Random convex hull peeling

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Outline

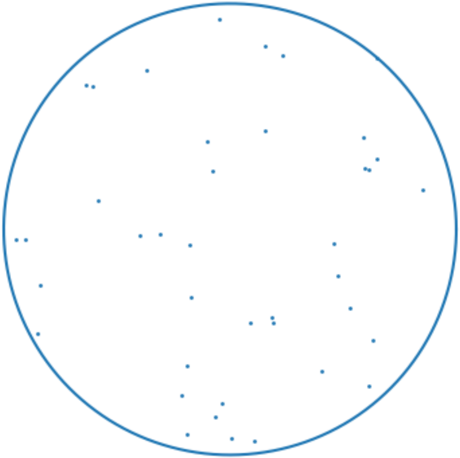
1. Convex hull peeling
2. Earlier results on the convex hull peeling
3. k-faces of the convex hull peeling

1 Convex hull peeling

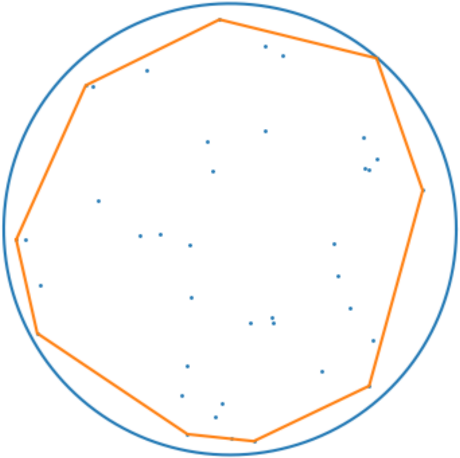
2 Earlier results on the convex hull peeling

3 k-faces of the convex hull peeling

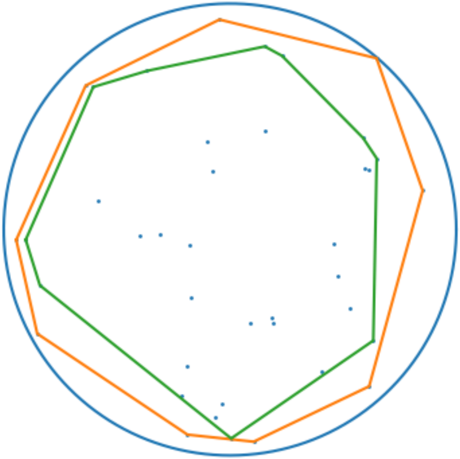
Convex hull peeling



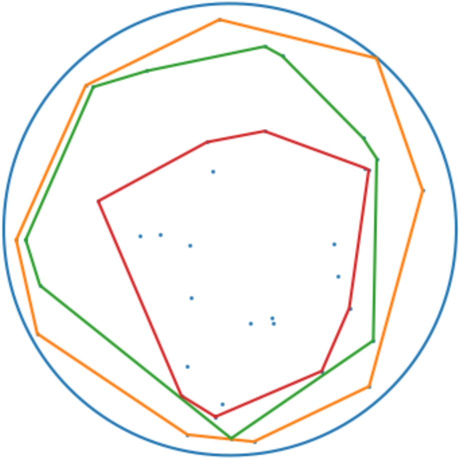
Convex hull peeling



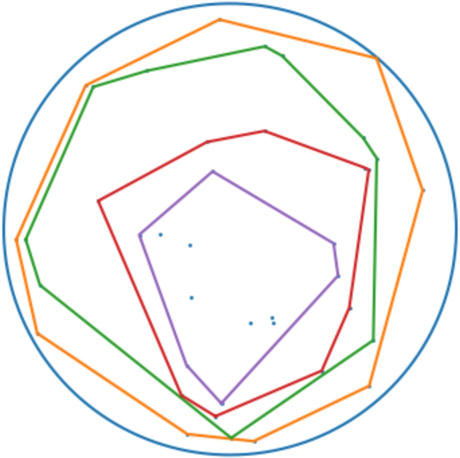
Convex hull peeling



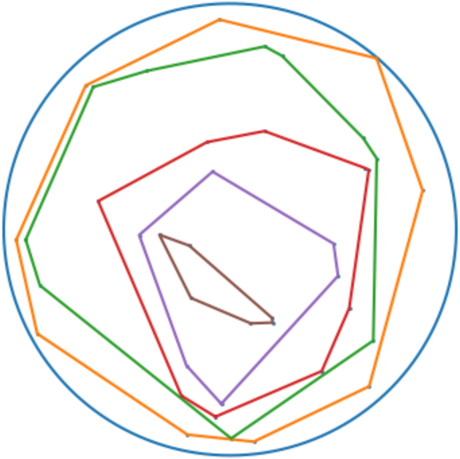
Convex hull peeling



Convex hull peeling



Convex hull peeling



Principle

Convex hull peeling

- Consider a set of points in a convex compact set of $\mathbb{R}^d$.
- Take the convex hull of these points.
- Remove the extreme points.
- Take the convex hull of the remaining points.
- Iterate until no point remains.

