Fluctuations of symmetric exclusion with open boundary

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Joint with

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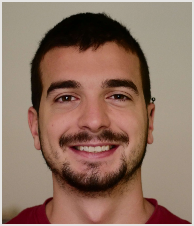


Joint with

Tertuliano Franco, Milton Jara, Otávio Menezes, Adriana Neumann

Outline of the mini-course:

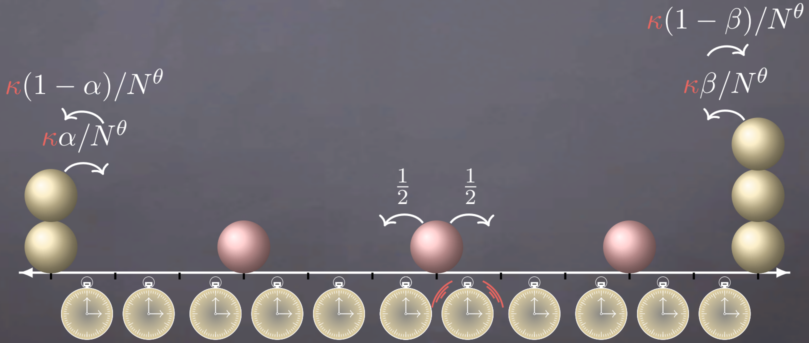
- ♣ We will analyse the fluctuations for an exclusion process in contact with stochastic reservoirs when jumps are:
  - ♣ Hydrodynamics (Lecture 1);
  - ♣ Fluctuations (Lecture 2).

Let us start with the simplest case: jumps to nearest-neighbors.

Now $\Lambda = [0,1]$ and $\Lambda_N = \{1, ..., N-1\}$. The state space of the Markov process is $\Omega_N = \{0,1\}^{\Lambda_N}$.

Lecture 1: Hydrodynamics

SSEP in contact with reservoirs



The dynamics:

- For $N \geq 1$ let $\Lambda_N = \{1, \ldots, N-1\}$.
- We denote the process by $\{\eta_t : t \geq 0\}$ which has state space $\Omega_N := \{0,1\}^{\Lambda_N}$.
- The infinitesimal generator $\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,b}$ is given on $f : \Omega_N \to \mathbb{R}$, by

$$(\mathcal{L}_{N,0} f)(\eta) = \sum_{x=1}^{N-2} \frac{1}{2}\Big(f(\eta^{x,x+1}) - f(\eta) \Big),$$

$$(\mathcal{L}_{N,b} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{x \in \{1, N-1\}} c_{r_x}(\eta(x)) \Big(f(\eta^x) - f(\eta) \Big),$$

where for $x = 1$ and $x = N-1$, $c_{r_x}(\eta(x)) = r_x(1-\eta(x)) + (1-r_x)\eta(x)$, $r_1 = \alpha$ and $r_{N-1} = \beta$.

Goal: analyse the impact of changing the strength of the reservoirs (by changing θ) on the macroscopic behavior of the system.

Invariant measures:

If $\alpha = \beta = \rho$ the Bernoulli product measures are invariant (equilibrium measures): $\nu_\rho(\eta : \eta(x) = 1) = \rho.$

If $\alpha \neq \beta$ the Bernoulli product measure is no longer invariant, but since we have a finite state irreducible Markov process there exists a UNIQUE invariant measure: the stationary measure (non-equilibrium) denoted by μ_{ss}.

By the matrix ansatz method one can get information about this measure. (Not in the long jumps case.)

Hydrodynamic Limit:

♣ **For $\eta \in \Omega_N$, let**

$$\pi_t^N(\eta, dq) = \frac{1}{N-1} \sum_{x=1}^{N-1} \eta_{tN^2}(x)\delta_{x/N}(dq),$$

be the *empirical measure. (Diffusive time scaling!)*

♣ **Assumption: fix $g : [0,1] \to [0,1]$ measurable and a sequence of probability measures $\{\mu_N\}_{N\geq 1}$ such that for every $H \in C([0,1])$,**

$$\tfrac{1}{N-1} \sum_{x=1}^{N-1} H(\tfrac{x}{N})\, \eta(x) \to_{N\to+\infty} \int_0^1 H(q)\, g(q) dq,$$

wrt μ_N. (μ_N is associated with $g(\cdot)$)

Hydrodynamic Limit:

♣ **Assumption: fix** $g : [0,1] \to [0,1]$ **measurable and a sequence of probability measures** $\{\mu_N\}_{N \geq 1}$ **such that for every** $H \in C([0,1])$**,**

$$\frac{1}{N-1} \sum_{x=1}^{N-1} H(\tfrac{x}{N})\, \eta(x) \to_{N \to +\infty} \int_0^1 H(q)\, g(q) dq,$$

