Stationary measures for the KPZ equation

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One dimensional KPZ equation

The [Kardar-Parisi-Zhang] equation is a nonlinear stochastic PDE describing the time evolution of a function $h(t,\cdot) \in C(\mathbb{X},\mathbb{R})$, on a spatial domain $\mathbb{X} \subset \mathbb{R}$,

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}\big(\partial_x h\big)^2 + \xi, \quad t \geqslant 0, \quad x \in \mathbb{X}.$$

Question: Are there stationary measures ? Can one classify them ? How to describe them precisely ?
The answer depends on the spatial domain $\mathbb{X}$.

We may consider

- The whole line $\mathbb{X} = \mathbb{R}$
- Periodic boundary conditions $\mathbb{X} = \mathbb{R}/\mathbb{Z}$
- An interval $\mathbb{X} = [0, L]$ with boundary conditions
- A half-line $\mathbb{X} = \mathbb{R}_+$

The simple case: $\mathbb{R}$ ou $\mathbb{R}/\mathbb{Z}$

- Assume that $\mathbb{X} = \mathbb{R}$. For a large class of initial conditions, $h(t,x) \sim \frac{-t}{24}$, so we do not expect that the law of $h(t,x)$ can be stationary in time.

  No stationary measures in $C(\mathbb{R},\mathbb{R})$ but there exist stationary measures for the law of spatial increments.

- If $h(0,x) = B_x^{(\mu)}$ a Brownian motion with drift μ, then for all time $t > 0$, as processes in x,

$$h(t,x) - h(t,0) \overset{(d)}{=} B_x^{(\mu)}.$$

  [Bertini-Giacomin 1997, Funaki-Quastel 2014].

- On the torus $\mathbb{R}/\mathbb{Z}$, the Brownian motion is the unique invariant measure [Hairer-Mattingly 2016].

Plan of the talk

1. How to find stationary measures on $\mathbb{R}$?
   Discretize!

2. KPZ equation on a segment and its invariant measures
   arXiv:2105.15178 joint with Pierre Le Doussal

3. Matrix Product Ansatz
   arXiv:2209.03131 joint with Pierre Le Doussal

KPZ equation on $\mathbb{R}$

Solutions of the KPZ equation

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}\big(\partial_x h\big)^2 + \xi, \quad t \geqslant 0, x \in \mathbb{R}$$

are defined through the Cole-Hopf transform $h := \log(Z)$ where Z solves the multiplicative noise stochastic heat equation

$$\partial_t Z(t,x) = \tfrac{1}{2}\Delta\ Z(t,x) + Z(t,x)\ \xi(t,x), \quad t > 0, x \in \mathbb{R},$$

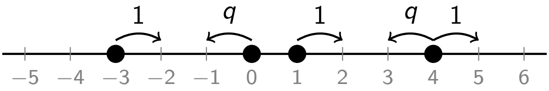
where ξ is a space-time Gaussian white noise. In dimension 1, a solution $Z(t,x)$ solves

$$Z(t,x) = \int_{\mathbb{R}} dy Z_0(y) p_t(y,x) + \int_0^t ds \int_{\mathbb{R}} d_y p_{t-s}(y,x) Z(s,y)\xi(s,y),$$

where $p_t(y,x)$ is the standard heat kernel.

ASEP

ASEP (asymmetric simple exclusion process) is a continuous Markov process on $\{0,1\}^{\mathbb{Z}}$, whose transition rates depend on an asymmetry parameter q.



- For any $\varrho \in [0,1]$, the measure $\mathrm{Ber}(\varrho)^{\otimes \mathbb{Z}}$ is invariant.
- Define a height function $H(t,x)$ so that

$$H(t,x) - H(t,x-1) = \begin{cases} 1 & \text{if site } x \text{ is occupied.} \\ -1 & \text{if site } x \text{ is empty.} \end{cases}$$

and $H(t,0)$ is the number of particles which have crossed the origin.

