The explicit expression of the distribution of the number of sides of the typical Poisson-Voronoi cell. *

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Abstract

In this paper, we give an explicit expression for the distribution of the number of sides (or equivalently vertices) of the typical cell of a two-dimensional Poisson-Voronoi tessellation. We use this formula to give a table of numerical values of the distribution function.

1 Introduction and principal result.

Consider $\Phi = \{x_n; n \geq 1\}$ a homogeneous Poisson point process in $\mathbb{R}^2$, with the two-dimensional Lebesgue measure V_2 for intensity measure. The set of cells

$$C(x) = \{y \in \mathbb{R}^2; ||y - x|| \leq ||y - x'||, x' \in \Phi\}, \quad x \in \Phi,$$

(which are almost surely bounded polygons) is the well-known *Poisson-Voronoi tessellation* of $\mathbb{R}^2$. Introduced by Meijering [12] and Gilbert [4] as a model of crystal aggregates, it provides now models for many natural phenomena such as thermal conductivity [10], telecommunications [1], astrophysics [18] and ecology [16]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [17] and Okabe et al. [15].

In order to describe the statistical properties of the tessellation, the notion of *typical cell* $\mathcal{C}$ in the Palm sense is commonly used [14]. Consider the space $\mathcal{K}$ of convex compact sets of $\mathbb{R}^2$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set $B \subset \mathbb{R}^2$ such that $0 < V_2(B) < +\infty$. Applying Campbell's theorem (cf. [14], p. 51) to the function $f(C, x) = h(C) \cdot \mathbf{1}_B(x)$, $C \in \mathcal{K}$, $x \in \mathbb{R}^2$, where $h : \mathcal{K} \longrightarrow \mathbb{R}$ is bounded and measurable, we get the following formula describing the typical cell $\mathcal{C}$:

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_2(B)}\mathbf{E}\sum_{x \in B \cap \Phi} h(C(x) - x).$$

Consider now the cell

$$C(0) = \{y \in \mathbb{R}^2; ||y|| \leq ||y - x||, x \in \Phi\}$$

obtained when the origin is added to the point process Φ. It is well known [14] that $C(0)$ and $\mathcal{C}$ are equal in law. From now on, we will use $C(0)$ as a realization of the typical cell $\mathcal{C}$.

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The explicit distributions of the main geometrical characteristics of the typical cell, as the area or the perimeter, are mainly unknown. Nevertheless, the law of the largest disk centered at the origin and contained in $C(0)$ is easy to obtain and the distribution of the radius of the smallest disk centered at the origin and containing $C(0)$ has been recently calculated [2].

Let $N_0(\mathcal{C})$ denote the number of sides (or equivalently vertices) of the typical cell $\mathcal{C}$. Miles and Maillardet [13] obtained integral formulas for $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $k \geq 3$, but did not precisely evaluate them. These probabilities have been often estimated by simulation (see for example [15], Table 5.5.13). Moreover, an explicit integral formula has been recently obtained by Hayen and Quine [7] for $\mathbf{P}\{N_0(\mathcal{C}) = 3\}$. However, it seems to be difficult to generalize their method for getting $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $k \geq 4$.

In this paper, we use a technique based on the famous formula due to Slivnyak (see for example [14]) to provide an explicit expression of the distribution of $N_0(\mathcal{C})$.

Theorem 1 *For every $k \geq 3$, we have*

$$\mathbf{P}\{N_0(\mathcal{C}) = k\} = \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \int \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) p_i dp_i, \quad (1)$$

where σ_k is the (normalized) uniform measure on the simplex

$$\{(\delta_1, \cdots, \delta_k) \in [0, 2\pi]; \sum_{i=1}^{k} \delta_i = 2\pi\}, \quad (2)$$

with

$$B = \{(p, q, r, \alpha, \beta) \in (\mathbb{R}_+)^3 \times (0, \pi)^2; p \sin(\beta) + r \sin(\alpha) \geq q \sin(\alpha + \beta)\}, \quad (3)$$

with for every $\delta \in (0, \pi)$, $p, q \geq 0$,

$$H(\delta, p, q) = \frac{1}{2\sin^2(\delta)} \left\{ (p^2 + q^2 - 2pq\cos(\delta))\frac{\delta}{2} + pq\sin(\delta) - \frac{p^2}{4}\sin(2\delta) - \frac{q^2}{4}\sin(2\delta) \right\}, \quad (4)$$

and with the conventions $p_0 = p_k$, $p_{k+1} = p_1$, and $\delta_0 = \delta_k$.

