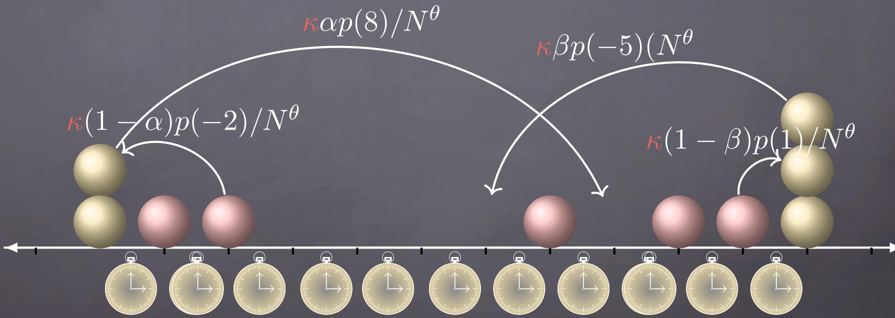
Exclusion in contact with infinitely many reservoirs



The finite variance case

If jumps are arbitrarily large?

Let $p(\cdot)$ be a translation invariant transition probability given at $z \in \mathbb{Z}$ by

$$p(z) = \begin{cases} \frac{c_\gamma}{|z|^{\gamma+1}}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where c_γ is a normalizing constant. Since $p(\cdot)$ is symmetric it is mean zero, that is:

$$\sum_{z \in \mathbb{Z}} z p(z) = 0$$

and take (by now) $\gamma > 2$ so that its variance is finite

$$\sigma_\gamma^2 = \sum_{z \in \mathbb{Z}} z^2 p(z) < \infty.$$

The infinitesimal generator:

$\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,r} + \mathcal{L}_{N,\ell}$ where

$$(\mathcal{L}_{N,0} f)(\eta) = \frac{1}{2} \sum_{x,y \in \Lambda_N} p(x-y)[f(\eta^{x,y}) - f(\eta)],$$

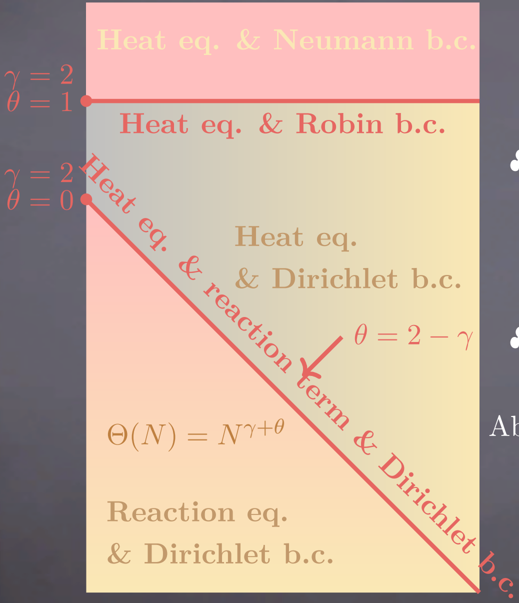
$$(\mathcal{L}_{N,\ell} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{\substack{x \in \Lambda_N \\ y \le 0}} p(x-y) c_x(\eta;\alpha)[f(\eta^x) - f(\eta)],$$

$$(\mathcal{L}_{N,r} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{\substack{x \in \Lambda_N \\ y \ge N}} p(x-y) c_x(\eta;\beta)[f(\eta^x) - f(\eta)]$$

where

$$c_x(\eta;\alpha) := (1-\eta_x)\alpha + (1-\alpha)\eta_x.$$

$$c_x(\eta;\beta) := (1-\eta_x)\beta + (1-\beta)\eta_x.$$



- **Heat equation:**

  $$\partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \rho_t(q)$$

  - $\theta = 1$ **Robin b.c.:**

    $$\partial_q \rho_t(0) = \frac{2m\kappa}{\sigma^2}(\rho_t(0) - \alpha),$$
    $$\partial_q \rho_t(1) = \frac{2m\kappa}{\sigma^2}(\beta - \rho_t(1)),$$

- **Reaction-diffusion eq.:**

  $$\partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \rho_t(q) + \kappa(V_0(q) - V_1(q)\rho_t(q))$$

- **Reaction equation:**

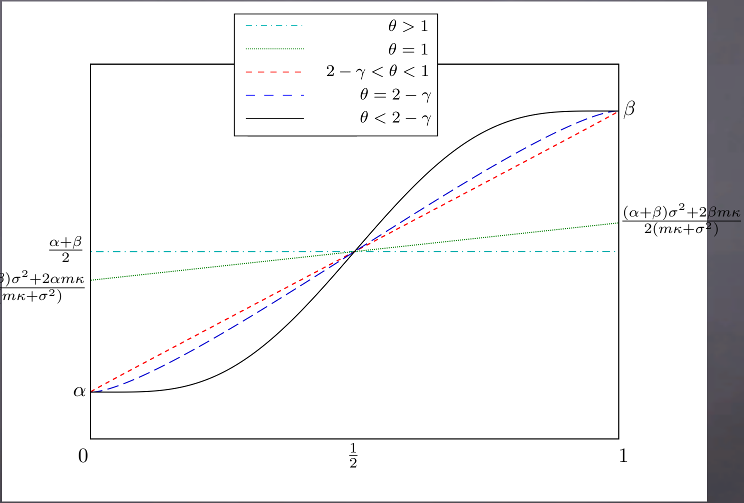
  $$\partial_t \rho_t(q) = \kappa(V_0(q) - V_1(q)\rho_t(q))$$

