Cutoff Profile of ASEP on the segment

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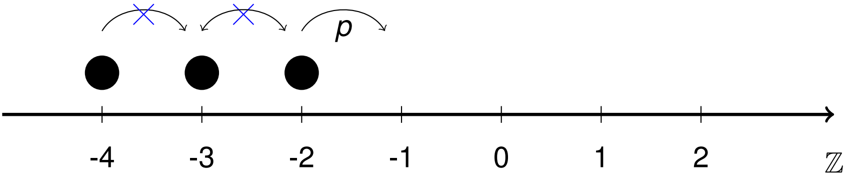
Rencontre de Probabilités

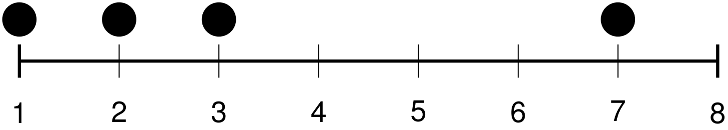
Asymmetric simple exclusion process (ASEP)



Dynamics:

- There is at most one particle per site, at rate one particles jump one step to the right or left if possible
- jumps to the right (resp. left) have probability $p > 1/2$, (resp. $q = 1 - p$), so there is a drift to the right
- for $p = 1$ we obtain the totally ASEP (TASEP)
- Markov process $(\eta_t, t \geq 0)$ with state space $\Omega = \{0, 1\}^{\mathbb{Z}}$

ASEP on the segment



- put k particles on the segment $[1;N] = \{1, \ldots, N\}$
- jumps out of the segment are forbidden
- Markov chain with stationary measure $\pi_{N,k}$
- P_t^η = law of the ASEP started from η at time t

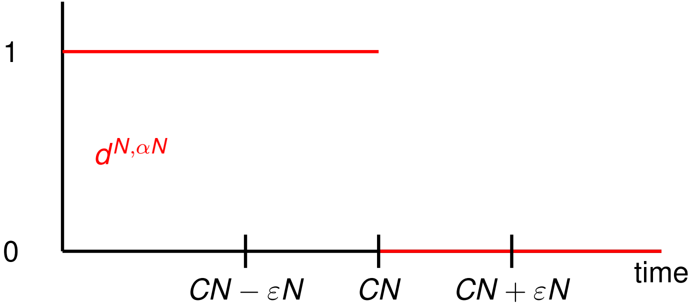
Goal: Understand how ASEP mixes to equilibrium w.r.t.

$$d^{N,k}(t) = \max_\eta \|P_t^\eta - \pi_{N,k}\|_{\mathrm{TV}} \in [0,1]$$

as $N \to \infty$ and $k = \alpha N, \alpha \in (0,1)$.

Cutoff

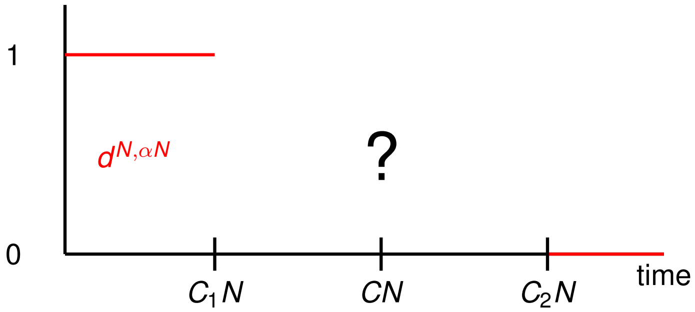
- Often $d^{N,\alpha N}$ drops from 1 to 0 abruptly, i.e. cutoff holds: At time $CN \pm \varepsilon N$, $d^{N,\alpha N}$ goes to 1 resp. 0 for any $\varepsilon > 0$.



