Slow-fast dynamics and noise-induced periodic behaviors for mean-field excitable systems

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In collaboration with E. Luçon (Université de Paris)

Consider a population of N interacting units in $\mathbb{R}^d$ with dynamics

$$dX_{i,t} = \delta F(X_{i,t})dt - K\left(X_{i,t} - \frac{1}{N}\sum_{j=1}^{N} X_{j,t}\right)dt + \sqrt{2}\sigma dB_{i,t},$$

where

- $\delta \geqslant 0$, $K = \mathrm{diag}(k_1, \ldots, k_d) > 0$, $\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d) > 0$,
- $(B_i)_{i=1\ldots N}$ family of standard independent Brownian motions,
- F smooth and one-sided Lipschitz : $(F(x) - F(y)) \cdot (x - y) \leqslant C|x - y|^2$.

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On any time interval $[0, T]$, the empirical measure $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_{i,t}}$ converges weakly to the solution of

$$\partial_t \mu_t = \nabla \cdot (\sigma^2 \nabla \mu_t) + \nabla \cdot \left(\mu_t K(x - \int_{\mathbb{R}^d} z d\mu_t(z)\right) - \delta \nabla \cdot (\mu_t F).$$

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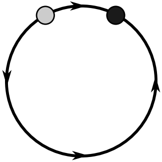
Aim

Prove that this PDE admits a periodic solution for some choices of F (in particular in cases when F defines an excitable dynamics) and δ small.

An excitable system :

- possesses a **stable rest position**.
- **threshold phenomenon** : after a sufficiently large perturbation, follows a complex trajectory before coming back to the rest state.



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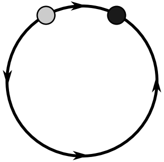

General observation :

A large population of noisy excitable systems in mean field interaction may possess a **synchronized periodic behavior**.

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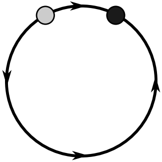

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A large population of noisy excitable systems in mean field interaction may possess a **synchronized periodic behavior**.

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Rigorous proof of this phenomenon ?

[Shinimoto, Kuramoto, 1986] Consider a population of N oscillators in $\mathcal{S} = \mathbb{R}/(2\pi\mathbb{Z})$ with dynamics

$$d\varphi_{i,t} = -\delta V'(\varphi_{i,t})dt - \frac{K}{N}\sum_{j=1}^{N}\sin(\varphi_{i,t} - \varphi_{j,t})dt + dB_{i,t}.$$

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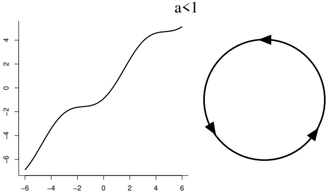
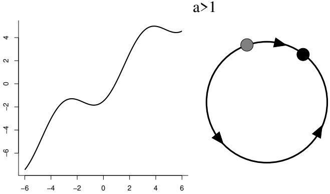
Example of potential : $V(\theta) = \theta - a\cos(\theta)$, $V'(\theta) = 1 + a\sin(\theta)$.





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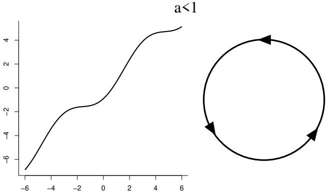

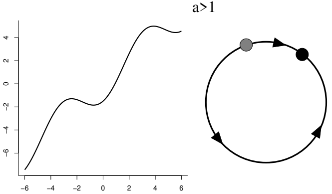

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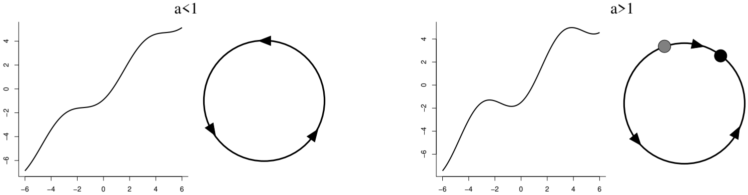

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For accurate choices of parameters (a may be larger than one) and δ small enough, this non-linear Fokker Planck PDE **admits a limit cycle**. [Giacomin, Pakdaman, Pellegrin and P., 2012]