Above

$$V_1(q) = \frac{c_\gamma}{\gamma}\Big(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma}\Big)$$

$$V_0(q) = \frac{c_\gamma}{\gamma}\Big(\frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma}\Big).$$

Stationary solutions:



Characterizing limit points:

A simple computation shows that

$$\begin{aligned}\Theta(N)\mathcal{L}_N(\langle \pi_s^N, H\rangle) =& \frac{\Theta(N)}{N} \sum_{x,y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right]\eta_s(x) \\ &+ \frac{\kappa\Theta(N)}{N^{1+\theta}} \sum_{x\in\Lambda_N} (Hr_N^-)(\tfrac{x}{N})(\alpha - \eta_s(x)) \\ &+ \frac{\kappa\Theta(N)}{N^{1+\theta}} \sum_{x\in\Lambda_N} (Hr_N^+)(\tfrac{x}{N})(\beta - \eta_s(x)),\end{aligned}$$

where for all $x \in \Lambda_N$

$$r_N^-(\tfrac{x}{N}) = \sum_{y\geq x} p(y), \quad r_N^+(\tfrac{x}{N}) = \sum_{y\leq x-N} p(y).$$

Extend H to $\mathbb{R}$ in such a way that it remains two times continuously differentiable, and the first term at the RHS is

$$\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}(K_NH)(\tfrac{x}{N})\eta_s(x)$$

$$-\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}\sum_{y\leq 0}\left[H(\tfrac{y}{N})-H(\tfrac{x}{N})\right]p(x-y)\eta_s(x)$$

$$-\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}\sum_{y\geq N}\left[H(\tfrac{y}{N})-H(\tfrac{x}{N})\right]p(x-y)\eta_s(x)$$

where $(K_NH)(\tfrac{x}{N})=\sum_{y\in\mathbb{Z}}p(y-x)\left[H(\tfrac{y}{N})-H(\tfrac{x}{N})\right]$.

Let $H\in C_c^2(\mathbb{R})$, we have

$$\limsup_{N\to\infty}\sup_{x\in\Lambda_N}\left|N^2K_NH\left(\tfrac{x}{N}\right)-\frac{\sigma^2}{2}\Delta H(\tfrac{x}{N})\right|=0.$$

For $\Theta(N)=N^{\theta+\gamma}$ and $\theta<2-\gamma$ the first term above vanishes as $N\to\infty$.

The infinite variance case

What about $\gamma \in (0,2)$?

Let $(-\Delta)^{\gamma/2}$ be the fractional Laplacian of exponent $\gamma/2$ which is defined on the set of functions $H : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|H(q)|}{(1+|q|)^{1+\gamma}} du < \infty$$

by (provided the limit exists)

$$(-\Delta)^{\gamma/2} H\,(q) = c_\gamma \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathbf{1}_{|u-q| \geq \varepsilon} \, \frac{H(q) - H(u)}{|u-q|^{1+\gamma}} du.$$

Let $\mathbb{L}$ be the regional fractional Laplacian on $[0,1]$, whose action on functions $H \in C_c^\infty(0,1)$ is given by

$$\begin{aligned}(\mathbb{L}H)(q) &= -(-\Delta)^{\gamma/2} H\,(q) + V_1(q) H(q) \\ &= c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \mathbf{1}_{|u-q| \geq \varepsilon} \, \frac{H(u) - H(q)}{|u-q|^{1+\gamma}} dy, \quad q \in (0,1).\end{aligned}$$

Fractional Sobolev space:

The Sobolev space $\mathcal{H}^{\gamma/2}$ consists of all square integrable functions $g : (0,1) \to \mathbb{R}$ such that $\|g\|_{\gamma/2} < \infty$, with

$$\|g\|_{\gamma/2} := \langle g, g \rangle_{\gamma/2} = \frac{c_\gamma}{2} \iint\limits_{[0,1]^2} \frac{(g(u) - g(q))^2}{|u-q|^{1+\gamma}} dudq.$$

The space $L^2(0,T;\mathcal{H}^{\gamma/2})$ is the set of measurable functions $f : [0,T] \to \mathcal{H}^{\gamma/2}$ such that $\int_0^T \|f_t\|^2_{\mathcal{H}^{\gamma/2}} dt < \infty$ where $\|f_t\|^2_{\mathcal{H}^{\gamma/2}} := \|f_t\|^2 + \|f_t\|^2_{\gamma/2}$.