Convergence ASEP $\to$ KPZ

Let $\mathcal{Z}_t(x) = q^{\frac{1}{2}H(t,x)-\nu t}$, where $\nu = (1-\sqrt{q})^2$. For $q = e^{-\varepsilon}$, when $\varepsilon \to 0$

$$\mathcal{Z}_{\epsilon^{-4}t}(\epsilon^{-2}x) \Longrightarrow Z(t,x),$$

the solution of

$$\partial_t Z(t,x) = \tfrac{1}{2}\Delta Z(t,x) + Z(t,x)\,\xi(t,x).$$

ASEP height function converges to a solution of KPZ equation.
[Bertini-Giacomin 1997]

Rmk: Under $\mathrm{Ber}(\varrho)^{\otimes\mathbb{Z}}$, the height function converges to a Brownian motion (with drift), up to a global shift.

KPZ equation on a segment

Consider the KPZ equation on the segment $[0, L]$,

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}\big(\partial_x h\big)^2 + \xi.$$

For the solution to be unique, one needs to impose boundary conditions. Since $h(x,t) \sim -ct$ cannot be fixed, it is natural to impose a Newman type condition

$$\partial_x h(t,0) = A, \quad \partial_x h(t,L) = B.$$

Physically, $\partial_x h$ corresponds to the density in ASEP. These parameters also have a natural interpretation when $Z(t,x) = e^{h(t,x)}$ is viewed as the partition function of a directed polymer.

$h(t,x)$ is not differentiable...

Boundary conditions

$h = \log Z$ yields

$$\partial_t Z(t,x) = \tfrac{1}{2}\Delta Z(t,x) + Z(t,x)\,\xi(t,x).$$

On $Z(t,x)$, boundary conditions become

$$\partial_x Z(t,0) = AZ(t,0), \quad \partial_x Z(t,L) = BZ(t,L).$$

Definition ([Corwin-Shen 2016])

$h(t,x)$ solves the KPZ equation on $[0,L]$ with boundary parameters $u \in \mathbb{R}$ and $v \in \mathbb{R}$ if $h(t,x) = \log Z(t,x)$ and

$$Z(t,x) = \int_0^L dy Z_0(y) p_t^{u,v}(x,y) + \int_0^t ds \int_0^L d_y\, p_{t-s}^{u,v}(x,y) Z(s,y)\xi(s,y),$$

where $p_t^{u,v}(x,y)$ is the heat kernel on $[0,L]$ with boundary conditions

$$\begin{cases} \partial_x p_t^{u,v}(x=0,y) = (u-\tfrac{1}{2})p_t^{u,v}(0,y), \\ \partial_x p_t^{u,v}(x=L,y) = (-v+\tfrac{1}{2})p_t^{u,v}(L,y). \end{cases}$$

Stationary measures on a segment

Theorem ([Corwin-Knizel 2021])

1. *For* $u, v \in \mathbb{R}$ *such that* $u + v > 0$*, there exist a stationary process* $h^L_{u,v} \in C([0, L], \mathbb{R})$*. Its finite dimensional marginals* $(h^L_{u,v}(x_1), \dots, h^L_{u,v}(x_k))$ *are characterized by a Laplace transform formula*

$$\mathbb{E}\left[\prod_{i=1}^{k} e^{-s_i(h^L_{u,v}(x_i) - h^L_{u,v}(x_{i-1}))}\right] = \textit{Some formula.}$$

2. *When* $u + v = 0$*,* $h^L_{u,v}$ *is a Brownian motion with drift* $u = -v$*.*

Motivations for looking at a simpler characterization:

- The symmetry in u, v is not really apparent.
- The formula does not clearly degenerates to the Gaussian Laplace transform when $u + v \to 0$
- It's not clear how to extend to $u + v < 0$.

Reweighted Brownian motion by an exponential functional

Theorem ([Bryc-Kuznetsov-Wang-Wesołowski 2021], [B.- Le Doussal 2021])

The stationary process is such that

$$h^L_{u,v}(x) = W_x + Y_x - Y_0,$$

where

- W *is a Brownian motion with diffusion coefficient* $1/2$ *starting from* 0.
- Y *is independent from* W*, and its law is absolutely continuous w.r.t. to that of a Brownian* B*, with diffusion coeff.* $1/2$*, and free starting point. The Radon-Nikodym derivative is*

$$\frac{1}{\mathcal{Z}_{u,v}} e^{-2uY_0 - 2vY_L} e^{-\int_0^L e^{-2Y_s} ds}$$

Rq: Exchanging u et v has the same effect as reversing space.