- simple example of cutoff: ASEP with 1 particle on the segment
- no cutoff for ASEP with 1 particle on the circle $\mathbb{Z}_N$

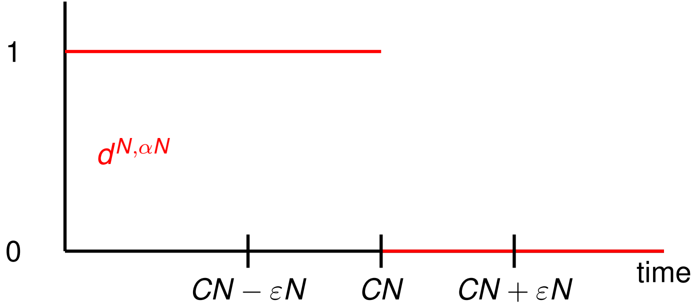
Precutoff and Cutoff for ASEP

1. [Benjamini et al. ’03, Trans. Math. Soc.] showed there is precutoff:



Precutoff and Cutoff for ASEP

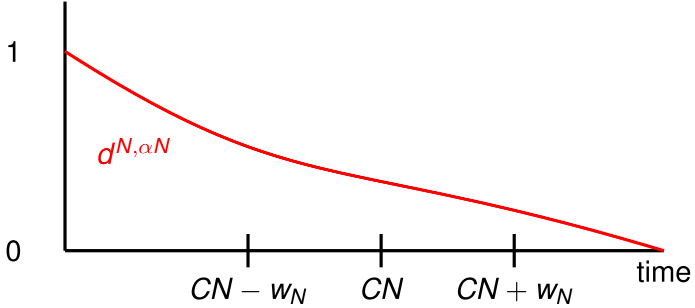
1. [Benjamini et al. '03, Trans. Math. Soc.] showed there is precutoff:
2. [Labbé - Lacoin '16, Ann. Prob.] showed cutoff at $C = \frac{(\sqrt{\alpha}+\sqrt{1-\alpha})^2}{p-q}$



Both papers show their result also for the multi-color ASEP

Cutoff window and profile

When we zoom in on the cutoff point, we expect to see a smooth transition in a window w_N of size $o(N)$:



The function which describes the smooth transition is the cutoff profile.

Cutoff profile

We prove the following:

Theorem (Bufetov-N. '22, PTRF '22)

Let $c \in \mathbb{R}$, *and let* $k_N = \alpha N, \alpha \in (0,1)$. *Then we have*

$$\lim_{N\to\infty} d^{N,\alpha N}\left(\frac{(\sqrt{\alpha}+\sqrt{1-\alpha})^2}{p-q}N + \frac{cN^{1/3}}{p-q}\right) = 1 - F_{\mathrm{GUE}}(cf(\alpha)),$$

where $f(\alpha) = \frac{(\alpha(1-\alpha))^{1/6}}{(\sqrt{\alpha}+\sqrt{1-\alpha})^{4/3}}$.

- F_{GUE} asymptotic law of the largest eigenvalue of a random matrix from the Gaussian Unitary Ensemble
- Cutoff window is $N^{1/3}$, -profile is $1 - F_{\mathrm{GUE}}$, Cutoff is reproven
- replace $\frac{(\sqrt{\alpha}+\sqrt{1-\alpha})^2}{p-q}N$ by $\frac{(\sqrt{k_N}+\sqrt{N-k_N})^2}{p-q}$ when $\frac{k_N}{N} \to \alpha$

Notation and bounds

For brevity, we denote the time point

$$g(k,c) := \frac{(\sqrt{k}+\sqrt{N-k})^2 + cN^{1/3}}{p-q}.$$

and always assume the number of particles k_N satisfies $k_N/N \to \alpha \in (0,1)$.

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and always assume the number of particles k_N satisfies $k_N/N \to \alpha \in (0,1)$.
We will separately show

$$\liminf_{N\to\infty} d^{N,k_N}(g(k_N,c)) \geq 1 - F_{\mathrm{GUE}}(cf(\alpha)),$$

and

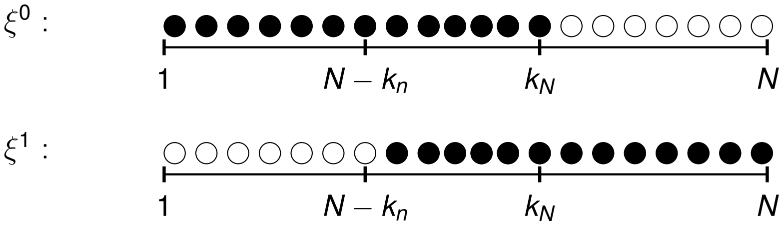
$$\limsup_{N\to\infty} d^{N,k_N}(g(k_N,c)) \leq 1 - F_{\mathrm{GUE}}(cf(\alpha)), \tag{1}$$

As is often the case, it is harder to show the upper bound (1), so we focus on (1).