Hy. Eq. $\theta < 0$ and $\gamma \in (0,2)$

Reaction equation with Dirichlet b.c.

$$\begin{cases} \partial_t \rho_t(u) = -\kappa \rho_t(u) V_1(u) + \kappa V_0(u), & (t,u) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta & t \in (0,T] \\ \rho_0(u) = g(u), & u \in (0,1), \end{cases}$$

if:

- ♣ $\rho \in L^2(0,T;\mathcal{H}^{\gamma/2})$.
- ♣ For $t \in [0,T]$ and $G \in C_c^{1,\infty}([0,T] \times (0,1))$ we have

$$\langle \rho_t, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s, \partial_s G_s \rangle \, ds$$
$$+ \kappa \int_0^t \left\langle \rho_s^k, G_s \right\rangle_{V_1} ds - \kappa \int_0^t \langle G_s, V_0 \rangle \, ds = 0.$$

Hy. Eq. $\theta = 0$ and $\gamma \in (0,2)$

Regional fractional reaction-diffusion equation with Dirichlet b.c.

$$\begin{cases} \partial_t \rho_t(u) = \mathbb{L}\rho_t(u) - \kappa \rho_t(u) V_1(u) + \kappa V_0(u), & (t,u) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta & t \in (0,T] \\ \rho_0(u) = g(u), & u \in (0,1), \end{cases}$$

if :

- ♣ $\rho \in L^2(0,T;\mathcal{H}^{\gamma/2})$.
- ♣ For $t \in [0,T]$ and $G \in C_c^{1,\infty}([0,T] \times (0,1))$ we have that

$$\langle \rho_t, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \Big\langle \rho_s, \Big(\partial_s + \mathbb{L}\Big) G_s \Big\rangle \, ds \\ + \kappa \int_0^t \langle \rho_s, V_1 G_s \rangle \, ds - \kappa \int_0^t \langle G_s, V_0 \rangle ds = 0.$$

Hy. Eq.: $\theta \in (0, \gamma - 1)$ and $\gamma \in (1, 2)$

Regional fractional diffusion equation with Dirichlet b.c.

$$\begin{cases} \partial_t \rho_t(u) = \mathbb{L}\rho_t(u), & (t,u) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, & t \in (0,T], \\ \rho_0(u) = g(u), & u \in (0,1), \end{cases}$$

if :

- ♣ $\rho \in L^2(0,T;\mathcal{H}^{\gamma/2})$.
- ♣ For $t \in [0,T]$ and $G \in C_c^{1,\infty}([0,T] \times (0,1))$ we have that

$$\langle \rho_t, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \Big\langle \rho_s, \Big(\partial_s + \mathbb{L}\Big) G_s \Big\rangle \, ds = 0.$$

Hy. Eq.: $\theta \geq \gamma - 1$ and $\gamma \in (1,2)$ or $\theta > 0$ and $\gamma \in (0,1)$.

Regional fractional diff. equation with frac. Robin/Neumann b.c.

$$\begin{cases} \partial_t \rho_t(u) = \mathbb{L}\rho_t(u), & (t,u) \in [0,T] \times (0,1), \\ \chi_\gamma (D^\gamma \rho_t)(0) = \hat{\kappa} m(\alpha - \rho_t(0)), & t \in (0,T], \\ \chi_\gamma (D^\gamma \rho_t)(1) = \hat{\kappa} m(\beta - \rho_t(1)), & t \in (0,T], \\ \rho_0(u) = g(u), & u \in (0,1), \end{cases}$$

if:

- ♣ $\rho \in L^2(0,T;\mathcal{H}^{\gamma/2})$.
- ♣ For $t \in [0,T]$ and $G \in C^{1,\infty}([0,T] \times (0,1))$ we have that

$$\langle \rho_t, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \Big\langle \rho_s, \Big(\partial_s + \mathbb{L}\Big) G_s \Big\rangle \, ds$$

$$- \hat{\kappa} m \int_0^t \{G_s(0)(\alpha - \rho_s(0)) + G_s(1)(\beta - \rho_s(1))\} ds = 0.$$

Integration by parts formula:

For $f : [0,1] \to \mathbb{R}$, let

$$(D^\gamma f)(0) := \lim_{u \to 0^+} f'(u) u^{2-\gamma}, \quad (D^\gamma f)(1) := \lim_{u \to 1^-} f'(u)(1-u)^{2-\gamma}.$$

Proposition : Guan, Ma '06

Let $\gamma \in (0,2)$ and $g \in C^2([0,1])$. Let $f : [0,1] \to \mathbb{R}$ be such that $u \to f(u)u^{1-\gamma}$ and $u \to f(u)(1-u)^{1-\gamma}$ are in $C^2([0,1])$. Then,

$$\langle -\mathbb{L}g, f \rangle = \chi_\gamma \left[g(1) D^\gamma f(1) - g(0) D^\gamma f(0) \right] + \langle -\mathbb{L}f, g \rangle$$

where χ_γ is a constant.

