Poissonian tessellations of the Euclidean space. An extension of a result of R. E. Miles. *

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Abstract

In 1973, R. E. Miles obtained an explicit characterization of the typical cell of the planar Poissonian tessellation, by means of the distributions of the indisc and the triangle circumscribed to the cell. In this paper, we propose a different proof, using the classical formula of Slivnyak for Poisson point processes. Not only the method is simple and rigorous, but it also extends the result of Miles to any dimension $d \geq 2$. We deduce from it some other properties of the geometrical characteristics of the typical cell.

Introduction.

Let Φ be a Poisson point process in $\mathbb{R}^d$, $d \geq 2$, of intensity measure

$$\mu(A) = \int_0^{+\infty} \int_{\mathbb{S}^{d-1}} \mathbf{1}_A(r, u) d\nu_d(u) dr, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where ν_d is the area measure of the unit-sphere $\mathbb{S}^{d-1}$.

Let us consider for all $x \in \mathbb{R}^d$, $H(x) = \{y \in \mathbb{R}^d; (y - x) \cdot x = 0\}$, ($x \cdot y$ being the usual scalar product). Then the set $\mathcal{H} = \{H(x); x \in \Phi\}$ divides the space into convex polyhedra that constitute the so-called *d-dimensional Poissonian tessellation*. This tessellation is isotropic, i.e. invariant in law by any isometric transformation of the Euclidean space.

This random object was used for the first time by S. A. Goudsmit [7] and by R. E. Miles ([10], [11] and [13]). In particular, it provides a model for the fibrous structure of sheets of paper.

Miles introduced in particular the notion of *empirical* (or *typical*) cell associated to the tessellation. Recent contributions about the law of the area of the typical cell and the famous D. G. Kendall conjecture were provided by A. Goldman [5] and I. N. Kovalenko [8]. The fundamental frequencies of the cells have been studied by A. Goldman [4] and have been useful to obtain informations about the frequencies of the cells of the Poisson-Voronoi tessellation [6]. Central limit theorems have been provided by K. Paroux [17] in this context. Besides, we have obtained the explicit distribution of the radius of the smallest disk centered at the origin containing the Crofton cell in the plane [2].

Miles obtained [13] in the two-dimensional case the explicit distributions of the indisk radius and the circumscribed triangle of the typical cell, which has provided numerous results concerning the area, the perimeter and the number of vertices. His method (see [13]) essentially relies on the concept of convex circuit in $\mathbb{R}^2$ and the ergodic properties of the tessellation (see also [3]); Nevertheless, it can not be generalized to any dimension.

In this work, we propose a new approach which is easier and can be extended to any dimension $d \geq 2$. It consists to use famous Slivnyak's formula which is a fundamental tool for the study of Poisson-Voronoi tessellations [15]. Nevertheless, to use this formula in our context, we need to describe the typical cell of R. E. Miles with a Palm procedure [16]. In the first section, we actually obtain such a description via the (stationary) point process of the centers of inballs of the cells constituting the tessellation. We then prove in the second section the principal result concerning the construction of the typical cell via the law of the inball radius and the circumscribed simplex. In the last section, we show how to obtain some new informations about the geometric characteristics of the typical cell by using Slivnyak's formula.

All the results presented here were announced in a previous note [1].

1 Preliminaries.

Let Φ be a Poisson point process in $\mathbb{R}^d$ of intensity measure

$$\mu(A) = \mathbf{E}\sum_{x\in\Phi}\mathbf{1}_A(x).$$

The process Φ is a measurable application which takes values in the space $\mathcal{M}_\sigma$ of the locally finite sets of $\mathbb{R}^d$ endowed with the cylindric σ-field $\mathcal{T}_c$ generated by the applications

$$\varphi_A : \begin{cases} \mathcal{M}_\sigma & \longrightarrow \quad \mathbb{N}\cup\{+\infty\} \\ \gamma & \longmapsto \quad \#(A\cap\gamma) \end{cases}, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