Hitting times I

Consider the configurations for ASEP on the segment

$$\xi^0 = \mathbf{1}_{[1;k_N]}, \quad \xi^1 = \mathbf{1}_{[N-k_N+1;N]}.$$



$\xi^{(0)}$ is intuitivly the 'worst' initial configuration.
Let $\mathfrak{h}$ the first time that ASEP started from ξ^0 reaches ξ^1 :

$$\mathfrak{h} = \inf\{t : \xi^0_t = \xi^1\}.$$

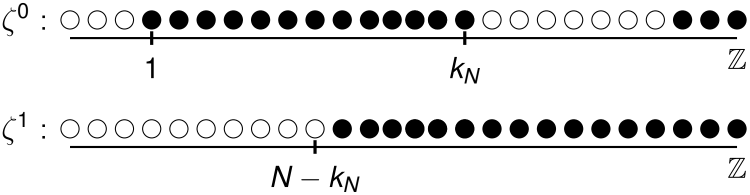
Then we have the inequality

$$d^{N,k_N}(t) \leq \mathbb{P}(\mathfrak{h} > t).$$

Hitting times II

Consider the configurations for ASEP on $\mathbb{Z}$

$$\zeta^0 = \mathbf{1}_{[1;k_N]} + \mathbf{1}_{\mathbb{Z}_{>N}}, \quad \zeta^1 = \mathbf{1}_{\mathbb{Z}_{>(N-k_N)}}.$$



Let $\mathfrak{H}$ the first time that ASEP started from ζ^0 reaches ζ^1 :

$$\mathfrak{H} = \inf\{t : \zeta^0_t = \zeta^1\}.$$

Then we have the inequalities

$$d^{N,k_N}(t) \leq \mathbb{P}(\mathfrak{h} > t) \leq \mathbb{P}(\mathfrak{H} > t)$$

Upper bound using hitting times

- The hitting time $\mathfrak{H}$ was already studied by [Benjamini et al '03].
- We will eventually show

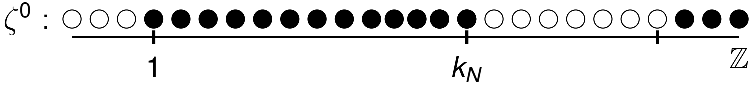
$$\lim_{N\to\infty} \mathbb{P}(\mathfrak{H} > g(k_N, c)) = 1 - F_{\mathrm{GUE}}(cf(\alpha)).$$

By the inequality $d^{N,k_N}(t) \leq \mathbb{P}(\mathfrak{H} > t)$, this will imply

$$\limsup_{N\to\infty} d^{N,k_N}(g(k_N, c)) \leq 1 - F_{\mathrm{GUE}}(cf(\alpha)).$$

Leftmost particle / rightmost hole

- Recall $\zeta^0 = \mathbf{1}_{[1;k_N]} + \mathbf{1}_{\mathbb{Z}_{>N}}$



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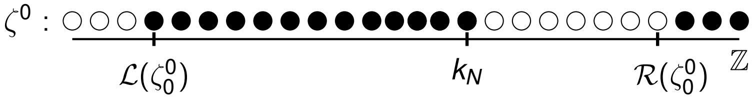


- To understand the hitting time $\mathfrak{H}$, we track the leftmost particle/rightmost hole of ζ_t^0:

$\mathcal{L}(\zeta_t^0) = \min\{i \in \mathbb{Z} : \zeta_t^0(i) = 1\} \quad \mathcal{R}(\zeta_t^0) = \max\{i \in \mathbb{Z} : \zeta_t^0(i) = 0\}.$