Characterizing limit points:

$$N^\gamma \mathcal{L}_N(\langle \pi_s^N, H\rangle) = \frac{N^\gamma}{N} \sum_{x,y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right]\eta_s(x)$$

$$+\frac{\kappa N^\gamma}{N}\sum_{x\in\Lambda_N}(Hr_N^-)(\tfrac{x}{N})(\alpha-\eta_s(x)) + \frac{\kappa N^\gamma}{N}\sum_{x\in\Lambda_N}(Hr_N^+)(\tfrac{x}{N})(\beta-\eta_s(x)).$$

For H with compact support in $[a, 1-a]$ for $a \in (0,1)$ we have

$$\lim_{N\to\infty}\left|N^\gamma \sum_{y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right] - (\mathbb{L}H)(\tfrac{x}{N})\right| = 0,$$

$$\lim_{N\to\infty}\left|N^\gamma (r_N^-)(\tfrac{x}{N}) - r^-(\tfrac{x}{N})\right| = 0,$$

$$\lim_{N\to\infty}\left|N^\gamma (r_N^+)(\tfrac{x}{N}) - r^+(\tfrac{x}{N})\right| = 0$$

uniformly in $[a, 1-a]$.

Characterizing limit points:

Thus, the first term on the right hand side above can be replaced by

$$\langle \pi_t^N, \mathbb{L}H \rangle \to \int_0^1 (\mathbb{L}H)(q)\rho_t(q)dq,$$

as N goes to ∞.
The other terms can be replaced by $\kappa\langle \alpha - \pi_t^N, Hr^- \rangle + \kappa\langle \beta - \pi_t^N, Hr^+ \rangle$ which converges to

$$\kappa \int_0^1 H(q)r^-(q)(\alpha - \rho_t(q))dq + \kappa \int_0^1 H(q)r^+(q)(\beta - \rho_t(q))dq$$
$$= \kappa \int_0^1 H(q)V_0(q)dq - \kappa \int_0^1 H(q)V_1(q)\rho_t(q)dq,$$

as N goes to ∞.

Uniqueness of weak solution:

To prove it we do the following. Let $\bar{\rho} = \rho^1 - \rho^2$, where ρ^1 and ρ^2 are two weak solutions starting from g. We have $\bar{\rho}_t(0) = \bar{\rho}_t(1) = 0$. Then,

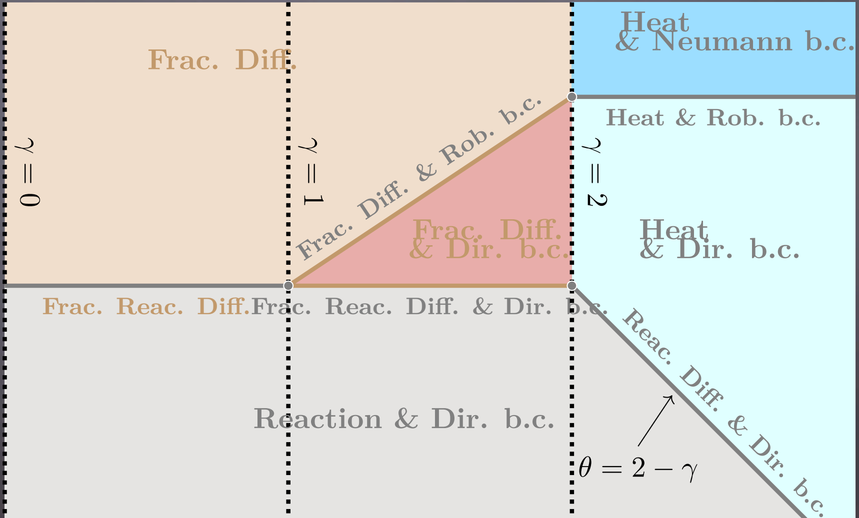
$$\langle \bar{\rho}_t, H_t \rangle - \int_0^t \langle \bar{\rho}_s, \Big(\partial_s + \mathbb{L} \Big) H_s \rangle ds + \kappa \int_0^t \langle V_1 H_s, \bar{\rho}_s \rangle ds = 0.$$

Take now $H_N(s,q) = \int_s^t G_N(r,q)\, dr$ where $(G_N)_{N \geq 0}$ is a sequence of functions in $C_c^{1,\infty}([0,T] \times (0,1))$ converging to $\bar{\rho}$. Plug H_N in the equation and take $N \to \infty$ to get

$$\int_0^t \int_0^1 \bar{\rho}_s^2(q)\, dq ds + \frac{1}{2} \Big\| \int_0^t \bar{\rho}_s ds \Big\|_{\gamma/2}^2 + \frac{\kappa}{2} \Big\| \int_0^t \bar{\rho}_s ds \Big\|_{V_1}^2 = 0.$$

From this we conclude the uniqueness.

Final picture:



Lecture 2: Fluctuations

The space of test functions: Let $\mathcal{S}_\theta$ denote the set of functions $H \in C^\infty([0,1])$ such that for any $k \in \mathbb{N} \cup \{0\}$ it holds that

(1) for $\theta < 1$: $\partial_u^{2k} H(0) = \partial_u^{2k} H(1) = 0$;

(2) for $\theta = 1$: $\partial_u^{2k+1} H(0) = \partial_u^{2k} H(0)$ and $\partial_u^{2k+1} H(1) = -\partial_u^{2k} H(1)$;

(3) for $\theta > 1$: $\partial_u^{2k+1} H(0) = \partial_u^{2k+1} H(1) = 0$.