For any positive measurable function f on the product space $(\mathbb{R}^d)^n \times \mathcal{M}_\sigma$, $n \in \mathbb{N}^*$, which is invariant by permutation of the first n coordinates, we have Slivnyak's formula (see for example [16]):

$$\mathbf{E}\sum_{\xi\in\overline{\Phi}^{(n)}} f(\xi,\Phi) = \frac{1}{n!}\int \mathbf{E}f\left(\xi,\Phi\cup\xi\right) d\overline{\mu}^{(n)}(\xi), \tag{1}$$

where $\overline{\Phi}^{(n)}$ denotes the space of sets of Φ with cardinal n, and where

$$d\overline{\mu}^{(n)}(\xi) = d\mu(\xi_1)\cdots d\mu(\xi_n), \quad \xi = \{\xi_1,\cdots\xi_n\} \in \overline{\Phi}^{(n)}.$$

Let us now take the measure

$$\mu(A) = \int_0^{+\infty}\int_{\mathbb{S}^{d-1}} \mathbf{1}_A(r,u)d\nu_d(u), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{2}$$

and consider the associated *Poissonian tessellation* [12]. We will note C_0 the cell of the tessellation containing the origin. We show [4] that the cell C_0 (called the *Crofton cell*) is a.s. well-defined. Let us denote by $\mathcal{C}_R$ the set of cells included in the open ball $B(R)$ centered at the origin, with radius $R > 0$ and $N_R = \#\mathcal{C}_R$. Besides, we consider the space $\mathcal{K}$ of the convex compact sets of $\mathbb{R}^d$ endowed with the usual Hausdorff topology and $h : \mathcal{K} \longrightarrow \mathbb{R}$ a translation-invariant, bounded measurable function.

We have the following result [4]:

Theorem 1 *The means*

$$\frac{1}{N_R}\sum_{C\in\mathcal{C}_R} h(C), \quad R > 0, \tag{3}$$

converge a.s. to a constant $\widetilde{\mathbf{E}}h$ *(the empirical mean) satisfying*

$$\widetilde{\mathbf{E}}h = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)}\mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right),$$

where V_d *denotes the d-dimensional Lebesgue measure.*

Let us define Ψ as the point process constituted with the centers of the inballs of the cells of the tessellation. The invariance by any translation of the Poissonian tessellation [4] implies that Ψ is a stationary (locally finite) point process. We fix a Borel set $B \subset \mathbb{R}^d$ verifying $0 < V_d(B) < +\infty$. The typical cell $\mathcal{C}$, in the Palm sense, then is defined by the following formula:

$$\mathbf{E}h(\mathcal{C}) = \frac{2^d}{\omega_d {\omega_{d-1}}^d} \frac{1}{V_d(B)} \mathbf{E} \sum_{z \in \Psi \cap B} h(C(z) - z), \tag{4}$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the set of bounded measurable functions and where ω_d is the Lebesgue measure of the unit-ball of $\mathbb{R}^d$ (the constant $\omega_d {\omega_{d-1}}^d / 2^d$ is the intensity of the process Ψ).

More generally, the definition of $\mathcal{C}$ implies that the following formula, called Campbell's formula, is satisfied:

$$\int \mathbf{E}f(\mathcal{C}, y)dy = \frac{2^d}{\omega_d {\omega_{d-1}}^d} \mathbf{E} \sum_{z \in \Psi} f(C(z) - z, z), \tag{5}$$

for all measurable function $f : \mathcal{K} \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$.
Repeating word for word the method of J. Møller described in [16], page 66 (applying to any stationary point process), we show the identity

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)} \mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right). \tag{6}$$

Comparing (6) with Theorem 1, we deduce that:

Theorem 2 *For all translation-invariant, bounded measurable function h, we have*

$$\widetilde{\mathbf{E}}h = \mathbf{E}h(\mathcal{C}).$$

2 The principal result.

For all $z \in \Psi$, we note $B(z)$ the inball of the cell $C(z)$ and $S(z)$ the simplex constructed with the support hyperplanes of $C(z)$ which are tangent to $B(z)$. Besides, let us consider

$$\Phi_z = \{x - \frac{x}{||x||^2}(x \cdot z); x \in \Phi, H(x) \cap B(z) = \emptyset\},$$

the point process associated to the hyperplanes which do not intersect $B(z)$ (taking the point z as new origin).
Then, denoting by $C_0(\Phi_z)$ the Crofton cell associated to the hyperplanes $H(y)$, $y \in \Phi_z$, we have easily

$$C(z) = S(z) \cap (z + C_0(\Phi_z)). \tag{7}$$

Let us consider a random couple $(\mathcal{S}, \widehat{\Phi})$, where $\mathcal{S}$ is a (random) simplex and $\widehat{\Phi}$ is a point process such that the joint distribution is given by the formula (analogous to (4)):

$$\mathbf{E}\{h(\mathcal{S})\mathbf{1}_{\{\widehat{\Phi} \cap A = \emptyset\}}\} = \frac{2^d}{\omega_d {\omega_{d-1}}^d V_d(B)} \mathbf{E} \sum_{z \in \Psi \cap B} h(S(z) - z)\mathbf{1}_{\{\Phi_z \cap A = \emptyset\}}, \tag{8}$$

satisfied for every positive measurable function $h : \mathcal{K} \longrightarrow \mathbb{R}_+$, and every fixed Borel set $A \in \mathbb{R}^d$;$B \subset \mathbb{R}^d$ such that $0 < V_d(B) < +\infty$.

Applying (8) to a function $g(\mathcal{S},\widehat{\Phi}) = h(\mathcal{S}\cap C_0(\widehat{\Phi}))$, where h is translation-invariant, we obtain thanks to (7), the equality

$$\mathbf{E}h(\mathcal{S}\cap C_0(\widehat{\Phi})) = \mathbf{E}h(\mathcal{C}),$$

which means

$$\mathcal{C} \stackrel{law}{=} \mathcal{S}\cap C_0(\widehat{\Phi}). \tag{9}$$

It then remains to determine the distribution of the couple $(\mathcal{S},\widehat{\Phi})$.
To this end, let us consider the set $\mathcal{A}$ of all the subsets $\xi\subset\mathbb{R}^d$, $\#\xi = d+1$, such that the associated hyperplanes $H(x)$, $x\in\xi$, form a $(d+1)$-simplex of $\mathbb{R}^d$. For all $\xi\in\mathcal{A}$, let us denote by $S(\xi)$ (resp. $B(\xi)$ and $c(\xi)$) the simplex associated to ξ (resp. the inball of $\mathcal{S}(\xi)$, and the center of this ball). Besides, we define

$$\mathcal{A}_\Phi = \{\xi\in\overline{\Phi}^{(d+1)}\cap\mathcal{A}; B(\xi)\cap H(x) = \emptyset \ \ \forall\, x\in\Phi\}.$$

The set $\mathcal{A}_\Phi$ is in one-to-one correspondence with the set of the points of Ψ via the application $\xi\longmapsto c(\xi)$; it is also in one-to-one correspondence with the set $\{S(z); z\in\Psi\}$ via the application S. Using (8), it implies that

$$\mathbf{E}\{h(\mathcal{S})\mathbf{1}_{\{\widehat{\Phi}\cap A=\emptyset\}}\} = \frac{2^d}{\omega_d{\omega_{d-1}}^d V_d(B)}\mathbf{E}\sum_{\xi\in\overline{\Phi}^{(d+1)}} h(S(\xi)-c(\xi))\mathbf{1}_B(c(\xi))\mathbf{1}_{\mathcal{A}_\Phi}(\xi)\mathbf{1}_{\{\Phi_{c(\xi)}\cap A=\emptyset\}}.$$