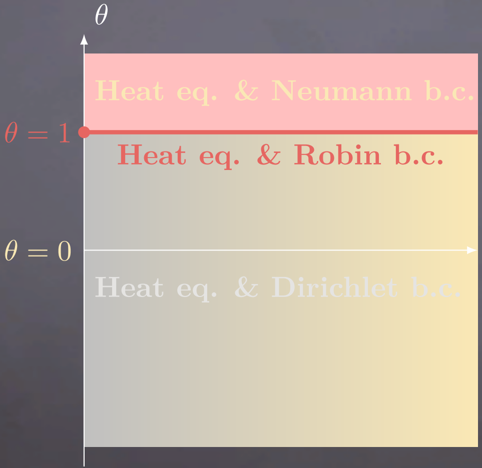
wrt μ_N**. (i.e.** $\pi_0^N(\eta, dq) \to_{N \to +\infty} g(q)dq$**)**

♣ **Then: for any** $t > 0$**,**

$$\pi_t^N(\eta, dq) \to_{N \to +\infty} \rho(t,q) dq,$$

wrt $\mu_N(t)$**, where** $\rho(t,q)$ **evolves according to a PDE, the hydrodynamic equation.**

Hydrodynamic eq. (Baldasso et al):



Heat equation:
$\partial_t \rho_t(q) = \frac{1}{2}\partial_q^2 \rho_t(q).$

- $\theta > 1$ **Neumann b.c.:** $\partial_q \rho_t(0) = \partial_q \rho_t(1) = 0.$
- $\theta = 1$ **Robin b.c.:** $\partial_q \rho_t(0) = \kappa(\rho_t(0) - \alpha),$ $\partial_q \rho_t(1) = \kappa(\beta - \rho_t(1)).$
- $\theta < 1$ **Dirichlet b.c.:** $\rho_t(0) = \alpha,\ \rho_t(1) = \beta.$

Hydrostatic Limit:

Theorem: Let μ_{ss} be the stationary measure for the process $\{\eta_t\}_{t\geq 0}$. Then, μ_{ss} is associated to $\bar{\rho} : [0,1] \to [0,1]$ given on $q \in (0,1)$ by

$$\bar{\rho}(q) = \begin{cases} (\beta - \alpha)q + \alpha\,;\, \theta < 1, \\ \frac{\kappa(\beta-\alpha)}{2+\kappa} q + \alpha + \frac{\beta-\alpha}{2+\kappa}\,;\, \theta = 1, \\ \frac{\beta+\alpha}{2}\,;\, \theta > 1. \end{cases}$$

$\bar{\rho}(\cdot)$ is a stationary solution of the hydrodynamic equation.

The proof:

Proof of the results?

Two things to do:

- ♣ Tightness of $\mathbb{Q}_N$, where $\mathbb{Q}_N$ is induced by $\mathbb{P}_{\mu_N}$ and the map

$$\pi_{\cdot}^N : \mathcal{D}([0,T],\Omega_N) \longrightarrow \mathcal{D}([0,T],\mathcal{M}_+)$$

- ♣ Characterization of limit points: limit points are concentrated on trajectories of measures that are absolutely continuous wrt the Lebesgue measure and the density is a weak solution of the corresponding PDE:

$$\mathbb{Q}(\pi_{\cdot} : \pi_t(dq) = \rho(t,q)dq \text{ and } \rho_t(q) \text{ is solution to the PDE}) = 1.$$

Let us focus on last item.

The notion of weak solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the HEDBC if:

♣ $\rho \in L^2(0,T;\mathcal{H}^1)$;

♣ ρ satisfies the weak formulation:

$$\begin{aligned}
&\int_0^1 \rho_t(q) H_t(q) - g(q) H_0(q)\, dq \\
&- \int_0^t \int_0^1 \rho_s(q) \Big(\frac{1}{2}\partial_q^2 + \partial_s\Big) H_s(q)\, ds\, dq \\
&+ \frac{1}{2} \int_0^t \beta \partial_q H_s(1) - \alpha \partial_q H_s(0)\, ds = 0,
\end{aligned}$$

for all $t \in [0,T]$ and any function $H \in C_0^{1,2}([0,T] \times [0,1])$.

