Cutoff Profile of ASEP on the segment

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Asymmetric simple exclusion process (ASEP)

Dynamics:
- There is at most one particle per site, at rate one particles jump one step to the right or left if possible
- jumps to the right (resp. left) have probability $p > 1/2$, (resp. $q = 1 - p$), so there is a drift to the right
- for $p = 1$ we obtain the totally ASEP (TASEP)
- Markov process $(\eta_t, t \geq 0)$ with state space $\Omega = \{0, 1\}^\mathbb{Z}$
ASEP on the segment

- put $k$ particles on the segment $[1; N] = \{1, \ldots, N\}$
- jumps out of the segment are forbidden
- Markov chain with stationary measure $\pi_{N,k}$
- $P^\eta_t$ = law of the ASEP started from $\eta$ at time $t$

Goal: Understand how ASEP mixes to equilibrium w.r.t. $d^{N,k}(t) = \max_{\eta} \| P^\eta_t - \pi_{N,k} \|_{TV} \in [0, 1]$ as $N \to \infty$ and $k = \alpha N$, $\alpha \in (0, 1)$. 
Often $d^{N,\alpha N}$ drops from 1 to 0 abruptly, i.e. cutoff holds: At time $CN \pm \epsilon N$, $d^{N,\alpha N}$ goes to 1 resp. 0 for any $\epsilon > 0$.

simple example of cutoff: ASEP with 1 particle on the segment

no cutoff for ASEP with 1 particle on the circle $\mathbb{Z}_N$
1. [Benjamini et al. ’03, Trans. Math. Soc.] showed there is precutoff:

\[ d^{N,\alpha N} \]
1. [Benjamini et al. ’03, Trans. Math. Soc.] showed there is precutoff:

2. [Labbé - Lacoin ’16, Ann. Prob.] showed cutoff at

\[ C = \frac{\left(\sqrt{\alpha} + \sqrt{1-\alpha}\right)^2}{p-q} \]

Both papers show their result also for the multi-color ASEP
When we zoom in on the cutoff point, we expect to see a smooth transition in a window $w_N$ of size $o(N)$:

The function which describes the smooth transition is the cutoff profile.
Cutoff profile

We prove the following:

**Theorem (Bufetov-N. ’22, PTRF ’22)**

*Let* $c \in \mathbb{R}$, *and let* $k_N = \alpha N$, $\alpha \in (0, 1)$. *Then we have*

$$
\lim_{N \to \infty} d^{N, \alpha N} \left( \frac{(\sqrt{\alpha} + \sqrt{1 - \alpha})^2}{p - q} N + \frac{cN^{1/3}}{p - q} \right) = 1 - F_{\text{GUE}}(cf(\alpha)),
$$

*where* $f(\alpha) = \frac{(\alpha(1-\alpha))^{1/6}}{(\sqrt{\alpha}+\sqrt{1-\alpha})^{4/3}}$.

- $F_{\text{GUE}}$ *asymptotic law of the largest eigenvalue of a random matrix from the Gaussian Unitary Ensemble*
- Cutoff window is $N^{1/3}$, -profile is $1 - F_{\text{GUE}}$, Cutoff is reproven
- replace $\frac{(\sqrt{\alpha}+\sqrt{1-\alpha})^2}{p-q} N$ by $\frac{(\sqrt{k_N}+\sqrt{N-k_N})^2}{p-q}$ when $\frac{k_N}{N} \to \alpha$
Notation and bounds

For brevity, we denote the time point

\[ g(k, c) := \frac{(\sqrt{k} + \sqrt{N - k})^2 + cN^{1/3}}{p - q}. \]

and always assume the number of particles \( k_N \) satisfies \( k_N/N \to \alpha \in (0, 1) \).
Notation and bounds

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and always assume the number of particles \( k_N \) satisfies \( k_N/N \to \alpha \in (0, 1) \).

We will separately show
\[
\liminf_{N \to \infty} d^{N,k_N} (g(k_N, c)) \geq 1 - F_{\text{GUE}}(cf(\alpha)),
\]
and
\[
\limsup_{N \to \infty} d^{N,k_N} (g(k_N, c)) \leq 1 - F_{\text{GUE}}(cf(\alpha)), \tag{1}
\]

As is often the case, it is harder to show the upper bound (1), so we focus on (1).
Hitting times I

Consider the configurations for ASEP on the segment

\[ \xi^0 = 1_{[1;k_N]}, \quad \xi^1 = 1_{[N-k_N+1;N]} . \]

\[ \xi^0 : \]

\[ \xi^1 : \]

\( \xi^{(0)} \) is intuitively the 'worst' initial configuration. Let \( h \) the first time that ASEP started from \( \xi^0 \) reaches \( \xi^1 \):

\[ h = \inf \{ t : \xi^0_t = \xi^1 \} . \]