Applying now Slivnyak's formula (1), we get from the preceding equality

$$\mathbf{E}\{h(\mathcal{S})\mathbf{1}_{\{\widehat{\Phi}\cap A=\emptyset\}}\} = \frac{2^d}{\omega_d{\omega_{d-1}}^d V_d(B)}\frac{1}{(d+1)!}\int h(S(\xi)-c(\xi))\mathbf{1}_B(c(\xi))\mathbf{1}_{\mathcal{A}}(\xi)$$
$$\mathbf{P}\{H(x)\cap B(\xi)=\emptyset,\ \forall x\in\Phi; \Phi_{c(\xi)}\cap A=\emptyset\}d\overline{\mu}^{(d+1)}(\xi) \tag{10}$$

It remains to explicit this integral via appropriated change of variables, and we deduce the principal result:

Theorem 3 **(1)** *Let us consider a ball of center zero and (random) radius R_I of law*

$$\mathbf{P}\{R_I\ge t\} = \exp\{-\sigma_d t\},\quad t\ge 0,$$

where $\sigma_d = \nu_d(\mathbb{S}^{d-1})$.
(2) *Let us construct a simplex $\mathcal{S}$ circumscribed to this ball such that the $(d+1)$ directions $U_0,\cdots,U_d\in\mathbb{S}^{d-1}$ from the center to the intersecting points are independent from R_I and have a joint distribution given by:*

$$[(U_0,\cdots,U_d)](\mathbf{P})(u) = \frac{d2^d}{(d+1){\sigma_d}^2{\omega_{d-1}}^d}\Delta(U_0,\cdots,U_d)\mathbf{1}_A(u)d\overline{\nu_d}^{(d+1)}(u), \tag{11}$$

where $d\overline{\nu_d}^{(d+1)}(u) = d\nu_d(u_0)...d\nu_d(u_d)$, $u=(u_0,...,u_d)\in(\mathbb{S}^{d-1})^{d+1}$,

$$A = \{(u_0,\cdots,u_d)\in(\mathbb{S}^{d-1})^{d+1}; \text{no half-sphere contains } u_0,\cdots,u_d\}, \tag{12}$$

and $\Delta(x_0,...,x_d)$, $x_0,...,x_d\in\mathbb{R}^d$, denotes the d-dimensional Lebesgue measure of the simplex with vertices at $x_0,\cdots,x_d$.
(3) *Let us take a point process $\widehat{\Phi}$ independent from $U_0,\cdots,U_d$ such that conditionally to $R_I=r$, $r>0$, $\widehat{\Phi}$ is distributed as a Poisson point process of intensity measure $\mathbf{1}_{B(r)^c}d\mu$.*
Then we have the following equality in law:

$$\mathcal{C}\stackrel{law}{=}\mathcal{S}\cap C_0(\widehat{\Phi}).$$

Proof. The following lemma provides a formula of change of variables of Blaschke-Petkantschin type (see for example [14]):

Lemma 1 *We have:*

$$\mathbf{1}_{\mathcal{A}}(\xi)d\overline{\mu}^{(d+1)}(\xi) = d!\Delta(u)\mathbf{1}_A(u)dzdRd\overline{\nu_d}^{(d+1)}(u).$$

Proof of Lemma 1. We first remark that $\mathcal{A}$ is in bijection with $\mathbb{R}^d \times \mathbb{R}_+^* \times A$ (where A is defined in (12)) by associating to each $\xi = \{\xi_0, \cdots, \xi_d\} \in \mathcal{A}$, $z = c(\xi)$, the radius R of $B(\xi)$ and the unit-directions $u_0, \cdots, u_d$ from $c(\xi)$ to $\xi_0, \cdots, \xi_d$ respectively.