Averaging out out the starting point

$$\frac{1}{\mathcal{Z}_{u,v}} e^{-2uY_0-2vY_L} e^{-\int_0^L e^{-2Y_s}ds},$$

Write $X_s = Y_s - Y_0$ and integrate over the starting point Y_0 using the identity

$$\frac{1}{\Gamma(u+v)} \int_0^\infty \mathrm{d}z\ z^{u+v-1} e^{-zA} = A^{-u-v}$$

(to be applied with $A = \int_0^L e^{-2(Y_s-Y_0)}ds$).

The law of $X_s = Y_s - Y_0$ is absolutely continuous w.r.t. that of a Brownian B with $B(0) = 0$ with Radon-Nikodym derivative

$$\frac{1}{\tilde{\mathcal{Z}}_{u,v}} e^{-2vX_L} \left(\int_0^L e^{-2X_s} ds \right)^{-u-v}.$$

When $u + v = 0$, X is a Brownian motion with drift $u = -v$.

Conjecture

We have seen that when $u + v \geqslant 0$,

$$h^L_{u,v}(x) \overset{(d)}{=} W_x + X_x$$

where the law of X has Radon-Nikodym derivative

$$e^{-2vX_L} \left(\int_0^L e^{-2X_s} ds \right)^{-u-v} .$$

This functional makes perfect sense and remains integrable even if $u + v < 0$.

Conjecture ([B.- Le Doussal, 2021])

For any $u, v \in \mathbb{R}$, there exists a unique stationary process $h_{u,v}$, whose distribution is analytic in u, v and the process $W_x + X_x$ is stationary for any $u, v \in \mathbb{R}$.

Initial proof of the Theorem

1. Stationary measures for ASEP on a finite domain with boundary conditions (reservoirs) are caracterized by the matrix product ansatz (MPA) [Derrida-Evans-Hakim-Pasquier 1993]. A representation of the MPA involving Askey-Wilson orthogonal polynomials was proposed in [Uchiyama-Sasamoto-Wadati, 2004].
2. Askey-Wilson polynomials arise in various areas, in particular quadratic harnesses [Bryc-Wesołowski 2010]
3. [Bryc-Wesołowski 2018] discovered that the Laplace transform of the stationary ASEP height function can be written in term of functionals of those quadratic harnesses.
4. The KPZ equation limit was studied in [Corwin-Shen 2016] and [Corwin-Knizel 2021] which led to Laplace transform formula for the KPZ equation stationary measures.
5. Laplace transform inversion was performed in [Bryc-Kuznetsov-Wang-Wesołowski 2021] and in [B.- Le Doussal 2021]. Both use essentially the same underlying machinery, the spectral theory of the operator

$$\frac{-1}{4}\frac{d^2}{dx^2} + e^{-2x}.$$

A corollary about uniqueness

Using the theory of regularity structures:

Theorem ([Knizel-Matetski 2022])

For any $u, v \in \mathbb{R}$, the KPZ equation on $[0,1]$ is a Markov process on $\mathcal{C}^\alpha([0,1])$ for any $\alpha \in (0,1/2)$, which satisfies the strong Feller property.

Using the Brownian reweighting description of the stationary process $h^L_{u,v}$:

Corollary ([Knizel-Matetski 2022])

For any $u + v \geqslant 0$, the stationary measure is unique.

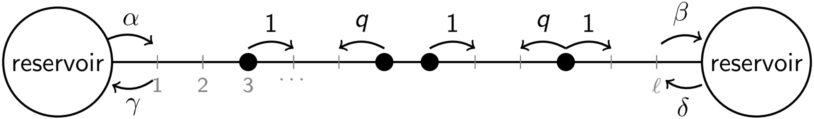
Another route to find KPZ stationary measures

[B.-Le Doussal 2022]

1. Use another representation of the Matrix Product Ansatz from [Enaud-Derrida, 2004]
2. Rewrite the steady-state as reweighted random walks, following [Derrida-Enaud-Lebowitz, 2004]
3. Take the scaling limit

Matrix product ansatz

Consider ASEP on $\{0,1\}^\ell$ with boundary parameters $\alpha, \beta, \gamma, \delta$.