Definition

n-th layer: boundary of the convex hull taken at step n.

Applications:

- Statistics: multivariate data analysis, outlier detection.
- Identification of fingerprints.
- Automatic map labeling.
- Computer vision.

Goal:

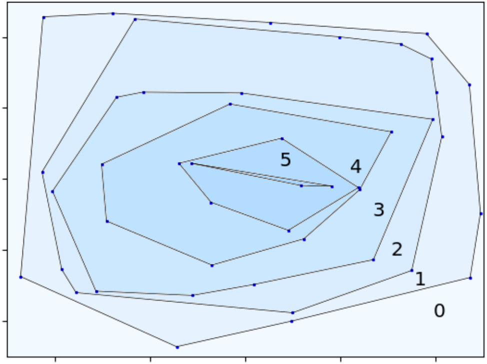
Obtaining asymptotic properties on these layers for a random point set whose size goes to infinity.

1 Convex hull peeling

2 Earlier results on the convex hull peeling

3 k-faces of the convex hull peeling

Convex height function



Asymptotics for the number of layers

Definition (Convex height function)

For $X \subseteq K$

$$h_X := \sum_{n \geq 1} \mathbb{1}_{\mathrm{int}(\mathrm{conv}_n(X))}.$$

K: convex body of $\mathbb{R}^d$,
$\mathcal{P}_\lambda$ Poisson p.p. of intensity measure λ times the Lebesgue measure on K, i.e. X_λ i.i.d. random points uniformly distributed in K where $X_\lambda \sim \mathrm{Poisson}(\lambda \mathrm{Vol}(K))$.

Theorem (Dalal 2004)

$$\mathbb{E}[\max h_{\mathcal{P}_\lambda}] = \Theta(\lambda^{2/(d+1)}).$$

Asymptotics for the number of layers

$\mathcal{P}_\lambda$ a Poisson p.p. in K with intensity measure $\lambda f(x)dx$ with $f > 0$ continuous.

Theorem (Calder-Smart 2020)

For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}[\sup_K |\lambda^{-2/(d+1)} h_{\mathcal{P}_\lambda} - \alpha h| > \varepsilon] \leq \exp(-\delta \lambda^\beta),$$

where $\alpha = \alpha(d) > 0$, $\beta = \beta(d) > 0$ and $h \in C(K)$ is the unique viscosity solution of

$$\begin{cases} \langle Dh, {}^t com(-D^2 h) Dh \rangle = f & \text{in } \mathring{K} \\ h = 0 & \text{on } \partial K. \end{cases}$$

For example in the unit ball $h(x) = C_d \left(1 - |x|^{\frac{2d}{d+1}}\right).$

Asymptotics for the number of layers

Corollary (Calder-Smart 2020)

$$\lambda^{-2/(d+1)} \max h_{\mathcal{P}_\lambda} \xrightarrow[\lambda\to\infty]{\text{a.s.}} \alpha \max h.$$

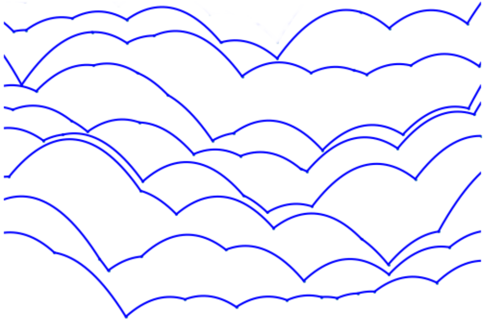
$(\max h_{\mathcal{P}_\lambda}/\lambda^{2/(d+1)})_{\lambda\geq 1}$ is uniformly integrable, it implies

Corollary

$$\lambda^{-2/(d+1)}\mathbb{E}[\max h_{\mathcal{P}_\lambda}] \xrightarrow[\lambda\to\infty]{} \alpha \max h.$$

Some ideas in Calder-Smart

- Game interpretation:
  $h_X(x) = \inf_{p\in\mathbb{R}^d\setminus\{0\}} \sup_{y\in p\cdot(y-x)>0}(h_X(y) + \mathbb{1}_X(y))$. It leads to the PDE.
- They study a parabolic model of peeling.



