Products of random matrices and the statistical mechanics of disordered systems

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Straight to the main issue

To be very concrete:

the talk is about the product of IID random matrices

$$M_n^\varepsilon := \begin{pmatrix} 1 & \varepsilon \\ \varepsilon Z_n & Z_n \end{pmatrix}$$

where $\varepsilon \in (-1, 1)$ and $\{Z_n\}_{n=1,2,\ldots}$ is an IID sequence of positive random variables with $\log Z_1 \in L^1$.

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More precisely we aim at the $\varepsilon \to 0$ behavior of the Lyapunov exponent

$$\mathcal{L}(\varepsilon) := \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \log \left\| M_n^\varepsilon M_{n-1}^\varepsilon \cdots M_1^\varepsilon \right\|$$

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where $\| \cdot \|$ is an arbitrary matrix norm.

Simple exercise: $\mathcal{L}(0) = \max(0, \mathbb{E} \log Z)$, but $\varepsilon = 0$ looks pathological...

Statistical mechanics origin of the question

Key reference for us

[DH83] B. Derrida and H. J. Hilhorst
Singular behaviour of certain infinite products of random 2×2 *matrices*
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$\hookrightarrow$ As it will be clear, we *exploit* [DH83] well beyond extracting from it the statmech motivation

Statistical mechanics origin of the question

Ising model with disordered external field: $d = 1$, $\{h_j\}_{j=1,2,\ldots}$ IID

$$\mathcal{H}_N(\sigma) := -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - \sum_{i=1}^{N} h_i \sigma_i$$

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The Gibbs measure $\exp(-\mathcal{H}_N(\sigma))/\mathcal{Z}_N$ with

$$\mathcal{Z}_N = \exp\left(\sum_{i=1}^{N} h_i + NJ\right) \mathrm{Tr} \prod_{i=1}^{N} \begin{pmatrix} 1 & e^{-2J} \\ e^{-2J} e^{-2h_i} & e^{-2h_i} \end{pmatrix}$$

and the matrix is of the desired form ($\varepsilon = e^{-2J}$ and $Z_i = e^{-2h_i}$) and the free energy density is the leading Lyapunov exponent apart for a trivial additive constant.

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The $\varepsilon \searrow 0$ limit corresponds to the fixed disorder – strong ferromagnetic interaction limit.

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- Quantum Ising chain with disordered external field and/or disordered interactions: mapping with Ising 2d with columnar disorder.
- Prototype for general models with 1d disorder: $\mathbb{P}(Z > 1) > 0$ and $\mathbb{P}(Z < 1) > 0$ is the signature of *frustration*.

Toward the result

Fundamental quantities

$$\mathcal{L}(\varepsilon) := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\log \left\| M_n^\varepsilon M_{n-1}^\varepsilon \cdots M_1^\varepsilon \right\| \quad \text{with } M_j^\varepsilon := \begin{pmatrix} 1 & \varepsilon \\ \varepsilon Z_j & Z_j \end{pmatrix}$$

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Existence of the limit and a number of facts like for example

$$\mathcal{L}(\varepsilon) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\log \left(M_n^\varepsilon M_{n-1}^\varepsilon \cdots M_1^\varepsilon \right)_{1,1}$$

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Important results: [Ruelle 79] $\mathcal{L}(\cdot)$ is analytic on $(-1,1) \setminus \{0\}$ and [Le Page 89] $\mathcal{L}(\cdot)$ is Hölder C^0 on $(-1,1)$ if $\mathbb{E}[\log Z] \neq 0$.

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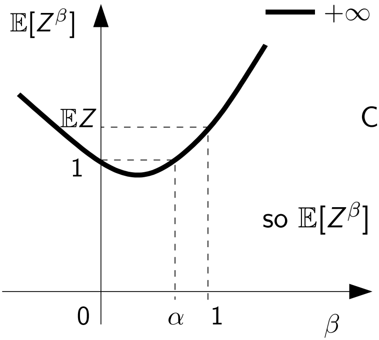

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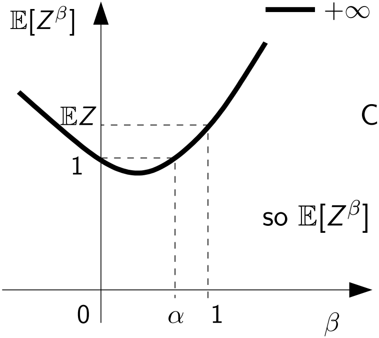

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so $\mathbb{E}[Z^\beta] = 1$ for $\beta = 0$ and $\beta = \alpha \in (0,1)$

$\alpha \in \mathbb{R}$ (or may not exist) but case $\alpha \leq 0$ is equivalent to $\alpha \geq 0$:

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon Z & Z \end{pmatrix} = Z \begin{pmatrix} 1/Z & \varepsilon/Z \\ \varepsilon & 1 \end{pmatrix}$$

Expected results (mostly [DH83])

For $\varepsilon \to 0$:

- If $\alpha \in (0,1)$ then

$$\mathcal{L}(\varepsilon) \sim C|\varepsilon|^{2\alpha}\,,$$

  with $C > 0$ semi-explicit.

