Hyperbolic Hydrodynamic Limits and Thermodynamic Entropy Solutions

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Introduction

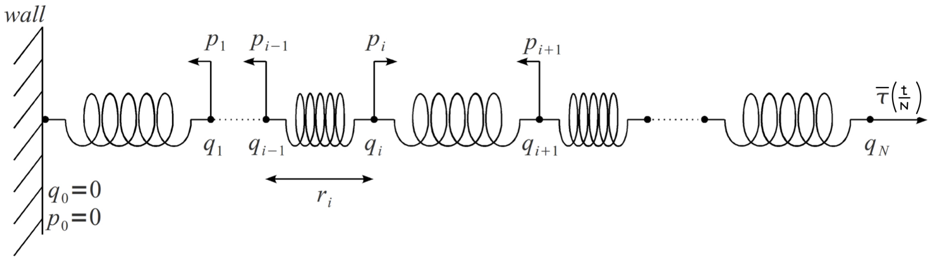
We construct weak physical solutions to a nonlinear system of conservation laws from the microscopic dynamics.

- What kind of weak solutions?
- What are the "physical" boundary conditions and how to incorporate in a weak sense?
- What about uniqueness? Maybe use entropy solutions?

Answer: we shall focus on L^2-valued **theremodynamic entropy solutions**, i.e. weak solutions with the boundary conditions intended in a weak sense, which obey the second law of thermodynamics.

Chain of oscillators - 1

Considered the following system of N anharmonic oscillators in $d = 1$.



Where $q_i(t) \in \mathbb{R}$ is the position of the particle i at time t and $p_i(t) \in \mathbb{R}$ is its momentum. Particle 0 is kept fixed at the orgin ($q_0 = p_0 = 0$) and at particle N is applied an external force $\bar{\tau}(\frac{t}{N})$.

We take all the masses equal to 1 and let the particles i and $i-1$ interact via a potential energy $V(r_i)$, where $r_i := q_i - q_{i-1}$.Thus the dynamics is governed by the Hamiltonian

$$\mathcal{H}_N := \frac{p_0^2}{2} + \sum_{i=1}^{N} \left\{ \frac{p_i^2}{2} + V(q_i - q_{i-1}) \right\} - \bar{\tau}\Big(\frac{t}{N}\Big) q_N = \sum_{i=1}^{N} \left\{ \frac{p_i^2}{2} + V(r_i) - \bar{\tau}\Big(\frac{t}{N}\Big) r_i \right\}.$$

Chain of oscillators - 2

We perturb the Hamiltonian dynamics with a stochastic noise in such a way that

1. The total **length** and **momentum** of the chain are conserved (away from the boundary);
2. The energy is **not** conserved and the temperature of the chain is kept to the constant β^{-1}.
3. Mixed Dirichlet-Neumann boundary conditions are enforced.

We also rescale time by the same "space scale" N (hyperbolic space-time scaling) so that the stochastic dynamics of $\{r_i(Nt), p_i(Nt)\}_{i=1}^{N}$ is generated by

$$L_N^{\bar{\tau}(t)} := N A_N + N \sigma_N (S_N + S_N^b)$$

where A_N is the generator of the Hamiltonian dynamics (Liouville generator), S_N is the generator of the noise in the bulk, S_N^b generate the noise at the boundary and σ_N tunes the strength of the noise. σ_N blows up as $N \to \infty$, so we have "a lot" of noise, also at the boundaries.

Viscous HDL - 1

The case $\sigma_N = N\varepsilon$, for $\varepsilon > 0$ fixed has been treated in [M. '21], where we show that the limiting system is the following viscous system of conservation laws

$$\begin{cases} \partial_t r^\varepsilon - \partial_x p^\varepsilon = \varepsilon \partial_{xx} \tau(\beta, r^\varepsilon) \\ \partial_t p^\varepsilon - \partial_x \tau(\beta, r^\varepsilon) = \varepsilon \partial_{xx} p^\varepsilon \end{cases} \tag{1}$$

with boundary conditions

$$p^\varepsilon(t,0) = 0, \qquad \tau(\beta, r^\varepsilon(t,1)) = \bar{\tau}(t), \qquad \partial_x p^\varepsilon(t,1) = 0, \qquad \partial_x r^\varepsilon(t,0) = 0.$$

Here and below τ will denote the macroscopic tension (i.e. the pressure) and it will depend in a non-trivial way from the interaction $V(r)$, namely τ is the expectation of V' w.r.t. a certain Gibbs measure.

The proof is done via the relative entropy method, which requires existence of smooth solutions of (1). Such existence has been proven in [M., Alasio '19] globally in time.

Viscous HDL - 2

The boundary conditions are inherited by the microscopic model and are such that the macroscopic Clausius inequality holds. Define the local free energy as

$$\mathcal{F}(t,x;\beta) := \int_0^{r(t,x)} \tau(\beta,\rho)\mathrm{d}\rho + \frac{1}{2}p(t,x)^2.$$

Then, assuming $\bar{\tau}(0)=\tau_0$, $\bar{\tau}(+\infty)=\tau_1$, the total free energy $F=\int_0^1 \mathcal{F}\mathrm{d}x$ satisfies the Clausius inequality

$$F(\beta,\tau_1) - F(\beta,\tau_0) \leqslant W, \tag{2}$$

where W is the total work done by the external force $\bar{\tau}(t)$.