Let us define for all $0 \le i \le d$, $p_i = ||\xi_i||$ and $v_i = \xi_i/p_i \in \mathbb{S}^{d-1}$ such that

$$d\mu(\xi_i) = \mathbf{1}_{p_i \ge 0} dp_i d\nu_d(v_i).$$

It is easy to prove that

$$p_i = |R + (z \cdot u_i)|, \quad v_i = \begin{cases} u_i & \text{if } ||z|| \le p_i \\ -u_i & \text{else} \end{cases}$$

So the jacobian of the one-to-one correspondence

$$(z, R, u_0, \cdots, u_d) \longmapsto (p_0, \cdots, p_d, v_0, \cdots, v_d)$$

is

$$J = \begin{vmatrix} A & B \\ C & D \end{vmatrix},$$

where $C = 0$, $D = I_{d(d+1)}$ is the unit-matrix,

$$A = \begin{pmatrix} {u_0}^t & 1 \\ \vdots & \vdots \\ {u_d}^t & 1 \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} z^t & 0 & \dots & 0 \\ 0 & z^t & \ddots & \vdots \\ \vdots & & & \\ & \ddots & \ddots & 0 \\ 0 & \dots & 0 & z^t \end{pmatrix}.$$

Consequently, $J = \det A = d!\Delta(u_0, \cdots, u_d)$.□

Let us go back to the proof of Theorem 3. Applying the preceding change of variables to the calculation of the integral in the equality (10), we obtain

$$\mathbf{E}\{h(\mathcal{S})\mathbf{1}_{\{\widehat{\Phi} \cap A = \emptyset\}}\} = \frac{2^d}{\omega_d {\omega_{d-1}}^d V_d(B)(d+1)} \int \mathcal{I}(u)\Delta(u)\mathbf{1}_A(u)d\overline{\nu_d}^{(d+1)}(u), \tag{13}$$

where

$$\mathcal{I}(u) = \int_0^{+\infty} h(S(Ru_0, \cdots, Ru_d))$$
$$\int_B \mathbf{P}\left\{H(x) \cap (z + B(R)) = \emptyset; \left(x - \frac{x}{||x||^2}(x \cdot z)\right) \notin A \; \forall x \in \Phi\right\} dzdR.$$

Let us notice that

$$\left\{(H(x)-z)\cap B(R)=\emptyset;\left(x-\frac{x}{||x||^2}(x\cdot z)\right)\not\in A\ \forall x\in\Phi\right\}$$
$$=\left\{H\left(x-\frac{x}{||x||^2}(x\cdot z)\right)\cap B(R)=\emptyset;\left(x-\frac{x}{||x||^2}(x\cdot z)\right)\not\in A\ \forall x\in\Phi\right\}. \quad (14)$$

Besides, the set $\left\{x-\frac{x}{||x||^2}(x\cdot z); x\in\Phi\right\}$ is distributed as Φ (see for example [5]). Consequently, we deduce from (14) and the Poissonian property of Φ that for all $z\in\mathbb{R}^d$,

$$\mathbf{P}\left\{(H(x)-z)\cap B(R)=\emptyset;\left(x-\frac{x}{||x||^2}(x\cdot z)\right)\not\in A\ \forall x\in\Phi\right\}$$
$$=\mathbf{P}\{\Phi\cap(B(R)\cup A)=\emptyset\}=\exp\{-\mu(A\setminus B(R))\}\exp\{-\sigma_d R\}.(15)$$

Inserting (15) in the equation (13), we get that:
- the inball radius of $\mathcal{S}$ is exponentially distributed with parameter σ_d.
- the directions from the origin to the points of tangency of $\mathcal{S}$ with its inball have a joint distribution given by (11) and are independent from the inball radius.
- Conditionally to the fact that the inball radius of $\mathcal{S}$ is equal to r, $r>0$, $\widehat{\Phi}$ is distributed as a Poisson point process of intensity measure $\mathbf{1}_{B(r)^c}d\mu$.

So the property (9) provides us the required construction of $\mathcal{C}$.□

3 Some consequences of the method concerning geometric characteristics of the Poissonian tessellation.

In this section, we show how to use the same method to obtain new (or not) informations about the typical cell.

As an example, let us denote by $N_k(\mathcal{C})$ (resp. $V_k(\mathcal{C})$) the number of k-dimensional faces (resp. the k-dimensional Hausdorff measure) of the typical cell $\mathcal{C}$, $0\le k\le d$.