Remark 1. The method used in the proof of Theorem 1 also gives the joint distribution of the respective positions of the k lines bounding the typical cell $\mathcal{C}$ conditionally to the event $\{N_0(\mathcal{C}) = k\}$, $k \geq 3$. This provides in particular a new proof of the following result due to Zuyev [19]: conditionally to $\{N_0(\mathcal{C}) = k\}$, the area of the fundamental domain of $\mathcal{C}$ is Gamma distributed, of parameters (k,1). Besides, we can adapt the procedure in order to get the distribution of the number of sides of the Crofton cell of an isotropic Poisson line process in the plane. All these results will take place in a future paper [3].

Remark 2. We can rewrite $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $k \geq 3$, as an expectation . Let us consider a random vector $(\Delta_1, \cdots, \Delta_k)$ uniformly distributed on the simplex given in (2), and a vector $(Q_1, \cdots, Q_k)$ such that conditionally to $\{\Delta_1 = \delta_1, \cdots, \Delta_k = \delta_k\}$, $Q_1, \cdots, Q_k$ are independent and exponentially distributed, with respective parameters $f(\delta_1, \delta_k)$, $f(\delta_2, \delta_1), \cdots, f(\delta_k, \delta_{k-1})$ where

$$f(x, y) = \frac{1}{4\sin^2(x)} \left(x - \frac{\sin(2x)}{2} \right) + \frac{1}{4\sin^2(y)} \left(y - \frac{\sin(2y)}{2} \right), \quad x, y \in [0, 2\pi].$$

From (1), we deduce that for every $k \geq 3$,

$$\mathbf{P}\{N_0(\mathcal{C}) = k\} = \frac{(2\pi)^k}{k!} \mathbf{E} \left\{ \prod_{i=1}^{k} \frac{\exp\left(\frac{\cos(\Delta_i)\Delta_i - \sin(\Delta_i)}{2\sin^2(\Delta_i)} \sqrt{Q_i Q_{i+1}} \right)}{2f(\Delta_i, \Delta_{i-1})} \mathbf{1}_{E_i} \right\}, \quad (5)$$

k	3	4	5	6	7	8	9
$\mathbf{P}\{N_0(\mathcal{C}) = k\}$	0.011240	0.106838	0.25946	0.29473	0.19877	0.0897	0.0295

Table 1: Numerical values for $\mathbf{P}\{N_0(\mathcal{C}) = k\}$.

where the event E_i, $i \geq 1$, is defined by

$$E_i = \{(\sqrt{Q_{i-1}}, \sqrt{Q_i}, \sqrt{Q_{i+1}}, \Delta_{i-1}, \Delta_i) \in B\},$$

and with the conventions $Q_0 = Q_k$, $Q_{k+1} = Q_1$, and $\Delta_0 = \Delta_k$.

Numerical values for $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $3 \leq k \leq 9$, obtained by a Monte-Carlo procedure using (5) with MATLAB are listed in Table 1 (with an estimated error of 10^{-6} for $k = 3, 4$, 10^{-5} for $k = 5, 6, 7$ and 10^{-4} for $k = 8, 9$). We notice that the value for $k = 3$ is the same as in [6]. For $k = 4, 5$, they are close to the results obtained by simulation of the tessellation ([8], [11], [9], [15]) but the values for $k = 6, 7, 8$ differ significantly from the older ones.

The rest of the paper will be devoted to the proof of Theorem 1.

2 Proof of Theorem 1.

For every $x \in \mathbb{R}^2$, let us denote by $L(x)$ (respectively $\mathcal{D}(x)$) the bisecting line of the segment $[0, x]$ (respectively the half-plane containing 0 delimited by $L(x)$).

We then define for all $k \geq 3$, and $x_1, \cdots, x_k \in \mathbb{R}^2$, the domain

$$\mathcal{D}(x_1, \cdots, x_k) = \cap_{i=1}^k \mathcal{D}(x_k).$$

Besides, we consider the set of $(\mathbb{R}^2)^k$

$$A_k = \{(x_1, \cdots, x_k) \in (\mathbb{R}^2)^k; \mathcal{D}(x_1, \cdots, x_k) \text{ is a convex polygon with } k \text{ sides}\}, \tag{6}$$

and for every $(x_1, \cdots, x_k) \in A_k$, the Lebesgue measure

$$V(x_1, \cdots, x_k) = V_2\left[\cup_{x \in \mathcal{D}(x_1, \cdots, x_k)} D(x, ||x||)\right], \tag{7}$$

$D(y, r)$ being the disk centered at $y \in \mathbb{R}^2$ and of radius $r > 0$.