We describe the state of the system by $\tau \in \{0,1\}^\ell$. The stationary measure $\mathbb{P}$ can be written as [Derrida-Evans-Hakim-Pasquier 1993]

$$\mathbb{P}(\tau) = \frac{1}{Z_\ell} \langle w| \prod_{i=1}^{\ell} (\tau_i D + (1-\tau_i)E) \, |v\rangle$$

where

$$Z_\ell = \langle w| (E+D)^\ell \, |v\rangle$$

and E, D are infinite matrices, and $\langle w|, |v\rangle$ are row/column vectors such that

$$\begin{gathered} DE - qED = D + E \\ \langle w| (\alpha E - \gamma D) = \langle w| \\ (\beta D - \delta E) |v\rangle = |v\rangle \end{gathered}$$

Representations of the MPA

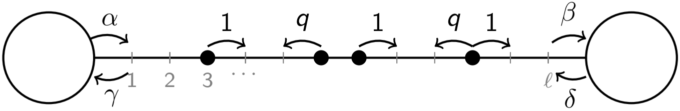
- Finding representations, i.e. matrices E, D and explicit vectors u, v satisfying the relations, is non trivial. Special cases are worked out in [Derrida-Evans-Hakim-Pasquier 1993].
- For TASEP, $q = \gamma = \delta = 0$, we may take

$$D = \begin{pmatrix} 1 & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ 0 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, E = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

  and easily find eigenvectors $\langle w| \, , |v\rangle$.
- [Sandow, 1995] proposed a representation in the most general case. The vectors $\langle w| \, , |v\rangle$ are complicated.
- Several families of orthogonal polynomials appear. In the most general case, [Uchiyama-Sasamoto-Wadati, December 2003] found a representation using Askey-Wilson orthogonal polynomials.
- Another very simple representation was proposed in [Enaud-Derrida, July 2003] and has been often overlooked in later papers on the subject.

Enaud-Derrida's representation

Enaud-Derrida found a very simple representation for any parameters $q, \alpha, \beta, \gamma, \delta$.



We may associate to these parameters density parameters ϱ_a, ϱ_b by imposing $j_a = (1-q)\varrho_a(1-\varrho_a) = \alpha(1-\varrho_a) - \gamma\varrho_a$ and similarly for ϱ_b.

In this talk, for simplicity, we impose further the conditions

$$\gamma = q(1-\alpha), \quad \delta = q(1-\beta)$$

so that the density parameters are related to the jump rates by

$$\varrho_a = \alpha, \quad 1 - \varrho_b = \beta.$$

Enaud-Derrida's representation

Let $[n]_q = \frac{1-q^n}{1-q}$ and

$$D = \begin{pmatrix} [1]_q & [1]_q & 0 & 0 & 0 & \cdots \\ 0 & [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & 0 & [3]_q & [3]_q & 0 & \cdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}, \quad E = \begin{pmatrix} [1]_q & 0 & 0 & 0 & \cdots \\ [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & [3]_q & [3]_q & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix}$$

Denoting by $\{|n\rangle\}_{n\geqslant 1}$ the vectors of the associated basis, let

$$\langle w| = \sum_{n\geqslant 1} \left(\frac{1-\varrho_a}{\varrho_a}\right)^n \langle n|, \qquad |v\rangle = \sum_{n\geqslant 1} \left(\frac{\varrho_b}{1-\varrho_b}\right)^n [n]_q \, |n\rangle .$$

Then, $E, D, \langle w|, |v\rangle$ satisfy

$$\begin{aligned} DE - qED &= D + E \\ \langle w| (\alpha E - \gamma D) &= \langle w| \\ (\beta D - \delta E) |v\rangle &= |v\rangle \end{aligned}$$

Sum over paths

Due to the bidiagonal structure, the normalization constant $Z_\ell = \langle w|\,(D+E)^\ell\,|v\rangle$ can be written as a sum over lattice paths $\vec{n} = (n_0, n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ of the form

$$Z_\ell = \sum_{\vec{n}} \Omega(\vec{n})$$

where

$$\Omega(\vec{n}) = \left(\frac{1-\varrho_a}{\varrho_a}\right)^{n_0} \left(\frac{\varrho_b}{1-\varrho_b}\right)^{n_\ell} \prod_{i=1}^{\ell} v(n_{i-1}, n_i) \prod_{i=0}^{\ell} [n_i]_q,$$

with

$$v(n,n') = \begin{cases} 2 & \text{if } n = n', \\ 1 & \text{if } |n-n'| = 1 \\ 0 & \text{else.} \end{cases}$$

- This introduces a natural probability measure on random walk paths $\vec{n}$. The stationary measure $\mathbb{P}(\tau)$ can be recovered from this measure.