1. Convex hull peeling
2. Earlier results on the convex hull peeling
3. *k*-faces of the convex hull peeling

Asymptotics for the number of k-faces of the first layer

$\mathcal{P}_\lambda$: Poisson point process of intensity λdx on K .
$N_{n,k,\lambda}$:= number of k-faces of the n-th layer of the peeling of $\mathcal{P}_\lambda$.

Case $K = \mathbb{B}^d$.

Theorem (Rényi-Sulanke 1963, Reitzner 2005)

$$\mathbb{E}[N_{1,k,\lambda}] \sim_{\lambda\to\infty} C_{1,k,d}\lambda^{\frac{d-1}{d+1}}.$$

Theorem (Calka, Schreiber, Yukich 2013)

$$Var[N_{1,k,\lambda}] \sim_{\lambda\to\infty} C'_{1,k,d}\lambda^{\frac{d-1}{d+1}}$$

$+$ *CLT.*

Case K simple polytope.

Theorem (Rényi-Sulanke 1963, Reitzner 2005)

$$\mathbb{E}[N_{1,k,\lambda}] \sim_{\lambda\to\infty} C''_{1,k,d}\log^{d-1}(\lambda).$$

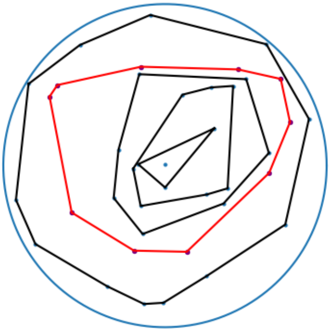
Theorem (Reitzner 2005, Calka, Yukich 2017)

$$Var[N_{1,k,\lambda}] \sim_{\lambda\to\infty} C'''_{1,k,d}\log^{d-1}(\lambda)$$

$+$ *CLT.*

k-faces of the convex hull peeling

$N_{n,k,\lambda}$:= number of k-faces of the n-th layer of the peeling of $\mathcal{P}_\lambda$.
n is a fixed integer and does not vary with λ.



$N_{2,0,\lambda}$: number of vertices of the red layer, $N_{2,1,\lambda}$: number of edges.

k-faces of the convex hull peeling in the unit ball

Case $K = \mathbb{B}^d$.

Theorem (Calka, Quilan)

$$\mathbb{E}[N_{n,k,\lambda}] \sim_{\lambda\to\infty} C_{n,k,d}\lambda^{\frac{d-1}{d+1}},$$

$$Var[N_{n,k,\lambda}] \sim_{\lambda\to\infty} C'_{n,k,d}\lambda^{\frac{d-1}{d+1}}$$

$+$ *CLT where* $C_{n,k,d}, C'_{n,k,d} > 0$.

Case K simple polytope.

Theorem (Calka, Quilan)

$$\mathbb{E}[N_{n,k,\lambda}] \sim_{\lambda\to\infty} C''_{n,k,d}\log^{d-1}(\lambda),$$

$$Var[N_{n,k,\lambda}] \sim_{\lambda\to\infty} C'''_{n,k,d}\log^{d-1}(\lambda).$$

Elements of proof: criterion layer n

We focus on $d = 2$ and $k = 0$.

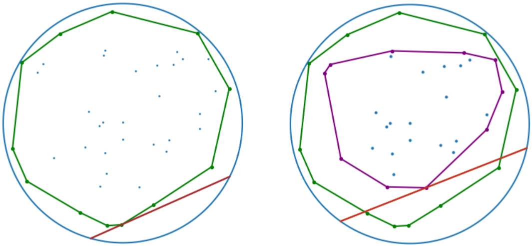
Lemma

x is extreme $\iff$ there exists a cap whose boundary contains x that contains no point of $\mathcal{P}_\lambda$.