Consider

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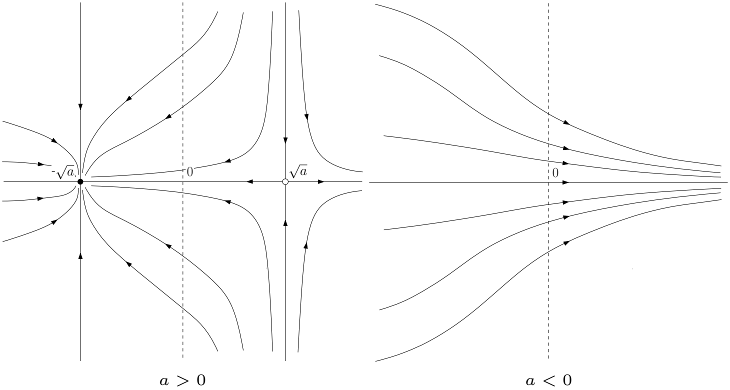
where $F(x,y) = \begin{pmatrix} x^2 - a \\ -by \end{pmatrix}$ with $a \in \mathbb{R}$, $b > 0$.



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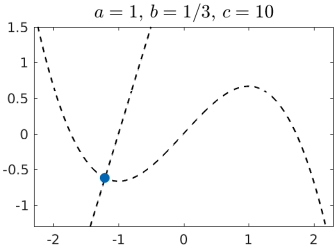

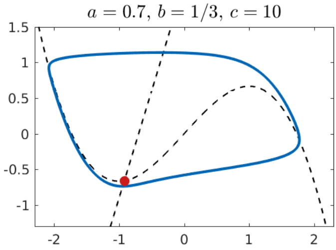

- [Scheutzow, 1985], [Touboul, Hermann, Faugeras, 2012] **noise-induced phenomena** for non-linear Fokker-Planck equations admitting **Gaussian solutions**.
- [Scheutzow, 1986] **existence of periodic solutions** for the mean-field Brusselator model (for large interaction, when each unit has a periodic behavior).
- [Giacomin, Pakdaman, Pellegrin and P., 2012] **noise-induced periodicity** for the *Active rotators* model.
- [Mischler, Quiñinao, Touboul, 2016] existence of stationary solutions for the **kinetic mean-field FitzHugh Nagumo model**, uniqueness and stability for small coupling.
- [Quiñinao, Touboul, 2018] for large coupling, the **kinetic mean-field FitzHugh Nagumo model** behaves as a single FitzHugh Nagumo unit.
- [Cormier, Tanré, Veltz, 2021] **existence of periodic solutions** for system of **integrate and fire neurons** in mean-field interaction.

Recall

$$\partial_t \mu_t = \nabla \cdot (\sigma^2 \nabla \mu_t) + \nabla \cdot \left(\mu_t K(x - \int_{\mathbb{R}^d} z d\mu_t(z) \right) - \delta \nabla \cdot (\mu_t F).$$

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Denote $m_t = \mathbb{E}[X_t] = \int x d\mu_t(x)$, and p_t the distribution of $X_t - m_t$.

(m_t, p_t) is solution of the system

$$\begin{cases} \dot{m}_t &= \delta \int F(x + m_t) dp_t(x) \\ \partial_t p_t &= \nabla \cdot (\sigma^2 \nabla p_t) + \nabla \cdot (p_t K x) + \nabla \cdot (p_t(\dot{m}_t - \delta F(x + m_t)) \end{cases},$$

which is a slow/fast system when $\delta \to 0$ with m_t the slow variable, p_t the fast one.

For $\delta = 0$ we get

$$\begin{cases} \dot{m}_t = 0 \\ \partial_t p_t = \nabla \cdot (\sigma^2 \nabla p_t) + \nabla \cdot (p_t K x) \end{cases} .$$

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In this case p_t is the distribution of the **Ornstein Uhlenbeck process**

$$dX_t = -KX_t dt + \sqrt{2}\sigma dB_t,$$

which has stationnary distribution $q \sim \mathcal{N}(0, \sigma^2 K^{-1})$, and satisfies in particular

$$\|p_t - q\|_{L^2(q^{-1})} \leqslant e^{-\min(k_1,\ldots,k_d)t} \|p_0 - q\|_{L^2(q^{-1})}$$

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Approximation for δ small :

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This corresponds to the approximation

$$\mu_t \approx \mathcal{N}(m_t, \sigma^2 K^{-1}), \quad \text{with} \quad \dot{m}_t \approx \delta F_{\sigma^2 K^{-1}}(m_t),$$

which reduces the problem to a d-dimensional dynamics.

