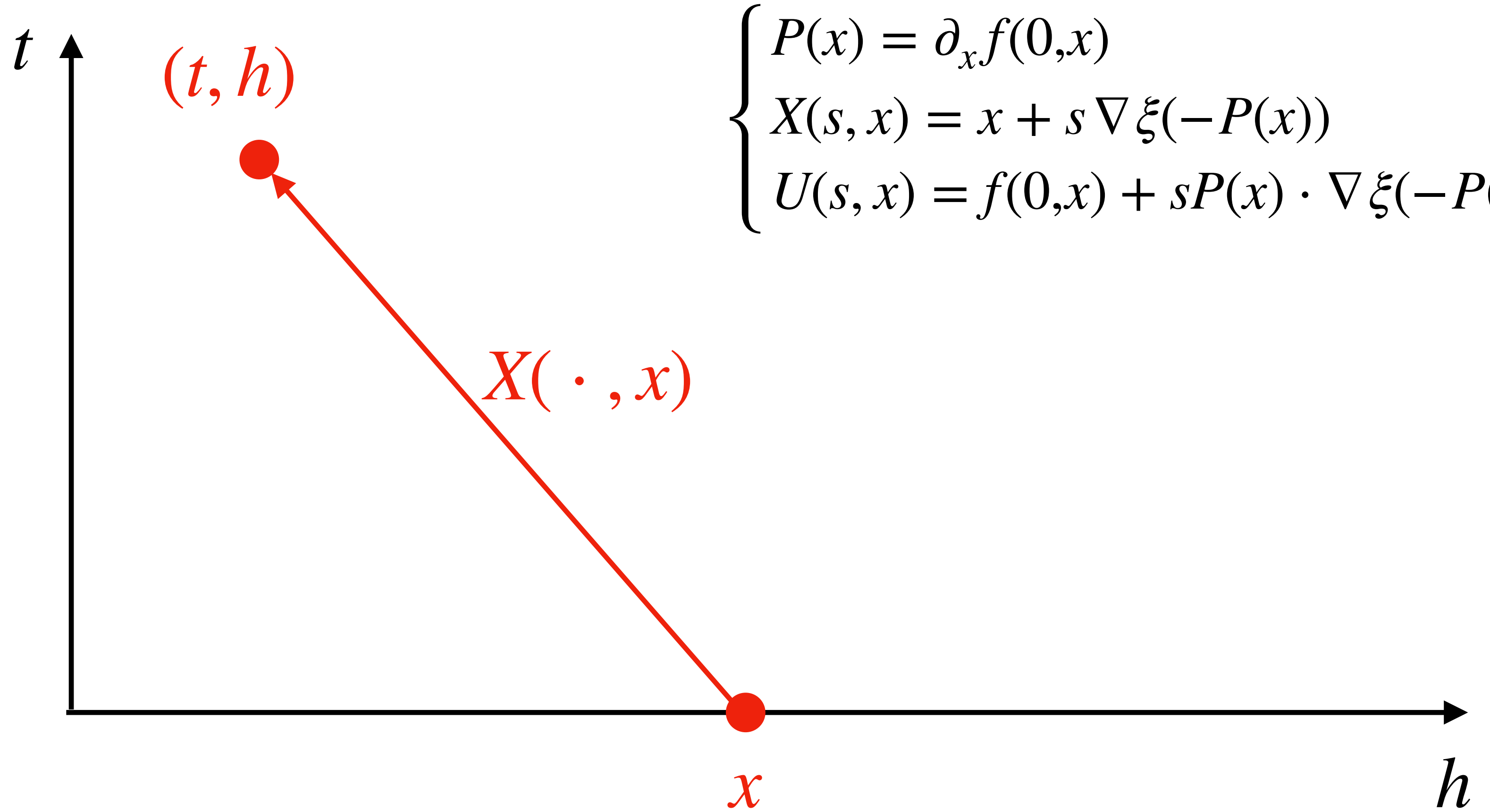

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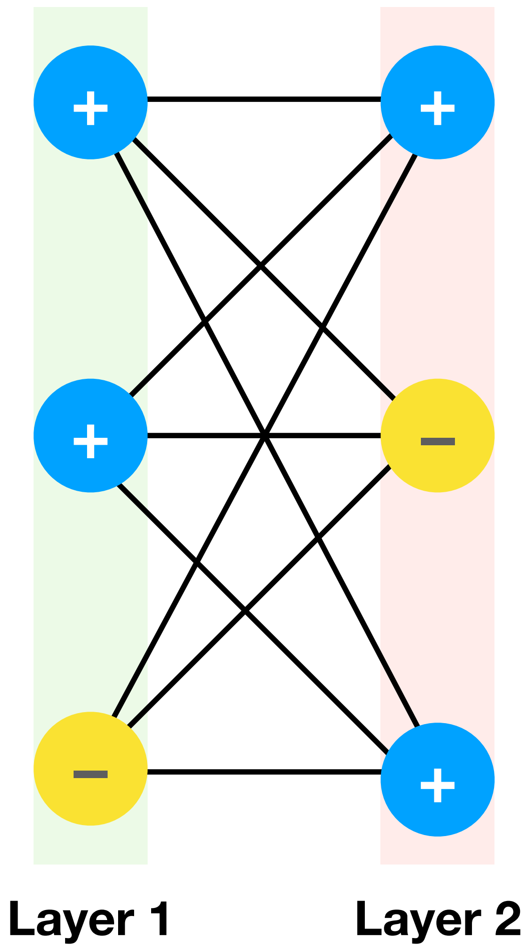

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