Open ASEP invariant measure

Following arguments similar as [Derrida-Enaud-Lebowitz 2004], one arrives at

Theorem ([B.-Le Doussal 2022])

Under the stationary measure $\mathbb{P}(\tau)$, ASEP height function $H(x) = \sum_{j=1}^{x}(2\tau_i - 1)$ is such that

$$(H(i))_{1\leqslant i\leqslant \ell} \overset{(d)}{=} (n_i - n_0 + m_i)_{1\leqslant i\leqslant \ell}\,,$$

where $(n_i, m_i)_{0\leqslant i\leqslant \ell}$ is a two dimensional random walk on $\mathbb{Z}^2$, starting from $(n_0, 0)$, distributed as

$$P(\vec{n}, \vec{m}) = \frac{\mathbb{1}_{n_0>0}}{4^{-\ell} Z_\ell} \left(\frac{1-\varrho_a}{\varrho_a}\right)^{n_0} \left(\frac{\varrho_b}{1-\varrho_b}\right)^{n_\ell} \prod_{i=0}^{\ell} [n_i]_q \times P^{SSRW}_{n_0,0}(\vec{n}, \vec{m}),$$

where $P^{SSRW}_{n_0,0}$ denotes the probability measure of the symmetric simple random walk (SSRW) on $\mathbb{Z}^2$ starting from $(n_0, 0)$.

Scaling limit

Under the scalings such that ASEP's height function converges to KPZ, in particular

$$\varrho_a = \frac{1}{2} + \frac{u}{\sqrt{\ell}}, \quad \varrho_b = \frac{1}{2} + \frac{-v}{\sqrt{\ell}}$$

we find that

$$\prod_{i=0}^{\ell} [n_i]_q \to e^{-\int_0^L e^{-2Y_s} ds}$$

$$\left(\frac{1-\varrho_a}{\varrho_a}\right)^{n_0} \left(\frac{\varrho_b}{1-\varrho_b}\right)^{n_\ell} \to e^{-2uY_0 - 2vY_L}$$

so that

$$m_i \Longrightarrow W_x$$

$$n_i \Longrightarrow Y_x$$

where W_x is a Brownian motion and Y_x is absolutely continuous to the Brownian measure with Radon Nikodym derivative

$$\frac{1}{\mathcal{Z}_{u,v}} e^{-2uY_0 - 2vY_L} e^{-\int_0^L e^{-2Y_s} ds}.$$

Related problems

- Stationary measures on $\mathbb{R}_+$ are $L \to +\infty$ limits of stationary measures on $[0, L]$ (studied by [Hariya-Yor 2004]). Proof via the log-gamma polymer and ideas from integrable probability [B.-Corwin 2022].

- Stationary measures of ASEP on $\mathbb{N}$ can also be computed via a variant of Derrida-Enaud's representation.

- Conjecture when $u + v < 0$. The analytic continuation cannot be made on ASEP (stationary measure is not analytic in ϱ_a, ϱ_b).

- Uniqueness holds on $\mathbb{R}/\mathbb{Z}$ and $[0, 1]$.
  Open problem: On unbounded domains, the full classification is conjectured in all cases but unproven.

- What is the connection between the KPZ equation and the operator

$$\frac{-1}{4} \frac{d^2}{dx^2} + e^{-2x}.$$

  What about higher dimensions?

Conclusion

Invariant measures for KPZ equation on a segment are given by re-weightings of the Brownian measure by exponential functionals.

One can rederive this result from a scaling limit of ASEP, using a remarkably simple representation of the matrix product ansatz due to Enaud and Derrida, that has been often overlooked.

Thank you