The density fluctuation field $\mathcal{Y}_\cdot^N$ is the time-trajectory of linear functionals acting on functions $H \in \mathcal{S}_\theta$ as

$$\mathcal{Y}_t^N(H) \;=\; \frac{1}{\sqrt{N-1}} \sum_{x=1}^{N-1} H(\tfrac{x}{N}) \Big(\eta_{tN^2}(x) - \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)] \Big) \,.$$

Operators:

For $\theta \geq 0$, let $-\Delta_\theta$ be the positive self-adjoint operator on $L^2[0,1]$, defined on $H \in \mathcal{S}_\theta$ by

$$\Delta_\theta H(u) = \begin{cases} \partial_u^2 H(u)\,, & \text{if } u \in (0,1), \\ \partial_u^2 H(0^+)\,, & \text{if } u = 0, \\ \partial_u^2 H(1^-)\,, & \text{if } u = 1. \end{cases}$$

Let $\nabla_\theta : \mathcal{S}_\theta \to C^\infty([0,1])$ be the operator given by

$$\nabla_\theta H(u) = \begin{cases} \partial_u H(u)\,, & \text{if } u \in (0,1), \\ \partial_u H(0^+)\,, & \text{if } u = 0, \\ \partial_u H(1^-)\,, & \text{if } u = 1. \end{cases}$$

Let $T_t^\theta : \mathcal{S}_\theta \to \mathcal{S}_\theta$ be the semigroup associated to the PDE with the corresponding boundary conditions with $\alpha = \beta = 0$.

$\underline{\mathscr{F}luctuations\text{: } \theta = 1}$

the initial state?

- For each $N \in \mathbb{N}$, the measure μ_N is associated to a measurable profile $\rho_0 : [0,1] \to [0,1]$ (This is the same condition for hydrodynamics!).
- For $\rho_0^N(x) = \mathbb{E}_{\mu_N}[\eta_0(x)]$

$$\max_{x \in \Lambda_N} \left| \rho_0^N(x) - \rho_0(\tfrac{x}{N}) \right| \lesssim \frac{1}{N}.$$

- For

$$\varphi_0^N(x,y) = \mathbb{E}_{\mu_N}[\eta(x)\eta(y)] - \rho_0^N(x)\rho_0^N(y)$$

it holds that

$$\max_{1 \le x < y \le N-1} \left| \varphi_0^N(x,y) \right| \lesssim \frac{1}{N}.$$

Examples - initial measures:

- If for a given measurable profile $\rho_0 : [0,1] \to [0,1]$, we take μ_N as the Bernoulli product measure given by

$$\mu_N\{\eta : \eta(x) = 1\} = \rho_0(\tfrac{x}{N})$$

  then all the conditions above are true.

- If μ_{ss} is the stationary measure, then all the conditions above are true, by choosing the profile ρ_0 as the stationary profile $\bar{\rho}$ given above.

$\theta = 1$:

For each $N \geq 1$, let $\mathbb{Q}_N$ be the probability measure on $\mathcal{D}([0,T], \mathcal{S}'_\theta)$ induced by $\mathcal{Y}_\cdot^N$ and μ_N.

Theorem

The sequence of measures $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ is tight on $\mathcal{D}([0,T], \mathcal{S}'_\theta)$ and all limit points $\mathbb{Q}$ are p.m. concentrated on paths $\mathcal{Y}_\cdot$ satisfying

$$\mathcal{Y}_t(H) \;=\; \mathcal{Y}_0(T_t^1 H) + \mathcal{W}_t(H)\,,$$

for any $H \in \mathcal{S}_\theta$. Above $\mathcal{W}_t(H)$ is a mean zero Gaussian variable of variance $\int_0^t \|\nabla_1 T_{t-r}^1 H\|^2_{L^{2,1}(\rho_r)} dr$, where $\rho(t,u)$ is the solution of the hydrodynamic equation. Moreover, $\mathbb{E}_{\mathbb{Q}}\Big[\mathcal{Y}_0(H)\,\mathcal{W}_t(G)\Big] = 0$ for all $H, G \in \mathcal{S}_\theta$.

Theorem

If $\{\mathcal{Y}_0^N\}_{N\in\mathbb{N}}$ converges, as $N \to \infty$, to a mean-zero Gaussian field $\mathcal{Y}_0$ with covariance given on $H, G \in \mathcal{S}_\theta$ by

$$\mathbb{E}\left[\mathcal{Y}_0(H)\mathcal{Y}_0(G)\right] := \sigma(H,G),$$

then, the sequence $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ converges, as $N \to \infty$, to a generalized Ornstein-Uhlenbeck process, which is the formal solution of: $\partial_t \mathcal{Y}_t = \Delta_1 \mathcal{Y}_t dt + \sqrt{2\chi(\rho_t)}\nabla_1 \mathcal{W}_t$, where $\mathcal{W}_t$ is a space-time white noise of unit variance. As a consequence, the covariance of the limit field $\mathcal{Y}_t$ is given on $H, G \in \mathcal{S}_\theta$ by

$$\mathbb{E}\left[\mathcal{Y}_t(H)\mathcal{Y}_s(G)\right] = \sigma(T_t^1 H, T_s^1 G) + \int_0^s \langle \nabla_1 T_{t-r}^1 H, \nabla_1 T_{s-r}^1 G\rangle_{L^{2,1}(\rho_r)} dr.$$