For an isothermal transformation (2) is equivalent to the second law of thermodynamics

$$S(\beta,\tau_1) - S(\beta,\tau_0) \geqslant 0. \tag{3}$$

Vanishing viscosity macroscopic limit - 1

In [M., Olla '20] we work entirely at the macroscopic level and we perform the limit $\varepsilon \to 0$ on the solutions $(r^\varepsilon, p^\varepsilon)$ of the viscous system. We obtain L^2-valued thermodynamic entropy solutions to the p-system

$$\begin{cases} \partial_t r - \partial_x p = 0 \\ \partial_t p - \partial_x \tau(r) = 0 \end{cases} \tag{4}$$

with boundary conditions

$$p(t,0) = 0, \quad \tau(r(t,1)) = \bar{\tau}(t). \tag{5}$$

Both the (4) and (5) must be intended in a weak sense.

Clausius inequality is also inherited from the viscous equation and it also holds in a weak sense.

Vanishing viscosity macroscopic limit - 2

The essential steps to perform the limit $\varepsilon \to 0$ for the system

$$\begin{cases} \partial_t r^\varepsilon - \partial_x p^\varepsilon = \varepsilon\, \partial_{xx} \tau(\beta, r^\varepsilon) \\ \partial_t p^\varepsilon - \partial_x \tau(\beta, r^\varepsilon) = \varepsilon\, \partial_{xx} p^\varepsilon \end{cases}$$

are the following. Fix $T > 0$

→ Prove that r^ε and p^ε are bounded in $L^2([0,T] \times [0,1])$. This implies that r^ε and p^ε admit (up to a subsequence) L^2-weak limits $r^\star, p^\star$.

→ Prove that $\sqrt{\varepsilon}\, \partial_x r^\varepsilon$ and $\sqrt{\varepsilon}\, \partial_x p^\varepsilon$ are bounded in $L^2([0,T] \times [0,1])$.

→ Pass to a weak form of the above system

→ Prove that $\tau(\beta, r^\varepsilon) \rightharpoonup \tau(\beta, r^\star)$

The last step is highly non-trivial due to the nonlinearity of τ. This is achieved via compensated compactness, and it only further requires regularity/decay properties of τ.

Vanishing viscosity HDL - 1

In [M., Olla '20] we perform the HDL directly with vanishing but strong viscosity. Namley we take

$$\lim_{N\to\infty} \frac{\sigma_N}{N} = \lim_{N\to\infty} \frac{N}{\sigma_N^2} = 0.$$

and obtain L^2-valued weak solution for the p-system.

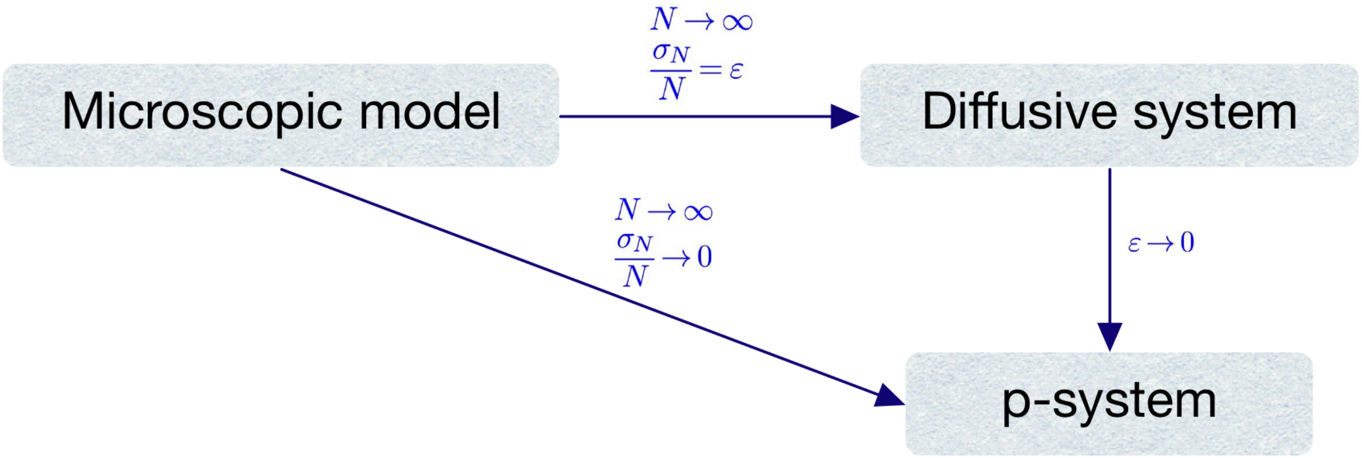
We also the macroscopic boundary conditions and deriving the Clausius inequality. The latter is obtained directly from the microscopic dynamics using the microscopic production of the relative entropy together with its variational formulation.

The idea here is to build two “empirical processes” $r_N(t,x)$ and $p_N(t,x)$ from mesoscopic blocks of size $K = K(N)$ and replicate the compensated compactness argument. This in turn means deriving the following properties

- $\rightarrow$ L^2 bounds for r_N, p_N (for free from the microscopic dynamics)
- $\rightarrow$ L^2 bounds for the “gradients” of r_N, p_N (two-block estimate with explicit bounds)
- $\rightarrow$ One-block estimate with explicit bounds

In order to obtain the desired one and two-block estimates we make a crucial use of a log-Sobolev inequality.

A summary


Microscopic model
Diffusive system
p-system
$N \to \infty$
$\frac{\sigma_N}{N} = \varepsilon$
$N \to \infty$
$\frac{\sigma_N}{N} \to 0$
$\varepsilon \to 0$

Thank you for your attention!