Another notion of solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the HEDBC if:

♣ $\rho \in L^2(0,T;\mathcal{H}^1)$;

♣ ρ satisfies the weak formulation:

$$\int_0^1 \rho_t(q) H_t(q)\, dq - \int_0^1 g(q) H_0(q)\, dq$$
$$- \int_0^t \int_0^1 \rho_s(q) \Big(\frac{1}{2}\partial_q^2 + \partial_s \Big) H_s(q)\, ds\, dq = 0,$$

for all $t \in [0,T]$ and any function $H \in C_c^{1,2}([0,T] \times [0,1])$;

♣ $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$, for $t \in (0,T]$.

The notion of weak solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the heat equation with Robin b.c. if:

♣ $\rho \in L^2(0,T;\mathcal{H}^1)$,

♣ ρ satisfies the weak formulation:

$$\begin{aligned}\int_0^1 \rho_t(q)H_t(q)dq - \int_0^1 g(q)H_0(q)\,dq \\ - \int_0^t \int_0^1 \rho_s(q)\Big(\frac{1}{2}\partial_q^2 + \partial_s\Big)H_s(q)\,ds\,dq \\ + \frac{1}{2}\int_0^t \{\rho_s(1)\partial_q H_s(1) - \rho_s(0)\partial_q H_s(0)\}\,ds \\ - \frac{\kappa}{2}\int_0^t \{H_s(0)(\alpha - \rho_s(0)) + H_s(1)(\beta - \rho_s(1))\}\,ds = 0,\end{aligned}$$

for all $t \in [0,T]$ and any function $H \in C^{1,2}([0,T] \times [0,1])$.

Characterizing limit points:

Dynkin's formula: Let $\{\eta_t\}_{t\geq 0}$ be a Markov process with generator $\mathcal{L}$ and with countable state space E. Let $F : \mathbb{R}^+ \times E \to \mathbb{R}$ be a bounded function such that

- $\forall \eta \in E, F(\cdot, \eta) \in C^2(\mathbb{R}^+)$,
- there exists a finite constant C, such that $\sup_{(s,\eta)} |\partial_s^j F(s,\eta)| \leq C$, for $j = 1, 2$.

For $t \geq 0$, let

$$M_t^F = F(t,\eta_t) - F(0,\eta_0) - \int_0^t (\partial_s + \mathcal{L})F(s,\eta_s)ds.$$

Then, $\{M_t^F\}_{t\geq 0}$ is a martingale wrt $\mathcal{F}_s = \sigma(\eta_s; s \leq t)$.

Characterizing limit points:

Let us fix a test function $H : [0,1] \to \mathbb{R}$ and apply Dynkin's formula with

$$F(t, \eta_t) = \langle \pi_t^N, H \rangle = \frac{1}{N-1} \sum_{x=1}^{N-1} \eta_{tN^2}(x) H\Big(\tfrac{x}{N}\Big).$$

Note that F does not depend on time only through the process $\eta_\cdot$. A simple computation shows that

$$\begin{aligned} N^2 \mathcal{L}_N \langle \pi_s^N, H \rangle = {} & \langle \pi_s^N, \frac{1}{2} \Delta_N H \rangle \\ & + \frac{1}{2} \nabla_N^+ H(0) \eta_{sN^2}(1) - \frac{1}{2} \nabla_N^- H(1) \eta_{sN^2}(N-1) \\ & + \kappa N^{1-\theta} H\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) \\ & + \kappa N^{1-\theta} H\Big(\tfrac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1)) \end{aligned}$$

$\theta \in [0, 1)$:

Take a function $H : [0, 1] \to \mathbb{R}$ such that $H(0) = H(1) = 0$ and then we get

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H\rangle ds$$
$$- \frac{1}{2}\int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1)ds + O(N^{-\theta}).$$

If we can replace $\eta_{sN^2}(1)$ by α and $\eta_{sN^2}(N-1)$ by β (this will be made rigorous ahead but only works for $\theta < 1$!) then above we have

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H\rangle ds$$
$$- \frac{1}{2}\int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}).$$

Compare with the PDE (note that H does not depend on time).

Still $\theta \in [0,1)$:

Take the expectation above to get

$$\frac{1}{N}\sum_{x=1}^{N-1} H\Big(\tfrac{x}{N}\Big)\Big(\rho_t^N(x) - \rho_0^N(x)\Big) - \int_0^t \frac{1}{N}\sum_{x=1}^{N-1}\frac{1}{2}\Delta_N H\Big(\tfrac{x}{N}\Big)\rho_s^N(x)ds$$
$$-\frac{1}{2}\int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}) = 0.$$

Assume that $\rho_t^N(x) \sim \rho_t(x/N)$ and take the limit in N to get

$$\int_0^1 \rho_t(q)H(q) - \rho_0(q)H(q)dq - \int_0^t\int_0^1 \frac{1}{2}\partial_q^2 H(q)\rho_s(q)dqds$$
$$-\frac{1}{2}\int_0^t \partial_q H(0)\alpha - \partial_q H(1)\beta ds = 0$$

Compare with the PDE (note that H does not depend on time).