Proposition 1 *For every $k \geq 3$,*

$$\mathbf{P}\{N_0(\mathcal{C}) = k\} = \frac{1}{k!} \int \exp\{-V(x_1, \cdots, x_k)\} \mathbf{1}_{A_k}(x_1, \cdots, x_k) dx_1 \cdots dx_k. \tag{8}$$

Proof. Using the equality in law $\mathcal{C} \stackrel{law}{=} C(0)$,

$$\begin{aligned} \mathbf{P}\{N_0(\mathcal{C}) = k\} &= \mathbf{E}\left\{ \sum_{\{x_1, \cdots, x_k\} \subset \Phi} \mathbf{1}_{A_k}(x_1, \cdots, x_k) \mathbf{1}_{\{\mathcal{D}(x_1, \cdots, x_k) = C(0)\}} \right\} \\ &= \mathbf{E}\left\{ \sum_{\{x_1, \cdots, x_k\} \subset \Phi} \mathbf{1}_{A_k}(x_1, \cdots, x_k) \mathbf{1}_{\{L(y) \cap \mathcal{D}(x_1, \cdots, x_k) = \emptyset \ \forall y \in \Phi \setminus \{x_1, \cdots, x_k\}\}} \right\}. \end{aligned}$$

Using Slivnyak's formula [14], we obtain

$$\begin{aligned}\mathbf{P}\{N_0(\mathcal{C})=k\} &= \frac{1}{k!}\int \mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{E}\left(\mathbf{1}_{\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\}}\right)dx_1\cdots dx_k \\ &= \frac{1}{k!}\int \mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{P}\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\}dx_1\cdots dx_k. \quad (9)\end{aligned}$$

We can easily verify that for any $z\in\mathbb{R}^2$,

$$L(z)\cap\mathcal{D}(x_1,\cdots,x_k)\neq\emptyset \Longleftrightarrow z\in\cup_{x\in\mathcal{D}(x_1\cdots,x_k)}D(x,||x||),$$

From this remark and the Poissonian property of Φ, we get

$$\begin{aligned}\mathbf{P}\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\} &= \mathbf{P}\{\Phi\cap\left[\cup_{x\in\mathcal{D}(x_1,\cdots,x_k)}D(x,||x||)\right]=\emptyset\} \\ &= e^{-V(x_1,\cdots,x_k)}. \quad (10)\end{aligned}$$

Inserting the equality (10) in (9), we deduce Proposition 1.□

Our next task is to understand analytically the set A_k, $k\geq 3$, and then calculate the area $V(x_1,\cdots,x_k)$ for $(x_1,\cdots,x_k)\in A_k$. To this end, let us denote by (p_i,θ_i), $p_i\geq 0$, $\theta_i\in[0,2\pi)$, the polar coordinates of x_i, $1\leq i\leq k$. Supposing $0\leq\theta_1\leq\theta_2\leq\cdots\leq\theta_n$, we define $\delta_i=\theta_{i+1}-\theta_i$ with the convention $\theta_{k+1}=\theta_1+2\pi$. In particular, $\sum_{i=1}^k\delta_i=2\pi$.

The following lemma gives a necessary and sufficient condition for a vector to be in the set A_k.

Lemma 1 *For every vector $(x_1,\cdots,x_k)\in(\mathbb{R}^2)^k$ such that $0\leq\theta_1\leq\theta_2\leq\cdots\leq\theta_n$,*

$$(x_1,\cdots,x_k)\in A_k \Longleftrightarrow \left\{\begin{array}{l} p_{i-1}\sin(\delta_i)+p_{i+1}\sin(\delta_{i-1})\geq p_i\sin(\delta_i+\delta_{i-1}) \\ 0\leq\delta_i\leq\pi.\end{array}\right. \quad \forall 1\leq i\leq k,$$

Proof. Let us first notice that having all the angles δ_i, $1\leq i\leq k$, less than π is clearly a necessary condition to obtain a bounded convex set. From now on, we suppose this is verified.