$\mathscr{From}$ $\mathscr{Bernoulli}$ $(\theta = 1)$:

Corollary [Local Gibbs state]:

Fix a Lipschitz profile $\rho_0 : [0,1] \to [0,1]$ and suppose to start the process from a Bernoulli product measure given by $\mu_N\{\eta : \eta(x) = 1\} = \rho_0(\frac{x}{N})$. Then, the previous theorem remains in force and the covariance in this case is given on $H, G \in \mathcal{S}_\theta$ by

$$
\begin{aligned}
E\left[\mathcal{Y}_t(H)\mathcal{Y}_s(G)\right] \;=\; & \int_0^1 \chi(\rho_0(u))\, H(u)G(u)\, du \\
& + \int_0^s \langle \nabla_1 T^1_{t-r} H, \nabla_1 T^1_{s-r} G\rangle_{L^{2,1}(\rho_r)} dr,
\end{aligned}
$$

where $\rho(t,u)$ is the solution of the hydrodynamic equation with initial condition given by $\rho_0(\cdot)$.

$\mathcal{S}tationary$ $(\theta = 1)$:

Theorem

Suppose to start the process from μ_{ss} with $\alpha \neq \beta$. Then, $\mathcal{Y}^N$ converges to the centered Gaussian field $\mathcal{Y}$ with covariance given on $H, G \in \mathcal{S}_\theta$ by:

$$\mathbb{E}_{\mu_{ss}}[\mathcal{Y}(H)\mathcal{Y}(G)] = \int_0^1 \chi(\overline{\rho}(u))H(u)G(u)\,du - \left(\tfrac{\beta-\alpha}{3}\right)^2 \int_0^1 [(-\Delta_1)^{-1}H(u)]G(u)\,du$$

where $\overline{\rho}(\cdot)$ is the stationary solution of the PDE.

$\mathscr{Associated\ martingales}$:

Let $H : [0,1] \to \mathbb{R}$ be a test function and note that

$$\mathscr{M}_t^N(H) \;:=\; \mathscr{Y}_t^N(H) - \mathscr{Y}_0^N(H) - \int_0^t N^2 \mathscr{L}_N \mathscr{Y}_s^N(H)\, ds$$

is a martingale where

$$\begin{aligned} N^2 \mathscr{L}_N \mathscr{Y}_s^n(H) \;&=\; \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \Delta_N H(\tfrac{x}{N}) \bar{\eta}_{sN^2}(x) \\ &\quad + \sqrt{N}\left[\nabla_N^+ H(0) - H(\tfrac{1}{N})\right] \bar{\eta}_{sN^2}(1) \\ &\quad + \sqrt{N}\left[H(\tfrac{N-1}{N}) + \nabla_N^- H(1)\right] \bar{\eta}_{sN^2}(N-1). \end{aligned}$$

Note that the second term at the right hand side of the previous expression is $\mathscr{Y}_s^N(\Delta_N H)$. Above, we have used the notation

$$\nabla_N^+ H(x) \;=\; N\left[H(\tfrac{x+1}{N}) - H(\tfrac{x}{N})\right],\;\; \nabla_N^- H(x) \;=\; N\left[H(\tfrac{x}{N}) - H(\tfrac{x-1}{N})\right].$$

The correlation estimate:

For each $x, y \in V_N \;=\; \{(x,y)\,;\, x, y \in \mathbb{N},\, 0 < x < y < N\}$ and $t \in [0,T]$, let

$$\varphi_t^N(x,y) \;=\; \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)\eta_{tN^2}(y)] - \rho_t^N(x)\rho_t^N(y)\,,$$

and set $\varphi_t^N(x,y) = 0$, for $x = 0$ or $y = N$, we set

Proposition:

If

$$\max_{x,y \in V_N} |\varphi_0^N(x,y)| \;\lesssim\; \frac{1}{N},$$

then

$$\sup_{t \geq 0} \max_{(x,y) \in V_N} |\varphi_t^N(x,y)| \;\lesssim\; \frac{1}{N}.$$

$\underline{\mathscr{Fluctuations} : \theta \neq 1}$

$\theta \neq 1$:

$$
\begin{aligned}
N^2 \mathscr{L}_N \mathscr{Y}_s^N(H) = & \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \Delta_N H\left(\tfrac{x}{N}\right)\left(\eta_{sN^2}(x) - \rho_s^N(x)\right) \\
& + \sqrt{N}\nabla_N^+ H(0)\bar{\eta}_{sN^2}(1) - \sqrt{N}\nabla_N^- H(1)\bar{\eta}_{sN^2}(N-1) \\
& - \frac{N^{3/2}}{N^\theta} H\left(\tfrac{1}{N}\right)\bar{\eta}_{sN^2}(1) - \frac{N^{3/2}}{N^\theta} H\left(\tfrac{N-1}{N}\right)\bar{\eta}_{sN^2}(N-1).
\end{aligned}
$$