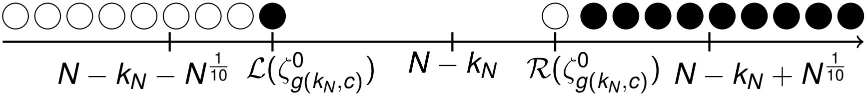
Note that deterministically for all t

$$\mathcal{L}(\zeta_t^0) - 1 \leq N - k_N \leq \mathcal{R}(\zeta_t^0)$$

and $\mathfrak{H}$ is precisely the first time that

$$\mathcal{L}(\zeta_{\mathfrak{H}}^0) - 1 = N - k_N = \mathcal{R}(\zeta_{\mathfrak{H}}^0)$$

Let $B_N(c)$ be the event that $\zeta^0_{g(k_N,c)}$ looks like this:



- when $B_N(c)$ happens, $\mathfrak{H}$ cannot be much bigger than $g(k_N, c)$, in particular we can show

$$\lim_{N\to\infty} \mathbb{P}(B_N(c)) = \lim_{N\to\infty} \mathbb{P}(\mathfrak{H} < g(k_N, c))$$

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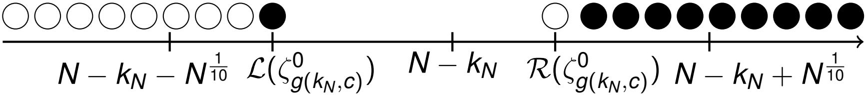

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We want to prove

$$\lim_{N\to\infty} \mathbb{P}(B_N(c)) = F_{\mathrm{GUE}}(cf(\alpha)).$$

- This will imply $\lim_{N\to\infty} \mathbb{P}(\mathfrak{H} > g(k_N, c)) = 1 - F_{\mathrm{GUE}}(cf(\alpha))$, and thus yield the upper bound for $d^{N,k_N}(g(k_N, c))$

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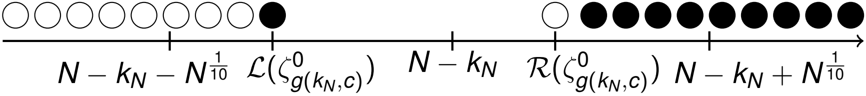


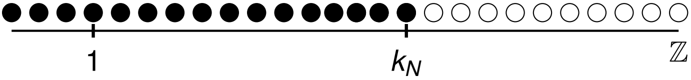
We want to prove

$$\lim_{N\to\infty} \mathbb{P}(B_N(c)) = F_{\mathrm{GUE}}(cf(\alpha)).$$

Main sources of this convergence are :

- non-standard CLT for ASEP
- algebraic identities for multi-color ASEP
- couplings to compare different ASEPs

Non-standard CLT for ASEP



We start ASEP on $\mathbb{Z}$ from $\mathbf{1}_{\mathbb{Z}_{\leq k_N}}$, ("step initial data") and denote

$$x_{k_N}(t) = \text{position at time } t \text{ of the particle that started in } 1.$$

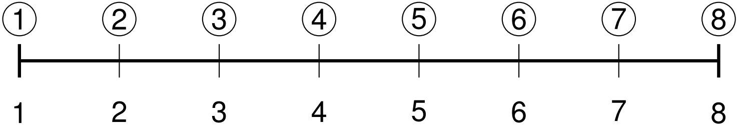
Theorem (Corollary of Johansson '00 (for p=1), Tracy-Widom '09)

We have for k_N with $k_N/N \to \alpha \in (0,1)$ that

$$\lim_{N\to\infty} \mathbb{P}\left(x_{k_N}(g(k_N,c)) \leq N - k_N\right) = 1 - F_{\mathrm{GUE}}(cf(\alpha)),$$

where $f(\alpha) = \frac{(\alpha(1-\alpha))^{1/6}}{(\sqrt{\alpha}+\sqrt{1-\alpha})^{4/3}}$.