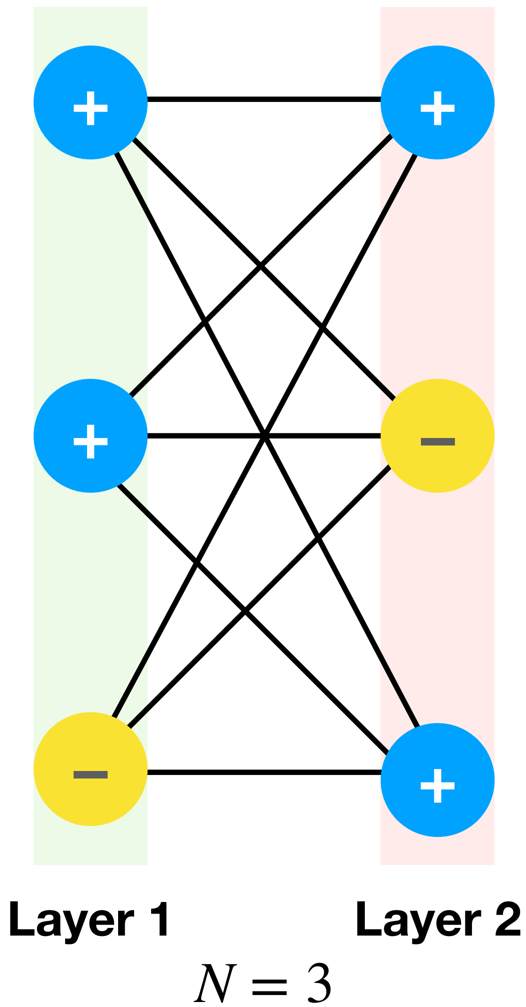

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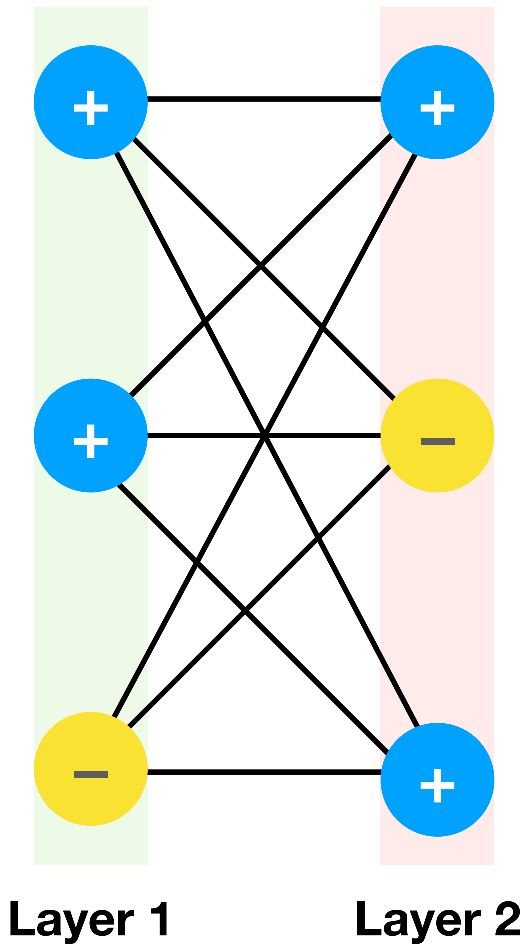