Theorem 4 *We have:*
(1) $\mathbf{E}V_k(\mathcal{C})=\frac{2^d\binom{d}{k}}{{\omega_{d-1}}^k\omega_k}$,
(2) $\mathbf{E}N_k(\mathcal{C})=2^{d-k}\binom{d}{k}$
(3) $\mathbf{P}\{N_{d-1}(\mathcal{C})=d+1\}=\frac{2^{d+1}}{d(d+1)\omega_d^2{\omega_{d-1}}^d}\int\frac{\Delta(u)}{b(u)}\mathbf{1}_A(u)d\overline{\nu_d}^{d+1}(u)$,
where $b(u)$, $u\in A$, denotes the mean width of the simplex which admits u as the set of its contact points with its inball.

Proof. **(1)** Let us first recall a well-known result due to R. E. Miles [12]:

Lemma 2 (Miles, 1969) *The intersection of a Poisson hyperplane process of $\mathbb{R}^d$ of intensity measure given by (2) with a affine sub-space of $\mathbb{R}^d$ of dimension k, $0\le k\le d$, is equal in law to a Poisson hyperplane process of $\mathbb{R}^k$ of intensity measure $\mu_{k,d}$ given by:*

$$\mu_{k,d}(A)=\frac{\omega_{d-1}}{\omega_{k-1}}\int_0^{+\infty}\int_{\mathbb{S}^{k-1}}\mathbf{1}_A(r,u)d\nu_k(u)dr,\quad A\in\mathcal{B}(\mathbb{R}^k). \quad (16)$$

Let us show the following intermediary lemma:

Lemma 3 *Let us consider for all $0 \leq k \leq d$, the measure Λ_k defined by*

$$\Lambda_k(B) = \mathbf{E} \sum_{F \in \mathcal{F}_k} V_k(B \cap F), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{F}_k$ denotes the set of all k-dimensional faces of the tessellation and V_k is the k-dimensional Hausdorff measure.
Then the equality

$$\Lambda_k = c_{k,d} \cdot V_d \tag{17}$$

is satisfied, $c_{k,d}$ being the mean k-dimensional measure of the tessellation per unit of volume such that

$$c_{k,d} = \binom{d}{k} \frac{\omega_d {\omega_{d-1}}^{d-k}}{\omega_k 2^{d-k}}. \tag{18}$$

Proof of Lemma 3. The result (17) comes from the fact that the measure Λ_k is invariant by any translation in $\mathbb{R}^d$, so is proportional to the Lebesgue measure of $\mathbb{R}^d$.

Let us show the result (18) by a reasoning of induction: it is clearly verified for $k = d$.
Let $0 \leq k \leq d-1$. Taking B equal to the unit-ball of $\mathbb{R}^d$, we have

$$\begin{aligned} \Lambda_k(B) &= c_{k,d}\omega_d \\ &= \mathbf{E} \sum_{\{x_1,\cdots,x_{d-k}\} \in \Phi} V_k(B \cap H(x_1) \cap \cdots \cap H(x_{d-k})). \end{aligned}$$

Applying Slivnyak's formula (1), we obtain

$$\begin{aligned} \Lambda_k(B) &= \frac{1}{(d-k)!} \int V_k(B \cap H(x_1) \cap \cdots \cap H(x_{d-k})) d\overline{\mu}^{(d-k)}(x) \qquad (19) \\ &= \frac{1}{(d-k)!} \int \int V_k\{[B \cap H(x_1) \cap \cdots \cap H(x_{d-1-k})] \cap H(x_{d-k})\} d\mu(x_{d-k}) d\overline{\mu}^{(d-1-k)}(x). \end{aligned}$$

Applying Lemma 2 to the section of the tessellation with the $(k+1)$-dimensional space $H(x_1) \cap \cdots H(x_{d-k-1})$, we obtain:

$$\Lambda_k(B) = \frac{1}{(d-k)} \omega_d c_{k+1,d} \frac{\omega_{d-1}}{\omega_k} c_{k,k+1}. \tag{20}$$