Multi-color ASEP

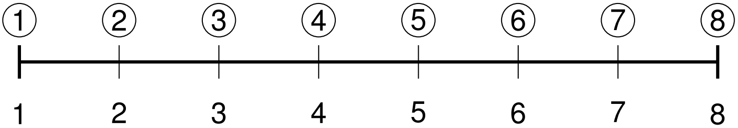


- we have N particles on $[1; N]$ with colors 1,. . . ,N
- each particle moves as in ASEP, but jumps to sites occupied by smaller color are impossible
- encoded by a permutation π mapping positions to colors
- invariant measure is the Mallows measure

$$\mathcal{M}(\pi) = \frac{(p/q)^{\#\mathrm{inv}(\pi)}}{Z_N},$$

where $\mathrm{inv}(\pi)$ are the inversions of π, for $q = 0$, $\mathcal{M}$ is the Dirac measure on $\pi(i) = N - i + 1$

Multi-color ASEP



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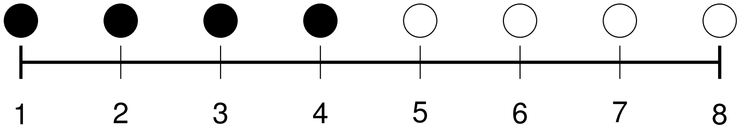
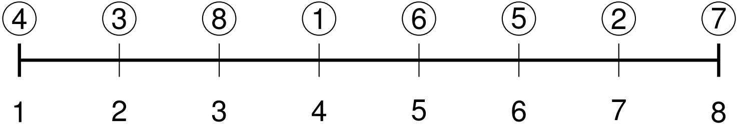


- we have N particles on $[1; N]$ with colors 1,...,N
- each particle moves as in ASEP, but jumps to sites occupied by smaller color are impossible
- encoded by a permutation π mapping positions to colors
- when we project down and only distinguish between particles of color smaller equal k and greater than k, we recover ASEP

Bringing into equilibrium

- Consider a permutation of $[a; b]$ and let $[c; d] \subseteq [a; b]$:
- Example : $[a; b] = [1; 8]$ and $[c; d] = [2; 4]$



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- more generally, it means to distribute the colors in $[c; d]$ according to the Mallows measure

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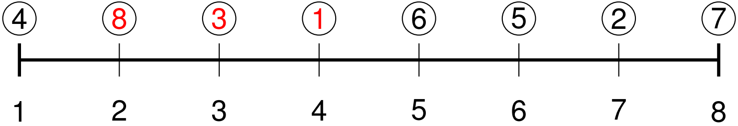

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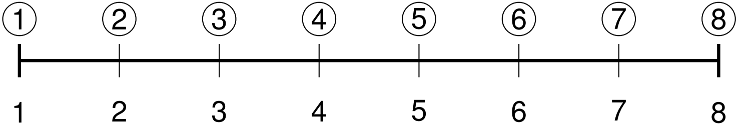
Color-position symmetry

- we start with the identity permutation



- bring into equilibrium $[c_1; d_1]$, then $[c_2; d_2]$
- here: $[c_1; d_1] = [2; 4]$ and $[c_2; d_2] = [4; 7]$

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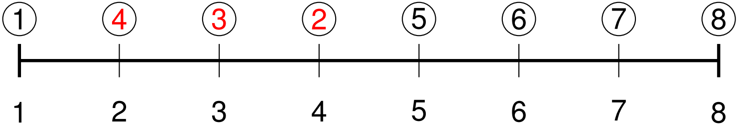

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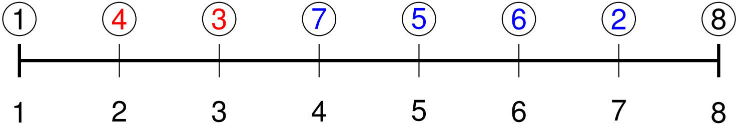

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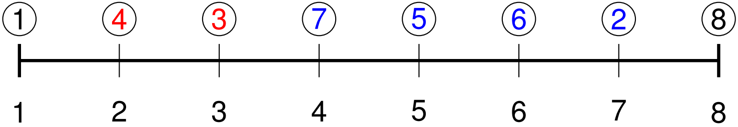

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- run the process up to time t
- this results in a permutation π_t

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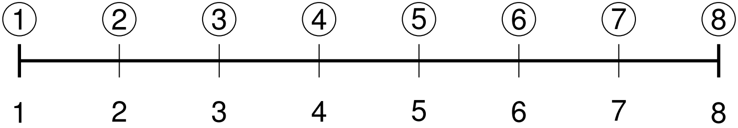