$\mathcal{D}(x_1,\cdots,x_k)$ is then a convex polygon with k sides if and only if for every $1\leq i\leq k$, the condition (C_i) is satisfied, i.e. the intersection point of $L(x_{i-1})$ and $L(x_{i+1})$ is not between $L(x_i)$ and the parallel line $L_0(x_i)$ containing the origin (with the conventions $x_0=x_k$ and $x_{k+1}=x_1$).

Because of the invariance by rotation, one may suppose that $L(x_i)$ has the equation $(x=p_i/2)$ and $L_0(x_i)$ is the y-axis. In that case, the equations of the lines $L(x_{i+1})$ and $L(x_{i-1})$ are respectively $(\cos(\delta_i)x+\sin(\delta_i)y-p_{i+1}/2=0)$ and $(\cos(\delta_{i-1})x-\sin(\delta_{i-1})y-p_{i-1}/2=0)$. We then obtain that the first coordinate of the intersection of these two lines is

$$X_{i-1,i+1}=(p_{i+1}\sin(\delta_{i-1})+p_{i-1}\sin(\delta_i))/(2\sin(\delta_i+\delta_{i-1})).$$

Consequently, (C_i) holds if and only if

$$\frac{p_{i+1}\sin(\delta_{i-1})+p_{i-1}\sin(\delta_i)}{2\sin(\delta_i+\delta_{i-1})}\in(-\infty,0]\cup[\frac{p_i}{2},+\infty),$$

or equivalently,

$$p_{i+1}\sin(\delta_{i-1})+p_{i-1}\sin(\delta_i)\geq p_i\sin(\delta_i+\delta_{i-1}).$$

This proves Lemma 1.□

It remains to determine the area $V(x_1,\cdots,x_k)$, $(x_1,\cdots,x_k)\in A_k$, defined in (7). This is the goal of the next lemma.

Lemma 2 *For every $(x_1, \cdots, x_k) \in A_k$, such that $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$,*

$$V(x_1, \cdots, x_k) = \sum_{i=1}^{k} \frac{1}{2\sin^2(\delta_i)} \left\{ (p_i^2 + p_{i+1}^2 - 2p_i p_{i+1}\cos(\delta_i))\frac{\delta_i}{2} + p_i p_{i+1}\sin(\delta_i) - \frac{p_i^2}{4}\sin(2\delta_i) - \frac{p_{i+1}^2}{4}\sin(2\delta_i) \right\}.$$

Proof. Supposing that $\mathcal{D}(x_1, \cdots, x_k)$ is a bounded convex set with k sides, let us define by L_θ (respectively l_θ) the support line of $\mathcal{D}(x_1, \cdots, x_k)$ orthogonal to the vector $u_\theta = (\cos(\theta), \sin(\theta))$ (respectively the distance between the origin and L_θ).

The boundary of $\cup_{x \in \mathcal{D}(x_1, \cdots, x_k)} D(x, ||x||)$ is constituted by portions of circles spanned by the points 0, x_i and x_{i+1}, $1 \leq i \leq n$ (with the convention $x_{n+1} = x_1$). We then notice that l_θ is the half length of the intersection line starting from the origin to this boundary in the direction of u_θ, $0 \leq \theta \leq 2\pi$. In other words, we have the following equivalence (see [5] for more details):

$$z \in \left[\cup_{x \in \mathcal{D}(x_1, \cdots, x_k)} D(x, ||x||)\right] \cap (\mathbb{R}_+ \cdot u_\theta) \Longleftrightarrow ||z|| \leq 2l_\theta.$$

Integrating in polar coordinates, we get that

$$\begin{aligned} V(x_1, \cdots, x_k) &= \int_0^{2\pi} \left(\int_0^{2l_\theta} r dr \right) d\theta \\ &= 2\int_0^{2\pi} {l_\theta}^2 d\theta. \end{aligned} \tag{11}$$

It remains to determine the distance l_θ. Denoting by $s_1, \cdots, s_k$ the consecutive vertices of the polygon $\mathcal{D}(x_1, \cdots, x_k)$, i.e. $s_i = L(x_{i+1}) \cap L(x_i)$, we have that

$$l_\theta = s_i \cdot u_\theta \quad \forall \theta \in (\theta_i, \theta_{i+1}), \tag{12}$$

where $\cdot$ denotes the usual scalar product of $\mathbb{R}^2$. Besides, by elementar geometric arguments,

$$s_i \cdot \frac{x_i}{||x_i||} = \frac{p_i}{2} \text{ and } s_i \cdot \frac{x_{i+1}}{||x_{i+1}||} = \frac{p_{i+1}}{2}. \tag{13}$$