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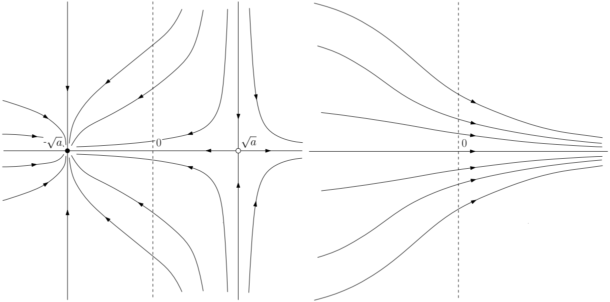

$a > 0$ $\quad$ $a < 0$

we get $F_{\sigma^2 K^{-1}}(m_x, m_y) = \left(m_x^2 - \left(a - \frac{\sigma_1^2}{k_1}\right), -b\, m_y\right).$

- For $F(v,w) = \left(v - \frac{v^3}{3} - w, \frac{1}{c}(v + a - bw)\right)$, we get

$$F_{\sigma^2 K^{-1}}(m_v, m_w) = \left(m_v \left(1 - \frac{\sigma_1^2}{k_1}\right) - \frac{m_v^3}{3} - m_w, \frac{1}{c}(m_v + a - b\, m_w)\right).$$

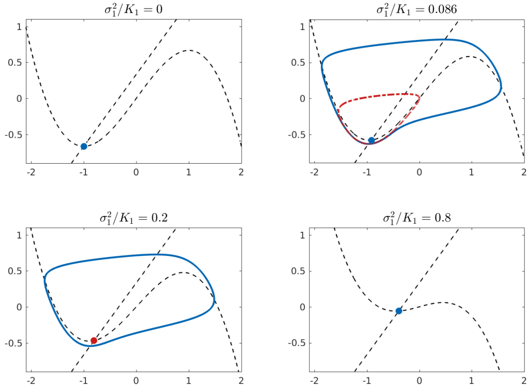

FIGURE. Dynamics of $\dot{m}_t = F_{\sigma^2 K^{-1}}(m_t)$, $a = \frac{1}{3}$, $b = 1$, $c = 10$.

Parameters : $N = 100000$, $k_1 = 1$, $k_2 = 1$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.03$, $\delta = 0.1$.

Hypotheses :

- $\max\left\{|F(x)|, \max_i |\partial_{x_i} F(x)|, \max_{i,j} |\partial^2_{x_i,x_j} F(x)|\right\} \leqslant e^{\varepsilon |x|^2}$,
- $F(x) \cdot K\sigma^{-2} x \leqslant C 1_{\{|x| \leqslant r\}} - c|x|^2$ and $\lim_{|x|\to\infty} \frac{|F(x)|}{F(x)\cdot K\sigma^{-2}x} = 0$,
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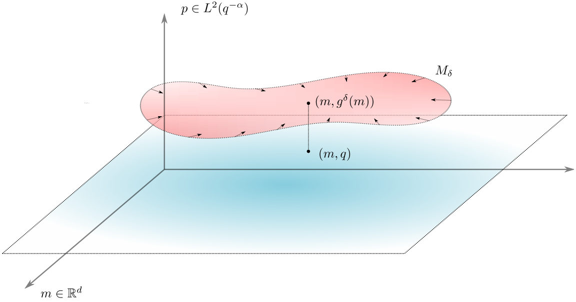
Result [Luçon, P., 2019] **:** existence of a positively invariant manifold $M_\delta = \{(m, g^\delta(m)) : m \in V\}$ in $\mathbb{R}^d \times L^2(q^{-\alpha})$.



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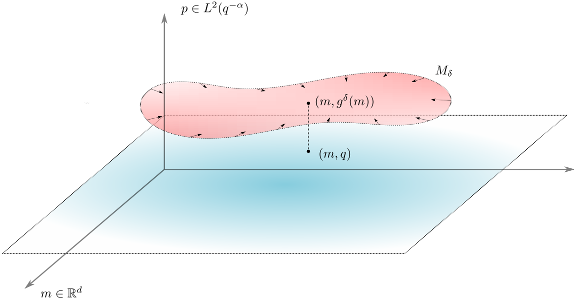

Idea : persistence of normally hyperbolic manifolds under perturbation [Fénichel, 1971], [Hirsh, Pugh, Shub, 1977], [Wiggins 1994], [Bates, Lu, Zeng, 1998], [Sell, You, 2002].