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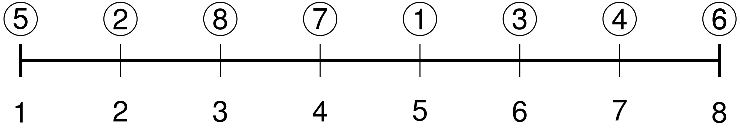

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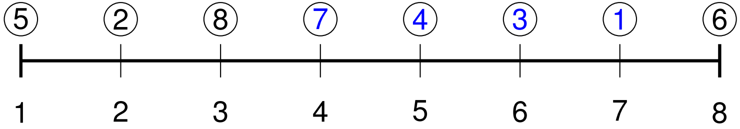

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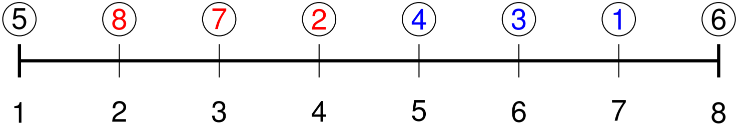

- run the process up to time t
- bring into equilibrium $[c_2; d_2] = [4; 7]$
- bring into equilibrium $[c_1; d_1] = [2; 4]$
- this results in a permutation $\hat{\pi}_t$

Color-position symmetry

We have the following unintuitive identity:

Proposition (Bufetov-N. ’22)

The permutations π_t^{-1} *and* $\hat{\pi}_t$ *are equal in law.*

- source of this identity is purely algebraic (using Hecke algebras, cf. [Bufetov ’21+], also [Borodin-Bufetov ’20])
- since π_t maps positions to colors, whereas $\hat{\pi}_t^{-1}$ maps colors to positions, this is a color-position symmetry

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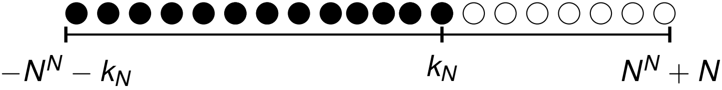
Main application:

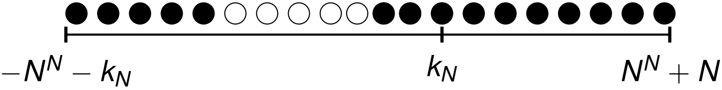
- $(\pi_t, t \geq 0)$ starts from a ’complicated’ permutation, which can be projected down to (a perturbation of) $\zeta^0 = \mathbf{1}_{[1;k_N]} + \mathbf{1}_{\mathbb{Z}_{>N}}$
- $(\hat{\pi}_t, t \geq 0)$ starts from the identity, which can be projected down to step initial data (CLT available)

The permutation π_0

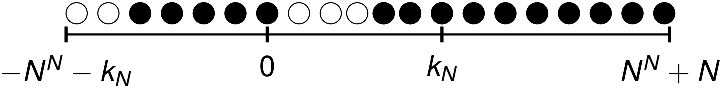
Consider ASEP on a very large segment with this step initial data:



We bring into equilibrium $[-N^N; N^N + N]$:



We bring into equilibrium $[-N^N - k_N; 0]$:



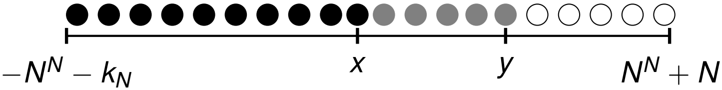
This is very close to a shift of ζ^0, we call it $\hat{\zeta}^0$

The permutation $\hat{\pi}_0$

- x,y free parameters
- in the identity, we project down on colors $\leq x$ (black), colors in $(x, y]$ (grey) and colors $> y$ (white)

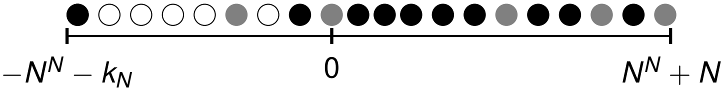
Our initial configuration thus is

$\mathfrak{D}_0$:



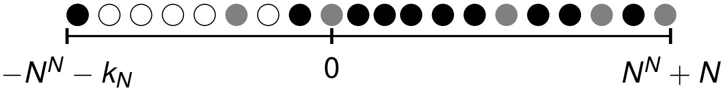
After time t, the process may look like this:

$\mathfrak{D}_t$:



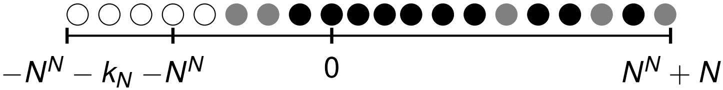
Bringing into equilibrium

$\mathfrak{D}_t$:



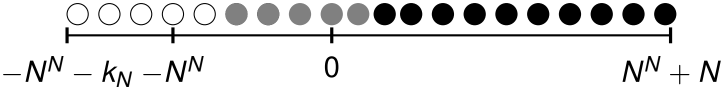
In $\mathfrak{D}_t$ we bring into equilibrium $[-N^N - k_N; 0]$:

$\tilde{\mathfrak{D}}_t$:



Afterwards, we bring into equilibrium $[-N^N; N^N + N]$:

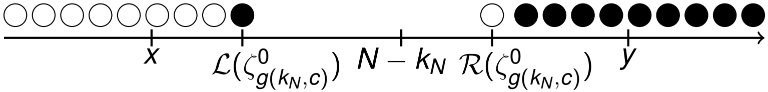
$\hat{\mathfrak{D}}_t$:



Reminders

$$x := N - k_N - N^{1/10} \quad y := N - k_N + N^{1/10}$$

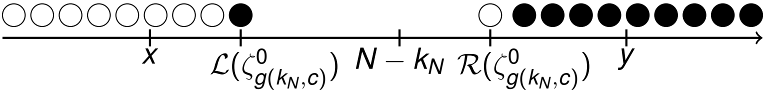
Recall $B_N(c)$ be the event that $\zeta^0_{g(k_N,c)}$ looks like this:



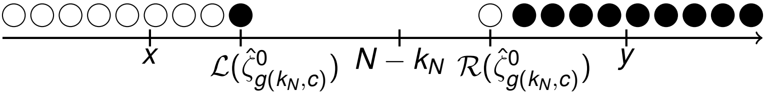
Reminders

$$x := N - k_N - N^{1/10} \quad y := N - k_N + N^{1/10}$$

Recall $B_N(c)$ be the event that $\zeta^0_{g(k_N,c)}$ looks like this:



We call $\hat{B}_N(c)$ be the event that $\hat{\zeta}^0_{g(k_N,c)}$ looks the same:



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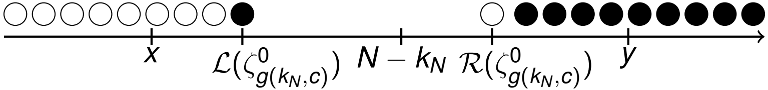

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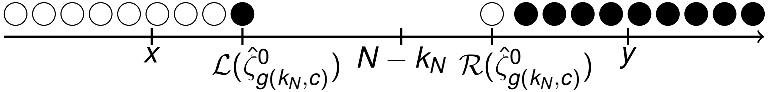
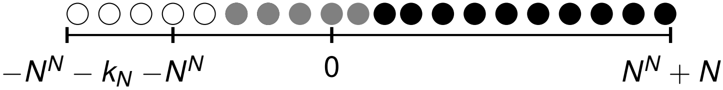


Recall $\hat{\mathfrak{D}}_t$ from the previous slide:

$\hat{\mathfrak{D}}_t$:



As corollary of the color-position symmetry, we have:

Proposition (Bufetov -N.)

We have

$$\mathbb{P}(\hat{B}_N(c)) = \mathbb{P}\Big(\textit{all black particles in } \hat{\mathfrak{D}}_t \textit{ are at positions} > 0, \\ \textit{all holes in } \hat{\mathfrak{D}}_t \textit{ are at positions} \leq 0\Big). \quad (2)$$

- the l.h.s. of (2) is very close to $\mathbb{P}(\mathfrak{H} < g(k_N, c))$
- the r.h.s. of (2) concerns two events involving only particles/holes
- using couplings, we can eventually compute the r.h.s. with the CLT for ASEP: It converges to $F_{\mathrm{GUE}}(cf(\alpha))$

Thank you for your attention !