Inserting the equalities (13) in (12), we obtain that for every $\theta \in (\theta_i, \theta_{i+1})$,

$$l_\theta = \frac{p_i}{2}\cos(\theta - \theta_i) + \left(\frac{1}{\sin(\delta_i)}\frac{p_{i+1}}{2} - \frac{\cos(\delta_i)}{\sin(\delta_i)}\frac{p_i}{2} \right) \sin(\theta - \theta_i) \tag{14}$$

Consequently, we combine the equality (11) with (14) to deduce

$$\begin{aligned} V(x_1, \cdots, x_k) &= 2\sum_{i=1}^{k} \int_0^{\delta_i} l_{\theta+\theta_i}^2 d\theta \\ &= 2\sum_{i=1}^{k} \int_0^{\delta_i} \left\{ \frac{p_i^2}{4}\cos^2(\theta) + \left(\frac{1}{\sin(\delta_i)}\frac{p_{i+1}}{2} - \frac{\cos(\delta_i)}{\sin(\delta_i)}\frac{p_i}{2} \right)^2 \sin^2(\theta) \right. \\ &\qquad \left. + \frac{p_i}{2}\left(\frac{1}{\sin(\delta_i)}\frac{p_{i+1}}{2} - \frac{\cos(\delta_i)}{\sin(\delta_i)}\frac{p_i}{2} \right)\sin(2\theta) \right\} d\theta \\ &= \sum_{i=1}^{k} \frac{1}{2\sin^2(\delta_i)} \left\{ (p_i^2 + p_{i+1}^2 - 2p_i p_{i+1}\cos(\delta_i))\frac{\delta_i}{2} + p_i p_{i+1}\sin(\delta_i) \right. \\ &\qquad \left. - \frac{p_i^2}{4}\sin(2\delta_i) - \frac{p_{i+1}^2}{4}\sin(2\delta_i) \right\}, \end{aligned}$$

which is the required result of Lemma 2.□

Proof of Theorem 1. Using polar coordinates in the integral of the equality (8) of Proposition 1, we obtain for every $k \geq 3$,

$$\begin{aligned}
\mathbf{P}\{N_0(\mathcal{C}) = k\} &= \frac{1}{k!} \int e^{-V(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k})} \mathbf{1}_{A_k}(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k}) \prod_{i=1}^{k} \mathbf{1}_{\{p_i \geq 0\}} \mathbf{1}_{\{0 \leq \theta_i \leq 2\pi\}} p_i dp_i d\theta_i \\
&= \int e^{-V(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k})} \mathbf{1}_{A_k}(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k}) \\
&\qquad \mathbf{1}_{\{0 \leq \theta_1 \leq \cdots \leq \theta_k \leq 2\pi\}} \prod_{i=1}^{k} \mathbf{1}_{\{p_i \geq 0\}} p_i dp_i d\theta_i. \quad (15)
\end{aligned}$$

Inserting then the results of Lemmas 1 and 2 in (15), and recalling the identity $\sum_{i=1}^{k} \delta_i = 2\pi$, we deduce that

$$\begin{aligned}
\mathbf{P}\{N_0(\mathcal{C}) = k\} &= \int \left[\int \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) p_i dp_i \right] \\
&\qquad \mathbf{1}_{\{\delta_1 + \cdots + \delta_{k-1} \leq 2\pi\}} \left[\int_0^{2\pi - \delta_1 - \cdots - \delta_{k-1}} d\theta_1 \right] d\delta_1 \cdots d\delta_{k-1} \\
&= \int \left[\int \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) p_i dp_i \right] \\
&\qquad \mathbf{1}_{\{\delta_1 + \cdots + \delta_{k-1} \leq 2\pi\}} \delta_k d\delta_1 \cdots d\delta_{k-1} \\
&= \frac{1}{k} \int \left[\int \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) p_i dp_i \right] \\
&\qquad \mathbf{1}_{\{\delta_1 + \cdots + \delta_{k-1} \leq 2\pi\}} (\delta_1 + \cdots + \delta_k) d\delta_1 \cdots d\delta_{k-1} \\
&= \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \int \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) p_i dp_i,
\end{aligned}$$

where the function H (resp. the set B) is defined by the equality (4) (resp. (3)).

This completes the proof of Theorem 1.□

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