For $x \in \{1, N-1\}$ and $t \in [0,T]$ it holds

$$
\mathbb{E}_{\mu_N}\left[\left(\int_0^t C_N^\theta(\eta_{sN^2}(x) - \rho_s^N(x))\, ds\right)^2\right] \lesssim (C_N^\theta)^2 \tfrac{N^\theta}{N^2}.
$$

Apply last result with $C_N^\theta = \sqrt{N}\mathbf{1}_{\{\theta<1\}} + N^{3/2-\theta}\mathbf{1}_{\{\theta>1\}}$.

The initial measures:

We fix an initial profile $\rho_0 : [0,1] \to [0,1]$ which is measurable and of class C^6, and we assume that

$$\max_{x \in \Lambda_N} |\rho_0^N(x) - \rho_0(\tfrac{x}{N})| \lesssim \tfrac{1}{N}.$$

Moreover, we also assume that

$$\max_{y \in \Lambda_N} |\varphi_0^N(x,y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, \ \theta \leq 1, \\ \frac{1}{N}, \ \theta \geq 1, \end{cases} \qquad \text{for } x = 1, N-1,$$

and that

$$\max_{(x,y) \in V_N} |\varphi_0^N(x,y)| \lesssim \tfrac{1}{N}.$$

The correlation estimate:

Proposition:

If

$$\max_{y\in\Lambda_N} |\varphi_0^N(x,y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, & \theta \le 1, \\ \frac{1}{N}, & \theta \ge 1, \end{cases} \qquad \text{for } x = 1, N-1,$$

$$\max_{(x,y)\in V_N} |\varphi_0^N(x,y)| \lesssim \frac{1}{N},$$

then,

$$\sup_{t\ge 0} \max_{y\in\Lambda_N} |\varphi_t^N(x,y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, & \theta \le 1, \\ \frac{1}{N}, & \theta \ge 1, \end{cases} \qquad \text{for } x = 1, N-1\,,$$

$$\sup_{t\ge 0} \max_{(x,y)\in V_N} |\varphi_t^N(x,y)| \lesssim \frac{1}{N}.$$

Ingredients for correlations

Show that $\varphi_t^N(x,y)$ is solution of

$$\begin{cases} \partial_t \varphi_t^N(x,y) = N^2 \mathcal{A}_N^\theta \varphi_t^N(x,y) - (\nabla_N^+ \rho_t^N(x))^2 \delta_{y=x+1}, (x,y) \in V_N, \\ \varphi_t^N(x,y) = 0, (x,y) \in \partial V_N, \\ \varphi_0^N(x,y) = \mathbb{E}_{\mu_N}[\eta_0(x)\eta_0(y)] - \rho_0^N(x)\rho_0^N(y), (x,y) \in V_N \cup \partial V_N, \end{cases}$$

where $\mathcal{A}_N^\theta$ acts on $f : V_N \cup \partial V_N \to \mathbb{R}$ as

$$(\mathcal{A}_N^\theta f)(u) = \sum_{v \in V_N} c_N^\theta(u,v)[f(v) - f(u)],$$

and it is the infinitesimal generator of the RW in $V_N \cup \partial V_N$ which is absorbed at ∂V_N. Above,

$$c_N^\theta(u,v) = \begin{cases} 1, & \text{if } \|u-v\| = 1 \text{ and } u,v \in V_N, \\ N^{-\theta}, & \text{if } \|u-v\| = 1 \text{ and } u \in V_N,\ v \in \partial V_N, \\ 0, & \text{otherwise.} \end{cases}$$

Ingredients for correlations

Show that $\rho_t^N(\cdot)$ is a solution of

$$\begin{cases} \partial_t \rho_t^N(x) \;=\; (N^2 \mathfrak{B}_N^\theta \rho_t^N)(x)\,, \;\; x \in \Lambda_N\,, \;\; t \geq 0\,, \\ \rho_t^N(0) \;= \alpha\,, \rho_t^N(N) \;=\; \beta\,, \;\; t \geq 0\,, \end{cases}$$

where $\mathfrak{B}_N^\theta$ acts on $f : \Lambda_N \cup \{0, N\} \to \mathbb{R}$ as

$$(\mathfrak{B}_N^\theta f)(x) \;=\; \sum_{y=0}^{N} \xi_{x,y}^{N,\theta}(f(y) - f(x)), \quad \text{for } x \in \Lambda_N$$

and it is the infinitesimal generator of the RW in $\overline{\Lambda}_N$ which is absorbed at the points $\{0, N\}$. Above

$$\xi_{x,y}^{N,\theta} \;=\; \begin{cases} 1\,, & \text{if } \; |y - x| = 1 \text{ and } x, y \in \Lambda_N, \\ N^{-\theta}\,, & \text{if } \; x = 1, y = 0 \text{ and } x = N-1\,, y = N, \\ 0\,, & \text{otherwise.} \end{cases}$$