$\theta < 0$:

Recall that the previous error blows up when $N \to \infty$. So now, we take a function $H : [0,1] \to \mathbb{R}$ with compact support and then we get

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H \rangle ds.$$

Again compare with the PDE but note that H does not depend on time.
In this case we do not see the Dirichlet boundary conditions and we need extra results to conclude.

$\theta = 1$:

Now, we take a function $H : [0,1] \to \mathbb{R}$ and we get

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H\rangle ds$$
$$- \frac{1}{2}\int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1) ds$$
$$- \frac{\kappa}{2}\int_0^t H\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) + H\Big(\tfrac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1)) ds.$$

If we can replace $\eta_{sN^2}(1)$ (resp. $\eta_{sN^2}(N-1)$) by its average in a box around 1 (resp. $N-1$) (this works for any $\theta \geq 1$):

$$\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) := \frac{1}{\epsilon N}\sum_{x=1}^{1+\epsilon N} \eta_{sN^2}(x), \quad \overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) := \frac{1}{\epsilon N}\sum_{x=N-1}^{N-1-\epsilon N} \eta_{sN^2}(x)$$

and noting that $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) \sim \rho_s(0)$ (resp. $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1) \sim \rho_s(1)$) we would get the terms in the PDE (compare).

$\theta > 1$:

Again we take a function $H : [0,1] \to \mathbb{R}$ and in this case the terms from the boundary vanish. So we get

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H\rangle ds$$
$$- \frac{1}{2}\int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1)ds + O(N^{1-\theta}).$$

As above, if we can replace $\eta_{sN^2}(1)$ (resp. $\eta_{sN^2}(N-1)$) by its average in a box around 1 (resp. $N-1$) and noting that $\overrightarrow{\eta}^{\epsilon N}_{sn^2}(1) \sim \rho_s(0)$ (resp. $\overrightarrow{\eta}^{\epsilon N}_{sN^2}(N-1) \sim \rho_s(1)$) we would get the terms in the PDE (compare).

Keystone ingredients: replacement lemmas

Recall that we need to prove that

For any $t > 0$, we have that:

- for $\theta < 1$

$$\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t (\eta_{sN^2}(1) - \alpha)\, ds\right|\right] = 0;$$

- for $\theta \geq 1$

$$\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t (\eta_{sN^2}(1) - \overrightarrow{\eta}^{\epsilon N}_{sN^2}(1))\, ds\right|\right] = 0;$$

and a similar result for $N-1$.

The empirical profile:

Fix an initial measure μ_N in Ω_N. For $x \in \Lambda_N$ and $t \geq 0$, let

$$\rho_t^N(x) \;=\; \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]\,.$$

We extend this definition to the boundary by setting

$$\rho_t^N(0) \;=\; \alpha \text{ and } \rho_t^N(N) \;=\; \beta\,, \text{ for all } t \geq 0\,.$$

A simple computation shows that $\rho_t^N(\cdot)$ is a solution of

$$\partial_t \rho_t^N(x) \;=\; N^2(\mathcal{B}_N \rho_t^N)(x)\,, \quad x \in \Lambda_N\,, \quad t \geq 0$$

where the operator $\mathcal{B}_N$ acts on functions $f : \Lambda_N \cup \{0, N\} \to \mathbb{R}$ as

$$\begin{aligned}
&N^2(\mathcal{B}_N f)(x) \;=\; \Delta_N f(x), \quad \text{for } x \in \{2, \cdots, N-2\},\\
&N^2(\mathcal{B}_N f)(1) = N^2(f(2) - f(1)) + \tfrac{\kappa N^2}{N^\theta}(f(0) - f(1)),\\
&N^2(\mathcal{B}_N f)(N\!-\!1) = N^2(f(N\!-\!2) - f(N\!-\!1)) + \tfrac{\kappa N^2}{N^\theta}(f(N) - f(N\!-\!1)).
\end{aligned}$$

Stationary empirical profile:

The stationary solution of the previous equation is given by

$$\rho_{ss}^N(x) = \mathbb{E}_{\mu_{ss}}[\eta_{tN^2}(x)] = a_N x + b_N$$

where $a_N = \frac{\kappa(\beta-\alpha)}{2N^\theta+\kappa(N-2)}$ and $b_N = a_N(\frac{N^\theta}{\kappa} - 1) + \alpha$, so that

$$\lim_{N\to\infty} \max_{x\in\Lambda_N} \left|\rho_{ss}^N(x) - \bar{\rho}(\tfrac{x}{N})\right| = 0$$

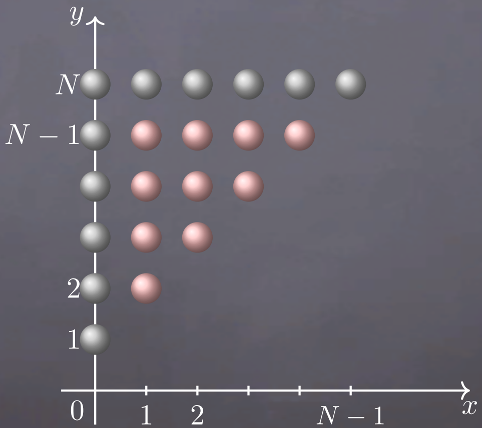
where

$$\bar{\rho}(q) = \begin{cases} (\beta-\alpha)q+\alpha\,;\,\theta<1, \\ \frac{\kappa(\beta-\alpha)}{2+\kappa}q+\alpha+\frac{\beta-\alpha}{2+\kappa}\,;\,\theta=1, \\ \frac{\beta+\alpha}{2}\,;\,\theta>1, \end{cases}$$

is a stationary solution of the hydrodynamic equation.

Stationary correlations:

Let $V_N = \{(x, y) \in \{0, \cdots, N\}^2 : 0 < x < y < N\}$, and its boundary $\partial V_N = \{(x, y) \in \{0, \cdots, N\}^2 : x = 0 \text{ or } y = N\}$.



Stationary correlations:

For $x < y \in V_N$, let $\varphi_t^N(x,y)$ the two point correlation function between the occupation sites at $x < y \in V_N$ is defined by

$$\varphi_t^N(x,y) = \mathbb{E}_{\mu_N}[(\eta_{tN^2}(x) - \rho_t^N(x))(\eta_{tN^2}(y) - \rho_t^N(y))].$$

Doing some simple, but long, computations we see that φ_t^N is a solution of

$$\begin{cases} \partial_s \varphi_s(x,y) = \Delta_V^N \varphi_s(x,y) + g_s^N(x,y) + f_s^N(x,y)\,, & (x,y) \in V_N, \\ \varphi_s(x,y) = 0\,, & (x,y) \in \partial V_N, \end{cases}$$

where the discrete laplacian $\Delta_{V_N}^N : V_N \cup \partial V_N \to \mathbb{R}$ is defined by

$$\begin{cases} (\Delta_V^N f)(x,y) \;=\; N^2(f(x+1,y) + f(x-1,y) + f(x,y-1) \\ \qquad\qquad\qquad +f(x,y+1) - 4f(x,y))\,, \quad \text{for } |x-y| > 1\,, \\ (\Delta_V^N f)(x,x+1) = N^2(f(x-1,x+1) + f(x,x+2) - 2f(x,x+1)) \\ (\Delta_V^N f)(x,y) = 0, \quad \text{if } (x,y) \in \partial V_N\,. \end{cases}$$

Stationary correlations:

Above

$$g_t^N(x,y) = -(\nabla_N^+ \rho_t^N(x))^2 \delta_{y=x+1},$$
$$\nabla_N^+ \rho_t^N(x) = N(\rho_t^N(x+1) - \rho_t^N(x))$$
$$f_s^N(x,y) = \Big(N^2 - \frac{N^2}{N^\theta}\Big)\varphi_t^N(x,y)\delta_{\{|y-x|=1,\ x=1 \text{ or } y=N-1\}}.$$

From simple, but long, computations we conclude that

$$\varphi_{ss}^N(x,y) = -\frac{(\alpha-\beta)^2(x+N^\theta-1)(N-y+N^\theta-1)}{(2N^\theta+N-2)^2(2N^\theta+N-3)}. \qquad (1)$$

from where it follows that

$$\max_{x<y} |\varphi_{ss}^N(x,y)| = \begin{cases} O\Big(\frac{N^\theta}{N^2}\Big),\ \theta<1, \\ O\Big(\frac{1}{N}\Big),\ \theta=1, \\ O\Big(\frac{1}{N^\theta}\Big),\ \theta>1, \end{cases} \quad \to_{N\to\infty} 0. \qquad (2)$$