Lemma

x on the n-th layer $\iff$ it verifies both conditions:

- *$\exists$ a cap C with $x \in \partial C$ that only contains points of layer $(n-1)$ at most.*
- *Any cap C with $x \in \partial C$ contains at least one point on layer $\geq (n-1)$.*

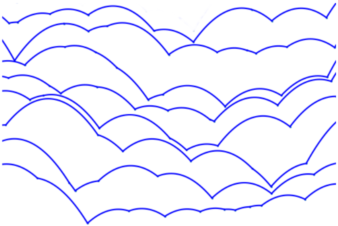


Elements of proof: rescaling

Case $K = \mathbb{B}^2$.

Definition

$$T^{(\lambda)} : \mathbb{B}^2 \to \lambda^{\frac{1}{3}}[-\pi, \pi] \times [0, \lambda^{\frac{2}{3}})$$
$$(r, \theta) \mapsto \left(\lambda^{\frac{1}{3}}\theta, \lambda^{\frac{2}{3}}(1 - r)\right).$$

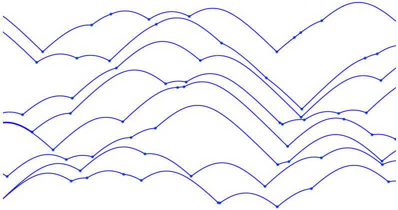


Case K simple polytope.

$V := \{(x, y) \in \mathbb{R}^2 : x = -y\}$ and p_V the orthogonal projection on V.

Definition

$$T^{(\lambda)} : [0, \infty)^2 \to V \times \mathbb{R}$$
$$(x,y) \mapsto (p_V(\log(x), \log(y)),$$
$$\frac{1}{2}(\log(\lambda) + \log(x) + \log(y))).$$



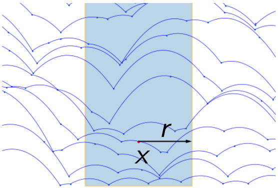
Elements of proof: stabilization

$C_x(r)$: vertical rectangle around x of half-width r in $\mathbb{R}^2$.
$\mathcal{P}^{(\lambda)}$: image of $\mathcal{P}_\lambda$ by $T^{(\lambda)}$.
$\Phi_n(\mathcal{P}^{(\lambda)})$: image of the n-th convex hull by $T^{(\lambda)}$.
$\xi_n(x, \mathcal{P}^{(\lambda)}) = \mathbb{1}_{\partial \Phi_n(\mathcal{P}^{(\lambda)})}(x)$.

Theorem (Stabilization)

There exist $c_1, c_2, \alpha > 0$ *such that*

$$\mathbb{P}(\xi_n(x, \mathcal{P}^{(\lambda)}) \neq \xi_n(x, \mathcal{P}^{(\lambda)} \cap C_x(r))) \leq c_1 \exp(-c_2 r^\alpha).$$



Elements of proof : Sandwiching

Problem : In the case where K is a simple polytope, the rescaling only works in a neighbourhood of a vertex of K.

$v : x \mapsto \min\{\mathrm{Vol}(H \cap K) : H \text{ halfspace containing } x\}$.
$K(v \geq t) := \{x \in \mathbb{R}^2 : v(x) \geq t\}$ the floating body of order t.
$s := c_1 \frac{1}{\lambda \log(\lambda)^\alpha}$ and $T^* := c_2 \frac{\log\log(\lambda)}{\lambda}$.

Theorem (Sandwiching)

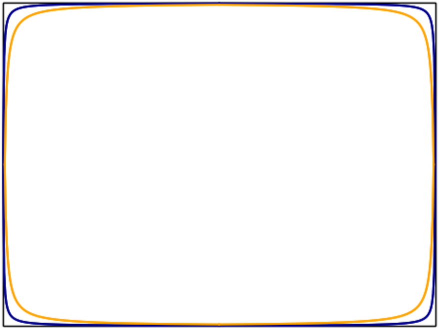
When K is a simple polytope, for λ large enough with probability higher than $1 - \log(\lambda)$:

$$\partial \mathrm{conv}_m(\mathcal{P}_\lambda) \subset K(v \geq s) \setminus K(v \geq T^*)$$

for all $1 \leq m \leq n$.

Generalization of Barany-Reitzner 2010.

Elements of proof : Sandwiching 2



Boundaries of two floating bodies in a square.

We prove that the contribution of points far from the vertices of K is negligible and that the contributions of points near different vertices are independent.

Open problems

- Evolution of $(C_{n,k,d})_n$?
- Other regimes for the layer number?
- Phase transition in the polytope case?
- Position of the layers as a function of the height in the rescaled model?
- More general Poisson point processes?

Thank you for your attention!