Then we have the inequality

\[ d^{N,k_N}(t) \leq \mathbb{P}(h > t) . \]
Consider the configurations for ASEP on $\mathbb{Z}$

$$\zeta^0 = 1_{[1:k_N]} + 1_{\mathbb{Z}_{>N}}, \quad \zeta^1 = 1_{\mathbb{Z}_{>(N-k_N)}}.$$  

Let $\mathcal{H}$ the first time that ASEP started from $\zeta^0$ reaches $\zeta^1$:

$$\mathcal{H} = \inf\{ t : \zeta^0_t = \zeta^1 \}.$$  

Then we have the inequalities

$$d^{N,k_N}(t) \leq \mathbb{P}(\mathcal{H} > t) \leq \mathbb{P}(\mathcal{H} > t).$$
The hitting time $\mathcal{H}$ was already studied by [Benjamini et al '03].

We will eventually show

$$\lim_{N \to \infty} \mathbb{P} (\mathcal{H} > g(k_N, c)) = 1 - F_{\text{GUE}}(cf(\alpha)).$$

By the inequality $d^{N,k_N}(t) \leq \mathbb{P}(\mathcal{H} > t)$, this will imply

$$\limsup_{N \to \infty} d^{N,k_N} (g(k_N, c)) \leq 1 - F_{\text{GUE}}(cf(\alpha)).$$
Recall \( \zeta^0 = 1_{[1:k_N]} + 1_{\mathbb{Z}_{>N}} \).
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To understand the hitting time $\mathcal{H}$, we track the leftmost particle/rightmost hole of $\zeta_0^t$:

$\mathcal{L}(\zeta_0^t) = \min\{i \in \mathbb{Z} : \zeta_0^t(i) = 1\}$ \hspace{1cm} $\mathcal{R}(\zeta_0^t) = \max\{i \in \mathbb{Z} : \zeta_0^t(i) = 0\}$.

Note that deterministically for all $t$

$$\mathcal{L}(\zeta_0^t) - 1 \leq N - k_N \leq \mathcal{R}(\zeta_0^t)$$

and $\mathcal{H}$ is precisely the first time that

$$\mathcal{L}(\zeta_{\mathcal{H}}^0) - 1 = N - k_N = \mathcal{R}(\zeta_{\mathcal{H}}^0)$$
Let $B_N(c)$ be the event that $\zeta_g^{0}(k_N,c)$ looks like this:

- when $B_N(c)$ happens, $\bar{g}$ cannot be much bigger than $g(k_N, c)$, in particular we can show

$$\lim_{N \to \infty} \mathbb{P}(B_N(c)) = \lim_{N \to \infty} \mathbb{P}(\bar{g} < g(k_N, c))$$
Let $B_N(c)$ be the event that $\zeta^0_{g(k_N,c)}$ looks like this:

\[
\begin{array}{ccccccc}
\circ \circ \circ \circ \circ \circ \circ \circ \bullet & N - k_N & N^{\frac{1}{10}} L(\zeta^0_{g(k_N,c)}) & N - k_N & R(\zeta^0_{g(k_N,c)}) & N - k_N + N^{\frac{1}{10}} \\
\end{array}
\]

- when $B_N(c)$ happens, $\tilde{\epsilon}$ cannot be much bigger than $g(k_N, c)$, in particular we can show

\[
\lim_{N \to \infty} P(B_N(c)) = \lim_{N \to \infty} P(\tilde{\epsilon} < g(k_N, c))
\]

We want to prove

\[
\lim_{N \to \infty} P(B_N(c)) = F_{\text{GUE}}(cf(\alpha)).
\]

- This will imply $\lim_{N \to \infty} P(\tilde{\epsilon} > g(k_N, c)) = 1 - F_{\text{GUE}}(cf(\alpha))$, and thus yield the upper bound for $d_{N,k_N}^N(g(k_N,c))$. 
Let $B_N(c)$ be the event that $\zeta_0^{g(k_N,c)}$ looks like this:

\[ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \bullet \circ \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \]

\[ N - k_N - N^{\frac{1}{10}} \quad \mathcal{L}(\zeta_0^{g(k_N,c)}) \quad N - k_N \quad \mathcal{R}(\zeta_0^{g(k_N,c)}) \quad N - k_N + N^{\frac{1}{10}} \]

We want to prove

\[ \lim_{N \to \infty} \mathbb{P}(B_N(c)) = F_{\text{GUE}}(\text{cf}(\alpha)). \]

Main sources of this convergence are:

- non-standard CLT for ASEP
- algebraic identities for multi-color ASEP
- couplings to compare different ASEPs
Non-standard CLT for ASEP

We start ASEP on $\mathbb{Z}$ from $1_{\mathbb{Z}_{\leq k_N}}$, ("step initial data") and denote $x_{k_N}(t) = \text{position at time } t \text{ of the particle that started in } 1$.