Ingredients for correlations:

The stationary solutions of the equations above are given by

$$\varphi_{ss}^N(x,y) = -\frac{(\alpha-\beta)^2(x+N^\theta-1)(N-y+N^\theta-1)}{(2N^\theta+N-2)^2(2N^\theta+N-3)}$$

and $\rho_{ss}^N(x) = \mathbb{E}_{\mu_{ss}}[\eta_{tN^2}(x)] = a_N x + b_N$, where

$$a_N = \frac{\beta-\alpha}{2N^\theta+(N-2)} \quad \text{and} \quad b_N = a_N(N^\theta-1)+\alpha.$$

The time spent by the 1-d RW at the points $x=1$ and $x=N-1$ is of order $O(\frac{N^\theta}{N^2})$ (good bound when $\theta<1$ but not when $\theta>1$). When $\theta>1$ we compare with the reflected RW and we prove that the time now is of order $O(\frac{1}{N})$. We need the same estimates in the 2-d setting for the time spent by the RW on the diagonal.

Equilibrium Fluctuations long range

Equilibrium fluctuations

The Bernoulli product measure on Ω_N

$$\nu_\rho^N(\eta \in \Omega_N : \eta(x) = 1) = \rho, \quad \forall x \in \Lambda_N,$$

with $\rho = \alpha = \beta$ is reversible.

The density fluctuation field $\mathcal{Y}_\cdot^N$ is the time-trajectory of linear functionals acting on functions $H \in \mathcal{S}_\theta$ as

$$\mathcal{Y}_t^N(H) \;=\; \frac{1}{\sqrt{N-1}} \sum_{x=1}^{N-1} H(\tfrac{x}{N}) \Big(\eta_{tN^2}(x) - \rho\Big)\,.$$

Space of test functions

$$\mathcal{S}_\theta = \begin{cases} \mathcal{S} & \text{if } \theta \le 2-\gamma; \\ \mathcal{S}_{Dir} & \text{if } 2-\gamma < \theta < 1; \\ \mathcal{S}_{Rob} & \text{if } \theta = 1; \\ \mathcal{S}_{Neu} & \text{if } \theta > 1; \end{cases}$$

$$\mathcal{S} := \Big\{ H \in C^\infty([0,1]) : H^{(i)}(0) = H^{(i)}(1) = 0,\ \forall i \in \mathbb{N} \Big\}$$

$$\mathcal{S}_{Dir} := \Big\{ H \in C^\infty([0,1]) : H^{(2i)}(0) = H^{(2i)}(1) = 0,\ \forall i \in \mathbb{N} \Big\}$$

$$\mathcal{S}_{Rob} := \Big\{ H \in C^\infty([0,1]) : H^{(2i+1)}(0) = -\frac{2m\kappa}{\sigma^2} H^{(2i)}(0), \\ H^{(2i+1)}(1) = \frac{2m\kappa}{\sigma^2} H^{(2i)}(1),\ \forall i \in \mathbb{N} \Big\}$$

$$\mathcal{S}_{Neu} := \Big\{ H \in C^\infty([0,1]) : H^{(2i+1)}(0) = H^{(2i+1)}(1) = 0,\ \forall i \in \mathbb{N} \Big\}$$

Result:

Proposition:

For $\gamma > 2$, the probability measure $Q_{\rho}^{\theta,N}$ on $\mathcal{D}([0,T],\mathcal{M}_+)$ associated to $\mathcal{Y}_{\cdot}^{N}$ converges as $N \to \infty$ to a probability measure concentrated on the formal solution of the SPDE

$$\partial_t \mathcal{Y}_t = \mathcal{A}_\theta \mathcal{Y}_t dt + \mathcal{B}_\theta dW_t,$$

associated to the two continuous martingales

$$M_t(H) = \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \int_0^t \mathcal{Y}_s(\mathcal{A}_\theta H) ds;$$
$$N_t(H) = M_t(H)^2 - t||H||_{L^2_{\theta,\rho}},$$

where $\mathcal{A}_\theta := \mathbf{1}_{\theta \geq 2-\gamma} \frac{\sigma^2}{2} \Delta + \mathbf{1}_{\theta \leq 2-\gamma} \kappa(r^+ + r^-)$,
$\mathcal{B}_\theta := \sqrt{2\chi(\rho)\sigma^2} \Big\{ \mathbf{1}_{\theta \geq 2-\gamma} \nabla + \mathbf{1}_{\theta \leq 2-\gamma} \sqrt{\frac{\kappa(r^- + r^+)}{2\sigma^2}} \Big\}$.

On the proof:

- Tightness, which implies the existence of limit points.
- Characterize the limit point by the use of Dynkin's martingale.
- Prove uniqueness of the solution of the martingale problems. (tricky part).
- For $\theta \geq 1$ the proof is similar to the nearest-neighbour case. Demanding: Other regimes.

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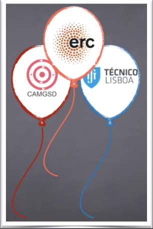
Thank you and...



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Thank you and...