Besides, applying (19) to $k = d-1$, we get that

$$\begin{aligned} c_{d-1,d} &= \frac{1}{\omega_d} \int_B V_{d-1}(B \cap H(tu)) dt d\nu_d(u) \\ &= \frac{\omega_{d-1}}{\omega_d} \sigma_d \int_0^1 (1-t^2)^{\frac{d-1}{2}} dt = \frac{\sigma_d}{2}. \end{aligned}$$

Consequently, inserting the equality $c_{k,k+1} = \sigma_{k+1}/2$ in (20), we deduce the following relation of induction:

$$c_{k,d} = \frac{k+1}{2(d-k)} \omega_{d-1} \frac{\omega_{k+1}}{\omega_k} c_{k+1,d},$$

which, after iteration, gives us the formula

$$c_{k,d} = \binom{d}{k} \frac{\omega_d {\omega_{d-1}}^{d-k}}{\omega_k 2^{d-k}}. \square$$

Let us go back to the determination of $\mathbf{E}V_k(\mathcal{C})$, $0 \le k \le d$: repeating an argument of Møller (see [16], page 62), we consider for all convex polyhedron P, $\mathcal{E}_k(P)$ the set of all k-dimensional faces of P. If $B \subset \mathbb{R}^d$ is a fixed Borel set such that $0 < V_d(B) < +\infty$, we then have

$$\begin{aligned} V_d(B)c_{k,d} &= \mathbf{E}\sum_{F\in\mathcal{F}_k} V_k(B\cap F) \\ &= \frac{1}{2^{d-k}}\mathbf{E}\sum_{z\in\Psi}\sum_{F\in\mathcal{E}_k(C(z))} V_k(B\cap F), \end{aligned}$$

the last equality being due to the fact that any k-dimensional face of the tessellation is in the boundary of exactly 2^{d-k} different cells.
Using formula (5) we obtain that

$$\begin{aligned} V_d(B)c_{k,d} &= \frac{\omega_d{\omega_{d-1}}^d}{2^d}\frac{1}{2^{d-k}}\int \mathbf{E}\sum_{F\in\mathcal{E}_k(\mathcal{C}+x)} V_k(B\cap F)dx \\ &= \frac{\omega_d{\omega_{d-1}}^d}{2^d}\frac{1}{2^{d-k}}\int \mathbf{E}\sum_{F\in\mathcal{E}_k(\mathcal{C})} V_k(B\cap (F+x))dx \\ &= \frac{\omega_d{\omega_{d-1}}^d}{2^d}\frac{1}{2^{d-k}}V_d(B)\mathbf{E}\sum_{F\in\mathcal{E}_k(\mathcal{C})} V_k(F), \end{aligned}$$

the last equality being deduced from Fubini's theorem. Consequently, we deduce that

$$\begin{aligned} \mathbf{E}V_k(\mathcal{C}) &= 2^{d-k}c_{k,d}\frac{2^d}{\omega_d{\omega_{d-1}}^d} \\ &= \frac{2^d\binom{d}{k}}{{\omega_{d-1}}^k\omega_k}. \end{aligned}$$

(2) We consider the proces Ψ_k, $0 \le k \le d$, of the centers of the inballs of the k-dimensional faces of the tessellation. Ψ_k is stationary and we will note λ_k its intensity. Besides, we define for any $z \in \Psi_k$, $F(z)$ as the unique k-dimensional face associated to z.
Then the typical k-dimensional face $\mathcal{C}_{k,d}$ associated to the tessellation is well-defined by the following formula:

$$\mathbf{E}h(\mathcal{C}_{k,d}) = \frac{1}{V_d(B)\lambda_k}\mathbf{E}\sum_{z\in\Psi_k\cap B} h(F(z)-z),$$

for all bounded measurable function h and any fixed Borel set $B \subset \mathbb{R}^d$, satisfying $0 < V_d(B) < +\infty$. Let us notice in particular that the typical d-face $\mathcal{C}_{d,d,}$ is the classical typical cell $\mathcal{C}$ associated to the tessellation. The following lemma gives a characterization of the law of the typical k-face and can be easily deduced from a joint use of Slivnyak's formula and Lemma 2:

Lemma 4 *$\mathcal{C}_{k,d}$ is equal in law to the typical cell of a Poissonian tessellation in $\mathbb{R}^k$ with intensity measure given by the formula (16).*

A direct consequence of the preceding lemma and the point (1) of Theorem 4 is that

$$\begin{aligned} \mathbf{E}V_k(\mathcal{C}_{k,d}) &= \frac{2^k}{\omega_k{\omega_{k-1}}^k}\times\left(\frac{\omega_{k-1}}{\omega_{d-1}}\right)^k \\ &= \frac{2^k}{\omega_k{\omega_{d-1}}^k}. \end{aligned} \tag{21}$$

Following the same method as Møller [16] (prop. 3.2.2) for Voronoi tessellations, we can show that:

$$\mathbf{E}V_k(\mathcal{C}_{k,d}) = \mathbf{E}V_k(\mathcal{C})/\mathbf{E}N_k(\mathcal{C}). \tag{22}$$

Consequently, from (21) and (22), we obtain:

$$\mathbf{E}N_k(\mathcal{C}) = 2^{d-k}\binom{d}{k}.$$

(3) Considering the construction of $\mathcal{C}$ obtained in Theorem 3, and denoting by Φ_r, $r > 0$, a Poisson point process of intensity measure $\mathbf{1}_{B(r)^c}d\mu$, we have

$$\mathbf{P}\{N_{d-1}(\mathcal{C}) = d+1\} = \frac{2^d}{(d+1)\omega_d\omega_{d-1}{}^d}\int \mathbf{P}\{\#(\Phi_r \cap S(Ru)) = 0\}e^{-\sigma_d R}\Delta(u)\mathbf{1}_A(u)dRd\overline{\nu_d}^{(d+1)}(u).$$

Let us then notice that it is well-known (see for example [4]) that

$$\mathbf{P}\{\#(\Phi_r \cap S(Ru)) = 0\} = \exp\left\{-R\left(\frac{\sigma_d}{2}b(u) - \sigma_d\right)\right\},$$

where $b(u)$ denotes the mean width of the simplex $S(u)$. Consequently, after an integration with respect to R, we deduce

$$\mathbf{P}\{N_{d-1}(\mathcal{C}) = d+1\} \quad = \quad \frac{2^{d+1}}{d(d+1)\omega_d^2\omega_{d-1}{}^d}\int \frac{\Delta(u)}{b(u)}\mathbf{1}_A(u)d\overline{\nu_d}^{d+1}(u).\square$$

Remark 1. Let us remark that it is possible to use the point (1) of Theorem 4 to obtain the expression of the intensity λ_d of the process Ψ of the centers of the inballs of the cells, and more precisely to show that λ_d has the same value as the intensity $c_{0,d}$ of the process of the vertices of the tessellation: actually, we could define the typical cell by associating to each cell constituting the tessellation its lowest vertex (which exists a.s.) and by replacing the process Ψ of the centers of the inballs by the process Λ of the lowest vertices. It is easy to notice that any vertex of the tessellation is the lowest vertex of exactly one cell, which means that Λ can be exactly identified to the process of the vertices of the tessellation, of intensity $c_{0,d}$. We can also remark that the two definitions of the typical cell provide the same law because in the two cases, the equality (6) is satisfied.

Repeating the argument of Møller [16] (page 62), we obtain that:

$$\mathbf{E}V_d(\mathcal{C}) = \lambda_d{}^{-1} = c_{0,d}{}^{-1}.$$

Remark 2. The formula (1) of Theorem 4 was previously obtained by Matheron [9]. The equalities (2) and (3) were given by R. E. Miles [12] when $d = 2, 3$; for any $d > 3$, they are new.

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