Theorem (Corollary of Johansson ’00 (for p=1), Tracy-Widom ’09)
We have for $k_N$ with $k_N/N \to \alpha \in (0, 1)$ that

$$
\lim_{N \to \infty} \mathbb{P} \left( x_{k_N}(g(k_N, c)) \leq N - k_N \right) = 1 - F_{\text{GUE}}(cf(\alpha)),
$$

where $f(\alpha) = \frac{(\alpha(1-\alpha))^{1/6}}{(\sqrt{\alpha} + \sqrt{1-\alpha})^{4/3}}$. 
Multi-color ASEP

we have $N$ particles on $[1; N]$ with colors $1, \ldots, N$

each particle moves as in ASEP, but jumps to sites occupied by smaller color are impossible

encoded by a permutation $\pi$ mapping positions to colors

invariant measure is the Mallows measure

$$\mathcal{M}(\pi) = \frac{(p/q)^{\#\text{inv}(\pi)}}{Z_N},$$

where $\text{inv}(\pi)$ are the inversions of $\pi$, for $q = 0$, $\mathcal{M}$ is the Dirac measure on $\pi(i) = N - i + 1$
we have $N$ particles on $[1; N]$ with colors $1, \ldots, N$.

- each particle moves as in ASEP, but jumps to sites occupied by smaller color are impossible.
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each particle moves as in ASEP, but jumps to sites occupied by smaller color are impossible

encoded by a permutation $\pi$ mapping positions to colors

when we project down and only distinguish between particles of color smaller equal $k$ and greater than $k$, we recover ASEP
Bringing into equilibrium

Consider a permutation of \([a; b]\) and let \([c; d] \subseteq [a; b]\):

- Example: \([a; b] = [1; 8]\) and \([c; d] = [2; 4]\)

![Diagram showing a sequence of colors from 1 to 8]

- for TASEP to bring into equilibrium \([c; d]\) means that we order the colors in \([c; d]\) in decreasing order.
- more generally, it means to distribute the colors in \([c; d]\) according to the Mallows measure.
Bringing into equilibrium

Consider a permutation of \([a; b]\) and let \([c; d] \subseteq [a; b]\):

Example: \([a; b] = [1; 8]\) and \([c; d] = [2; 4]\)

\[
\begin{array}{cccccccc}
4 & 8 & 3 & 1 & 6 & 5 & 2 & 7 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
\]

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More generally, it means to distribute the colors in \([c; d]\) according to the Mallows measure.
Color-position symmetry

we start with the identity permutation

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
\]

bring into equilibrium \([c_1; d_1]\), then \([c_2; d_2]\)

here: \([c_1; d_1] = [2; 4]\) and \([c_2; d_2] = [4; 7]\)
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\[\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}\]

bring into equilibrium \([c_1; d_1]\), then \([c_2; d_2]\)

here: \([c_1; d_1] = [2; 4]\) and \([c_2; d_2] = [4; 7]\)

run the process up to time \(t\)

this results in a permutation \(\pi_t\)
we start with the identity permutation

1 2 3 4 5 6 7 8

1 2 3 4 5 6 7 8
Color-position symmetry

- we start with the identity permutation

\[
\hat{\pi}_t
\]

- run the process up to time \( t \)
Color-position symmetry

-we start with the identity permutation

- run the process up to time $t$

- bring into equilibrium $[c_2; d_2] = [4; 7]$
Color-position symmetry

- we start with the identity permutation

- run the process up to time $t$
- bring into equilibrium $[c_2; d_2] = [4; 7]$
- bring into equilibrium $[c_1; d_1] = [2; 4]$
- this results in a permutation $\hat{\pi}_t$
Color-position symmetry

We have the following unintuitive identity:

**Proposition (Bufetov-N. ’22)**

The permutations $\pi_t^{-1}$ and $\hat{\pi}_t$ are equal in law.