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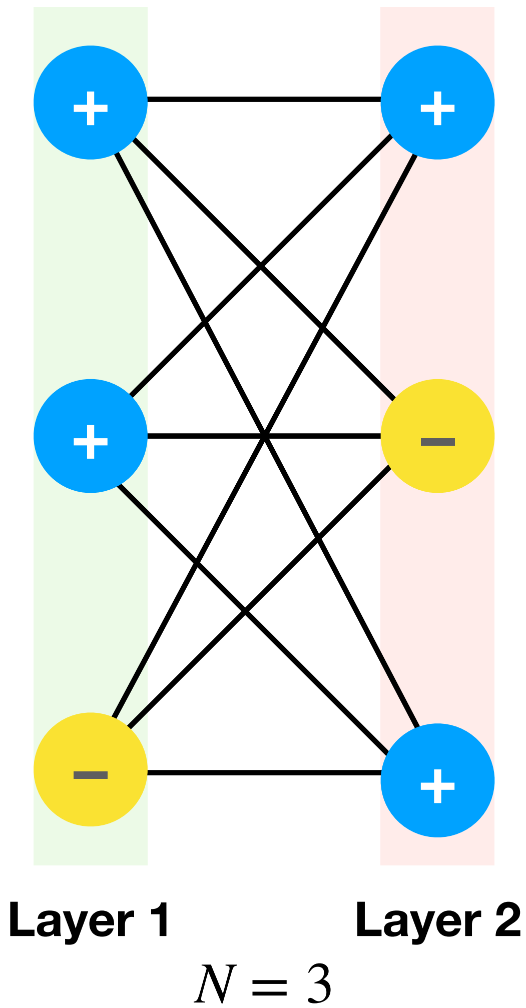

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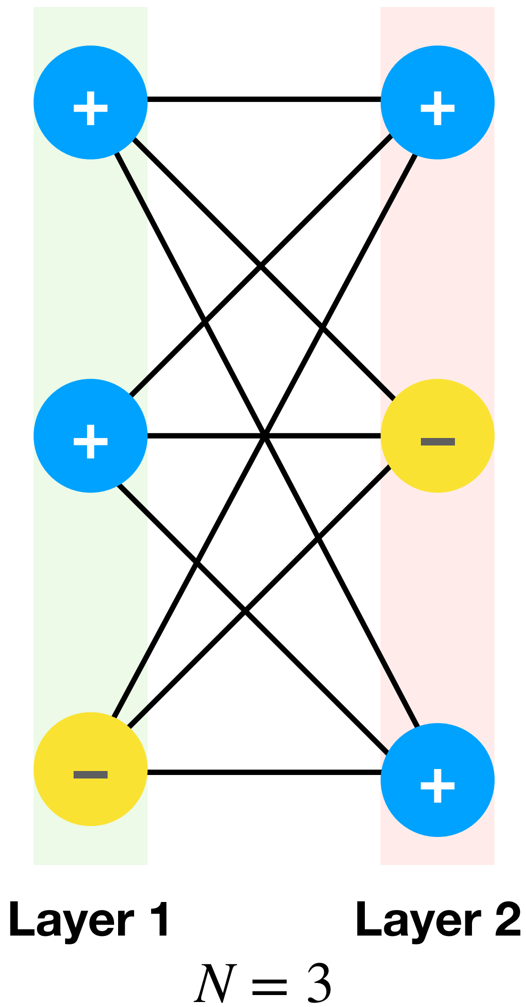

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Conjecture *In general case, even when ξ is non-convex,* $\lim_{N\to\infty} F_N = f$