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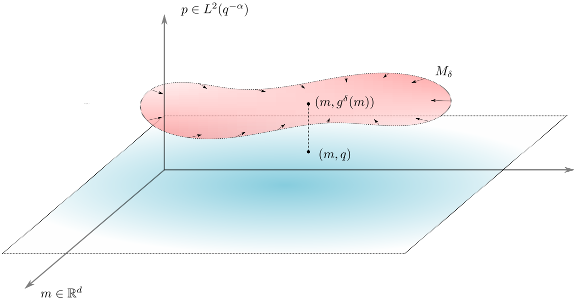

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If $p_0 = g^\delta(m_0) \in M_\delta$, then $p_t = g^\delta(m_t) \in M_\delta$ and $\dot{m}_t \approx \delta F_{\sigma^2 K^{-1}}(m_t)$.
$\rightarrow$ Existence of a periodic solution in M_δ when $\dot{m}_t = \delta F_{\sigma^2 K^{-1}}(m_t)$ has one in V.

Recall

$$dX_{i,t} = \delta F(X_{i,t})dt - K\left(X_{i,t} - \frac{1}{N}\sum_{j=1}^{N} X_{j,t}\right)dt + \sqrt{2}\sigma dB_{i,t},$$

and define $\nu_{N,t} = (m_{N,t}, p_{N,t})$ with

$$m_{N,t} = \frac{1}{N}\sum_{i=1}^{N} X_{i,t}, \quad p_{N,t} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_{i,t} - m_{N,t}}.$$

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On the time interval $[0, T]$ the process $(m_{N,t}, p_{N,t})$ converges weakly to (m_t, p_t) solution to

$$\left\{ \begin{array}{rcl} \dot{m}_t & = & \delta \int F(x + m_t)dp_t(x) \\ \partial_t p_t & = & \nabla \cdot (\sigma^2 \nabla p_t) + \nabla \cdot (p_t K x) + \nabla \cdot (p_t(\dot{m}_t - \delta F(x + m_t)) \end{array} \right. ,$$

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$$dX_{i,t} = \delta F(X_{i,t})dt - K\left(X_{i,t} - \frac{1}{N}\sum_{j=1}^{N} X_{j,t}\right)dt + \sqrt{2}\sigma dB_{i,t},$$

and define $\nu_{N,t} = (m_{N,t}, p_{N,t})$ with

$$m_{N,t} = \frac{1}{N}\sum_{i=1}^{N} X_{i,t}, \quad p_{N,t} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_{i,t}-m_{N,t}}.$$

On the time interval $[0,T]$ the process $(m_{N,t}, p_{N,t})$ converges weakly to (m_t, p_t) solution to

$$\left\{\begin{array}{lcl} \dot{m}_t & = & \delta \int F(x+m_t)dp_t(x) \\ \partial_t p_t & = & \nabla\cdot(\sigma^2\nabla p_t) + \nabla\cdot(p_t Kx) + \nabla\cdot(p_t(\dot{m}_t - \delta F(x+m_t))) \end{array}\right. ,$$

Hypotheses : that F and its derivatives bounded and $\dot{m}_t = \delta F_{K\sigma^{-2}}(m_t)$ admits a stable periodic solution.

Result : for δ small enough this limit PDE admits a stable periodic solution Γ and a C^2 isochron map Θ defined in a neighborhood of Γ [Luçon, P., 2021].

Theorem ([Luçon, P., 2021])

Suppose that, for some $\gamma > 0$*,*

$$\sup_{N \geqslant 1} \mathbb{E}\left[\left|\langle p_{N,0}, q^{\gamma}\rangle\right|\right] < \infty,$$

that for some r *taken large enough and* α *taken small enough (depending in particular on* γ*), for all* $\varepsilon > 0$

$$\mathbb{P}\left(\left\|\nu_{N,0} - \Gamma_{t_0}\right\|_{\mathbb{R}^d \times H_{q^\alpha}^{-r}} \leqslant \varepsilon\right) \underset{N\to\infty}{\longrightarrow} 1,$$

and that there exists a constant C_0 *such that*

$$\mathbb{P}\left(\left\|p_{N,0}\right\|_{H_{q^\alpha}^{-r+2}} \leqslant C_0\right) \underset{N\to\infty}{\longrightarrow} 1,$$

Then, for all $\varepsilon > 0$*,*

$$\mathbb{P}\left(\left\|\nu_{N,Nt} - \Gamma_{t_0+Nt+v_{N,t}}\right\|_{\mathbb{R}^d \times H_{q^\alpha}^{-r}} \leqslant \varepsilon\right) \underset{N\to\infty}{\longrightarrow} 1,$$

where $v_{N,0} = 0$ and $v_{N,t}$ converges weakly to $v_t = bt + aw_t$, with a and b constant depending on Γ, $D\Theta(\Gamma)$ and $D^2\Theta(\Gamma)$.

Open questions :

- other type of interaction (non linear interactions) ?
- random graphs ?

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Thank you for your attention.