- source of this identity is purely algebraic (using Hecke algebras, cf. [Bufetov ’21+], also [Borodin-Bufetov ’20])
- since $\pi_t$ maps positions to colors, whereas $\hat{\pi}_t^{-1}$ maps colors to positions, this is a color-position symmetry
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- since $\pi_t$ maps positions to colors, whereas $\hat{\pi}_t^{-1}$ maps colors to positions, this is a color-position symmetry

Main application:

- $(\pi_t, t \geq 0)$ starts from a ’complicated’ permutation, which can be projected down to (a perturbation of)
  $\zeta^0 = 1_{[1; k_N]} + 1_{\mathbb{Z}_{>N}}$
- $(\hat{\pi}_t, t \geq 0)$ starts from the identity, which can be projected down to step initial data (CLT available)
The permutation $\pi_0$

Consider ASEP on a very large segment with this step initial data:

We bring into equilibrium $[-N^N; N^N + N]$:

We bring into equilibrium $[-N^N - k_N; 0]$:

This is very close to a shift of $\zeta^0$, we call it $\hat{\zeta}^0$
The permutation $\hat{\pi}_0$

- $x, y$ free parameters
- In the identity, we project down on colors $\leq x$ (black), colors in $(x, y]$ (grey) and colors $> y$ (white)

Our initial configuration thus is

$$\mathcal{D}_0 : \quad \begin{array}{c}
\begin{array}{cccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ \\
\end{array}
\end{array}
$$

$$- N^N - k_N \quad x \quad y \quad N^N + N$$

After time $t$, the process may look like this:

$$\mathcal{D}_t : \quad \begin{array}{c}
\begin{array}{cccccccccccc}
\bullet & \circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ \\
\end{array}
\end{array}
$$

$$- N^N - k_N \quad 0 \quad N^N + N$$
Bringing into equilibrium

$\mathcal{D}_t : 
\begin{array}{c}
\text{\includegraphics[width=\textwidth]{image1.png}} \\
\end{array}$

$-N^N - k_N \quad 0 \quad N^N + N$

In $\mathcal{D}_t$ we bring into equilibrium $[-N^N - k_N; 0] :$

$\mathcal{\tilde{D}}_t : 
\begin{array}{c}
\text{\includegraphics[width=\textwidth]{image2.png}} \\
\end{array}$

$-N^N - k_N - N^N \quad 0 \quad N^N + N$

Afterwards, we bring into equilibrium $[-N^N; N^N + N] :$

$\mathcal{\hat{D}}_t : 
\begin{array}{c}
\text{\includegraphics[width=\textwidth]{image3.png}} \\
\end{array}$

$-N^N - k_N - N^N \quad 0 \quad N^N + N$
\( x := N - k_N - N^{1/10} \quad y := N - k_N + N^{1/10} \)

Recall \( B_N(c) \) be the event that \( \xi_0^{\zeta_g(k_N,c)} \) looks like this:

[Diagram showing a line with open and closed circles representing \( x, L(\xi_0^{\zeta_g(k_N,c)}), N - k_N, R(\xi_0^{\zeta_g(k_N,c)}), y \) with a specific pattern of open and closed circles.]
Reminders

\[ x := N - k_N - N^{1/10} \quad y := N - k_N + N^{1/10} \]

Recall \( B_N(c) \) be the event that \( \zeta^0_{g(k_N,c)} \) looks like this:

\[ \begin{array}{c}
\circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \bullet \\
\hline
x & \mathcal{L}(\zeta^0_{g(k_N,c)}) & N - k_N & \mathcal{R}(\zeta^0_{g(k_N,c)}) & y
\end{array} \]

We call \( \hat{B}_N(c) \) be the event that \( \hat{\zeta}^0_{g(k_N,c)} \) looks the same:

\[ \begin{array}{c}
\circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \bullet \\
\hline
x & \mathcal{L}(\hat{\zeta}^0_{g(k_N,c)}) & N - k_N & \mathcal{R}(\hat{\zeta}^0_{g(k_N,c)}) & y
\end{array} \]
$x := N - k_N - N^{1/10}$  $y := N - k_N + N^{1/10}$

Recall $B_N(c)$ be the event that $\zeta_{g(k_N,c)}^0$ looks like this:

We call $\hat{B}_N(c)$ be the event that $\hat{\zeta}_{g(k_N,c)}^0$ looks the same:

Recall $\hat{\mathcal{D}}_t$ from the previous slide:
As corollary of the color-position symmetry, we have:

**Proposition (Bufetov -N.)**

We have

\[
\mathbb{P}(\hat{B}_N(c)) = \mathbb{P}\left( \text{all black particles in } \hat{\mathcal{D}}_t \text{ are at positions } > 0, \text{ all holes in } \hat{\mathcal{D}}_t \text{ are at positions } \leq 0 \right). \tag{2}
\]

- the l.h.s. of (2) is very close to \( \mathbb{P}(\mathcal{H} < g(k_N, c)) \)
- the r.h.s. of (2) concerns two events involving only particles/holes
- using couplings, we can eventually compute the r.h.s. with the CLT for ASEP: It converges to \( F_{\text{GUE}}(cf(\alpha)) \)
Thank you for your attention!