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$$\mathcal{L}(\varepsilon) = c_1\varepsilon^2 + \ldots + c_{\lfloor\alpha\rfloor}\varepsilon^{2\lfloor\alpha\rfloor} + C|\varepsilon|^{2\alpha} + o\left(|\varepsilon|^{2\alpha}\right)$$
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Need conditions[DH83]: example of $Z \in \{0, z\}$ such that $\mathcal{L}(\varepsilon) \sim H(\log(1/|\varepsilon|)|\varepsilon|^{2\alpha}$, with $H(\cdot)$ periodic.

Mathematical results

Theorem (Genovese, G., Greenblatt 2017)

Assume $\alpha \in (0,1)$ *and*

1. *the support of the law of* Z *is bounded and bounded away from zero*
2. Z *has a* C^1 *density.*

Then there exists $C > 0$ *(DH83 expression) and* $\varkappa > 0$ *(explicit) s.t.*

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Theorem (Havret 2020)

Assume $\alpha \geq 1$ *and other mild conditions on Z. Then*

$$\mathcal{L}(\varepsilon) \;=\; c_1\varepsilon^2 + \ldots + c_{\lfloor\alpha\rfloor}\varepsilon^{2\lfloor\alpha\rfloor} + \mathrm{Rest}(\varepsilon)$$

with upper and lower bounds on $\mathrm{Rest}(\varepsilon) = o(\varepsilon^{2\lfloor\alpha\rfloor})$

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Theorem (G. and Greenblatt 2021)

Assume $\alpha = 0$ *and*

- $\mathbb{E}[Z^\delta] < \infty$ *for* δ *in neighborhood of* 0*;*
- Z *has a density and the density of* $\log Z$ *is uniformly Hölder* C^0*.*

Then there exist $\kappa_1 > 0$*,* $\kappa_2 \in \mathbb{R}$ *and* $\eta \in (0,1)$ *such that, for* $\varepsilon \to 0$*,*

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- Similar claim in [Nieuwenhuizen, Luck 86] assuming a special choice of law of Z without density, or with discontinuous densities (where one can push certain transform computations).
- [Derrida, priv. comm.]: [DH83] approach applies.

A formula for $\mathcal{L}(\varepsilon)$

Classical (Furstenberg) representation formula for the Lyapunov exponent in terms of the invariant probability of the Markov chain

$$\widehat{x}, \widehat{M_1^\varepsilon x}, \widehat{M_2^\varepsilon M_1^\varepsilon x}, \ldots$$

where $x \in \mathbb{R}^2$ (we can choose it in $\mathbb{R}^2_+$) and $\hat{x} = x/\|x\|$.

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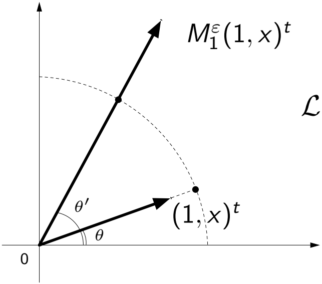

$$\mathcal{L}(\varepsilon) = \int \mathbb{E} \log \left(\frac{\left(M_1^\varepsilon v\right)_1}{v_1} \right) m_\varepsilon(\,\mathrm{d}v)$$

m_ε is the invariant probability

A formula for $\mathcal{L}(\varepsilon)$

We compute for $x > 0$

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon Z & Z \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 + \varepsilon x \\ Z(\varepsilon + x) \end{pmatrix}$$

so

$$\tan(\theta) = x \mapsto \tan(\theta') = Z \frac{\varepsilon + x}{1 + \varepsilon x} = Z \frac{\varepsilon + \tan(\theta)}{1 + \varepsilon \tan(\theta)}$$

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but we prefer to work with the variable $\sigma = \varepsilon \tan(\theta)$ so

$$\mathcal{L}(\varepsilon) = \int_0^{\infty} \log\left(1 + \sigma\right) m_\varepsilon^{(1)}(\,\mathrm{d}\sigma) \quad \text{with} \quad \sigma \overset{T_\varepsilon}{\mapsto} Z \frac{\varepsilon^2 + \sigma}{1 + \sigma}$$

Can we "find" the invariant probability $m_\varepsilon = m_\varepsilon^{(1)}$?

The MC $\sigma_1, \sigma_2, \ldots$ on $(0, \infty)$ defined by

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is very well behaved under mild hypotheses on Z (positive recurrent).

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$$\mathbb{E}\log Z < 0 \implies \prod_{j=1}^{\infty} Z_j \stackrel{\text{a.s.}}{=} 0$$

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one can extract $\mathcal{L}(\varepsilon) \longrightarrow 0$. Not enough!

The [DH83] idea

A two scale analysis ($\mathbb{E}\log Z < 0$):

- Regime I (away from 0): the random transformation is

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$$\widetilde{T}_\varepsilon(s) = Z\frac{1+s}{1+\varepsilon^2 s} \quad \text{and} \quad \widetilde{T}_0(s) = Z(1+s)$$

  and $\widetilde{T}_0$ has a unique invariant probability (density).

The [DH83] idea

A two scale analysis ($\mathbb{E} \log Z < 0$):

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DH83: piece together these two solutions, normalize, and compute!

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In [GGG17] we introduced a family of norms $|||\cdot|||_\beta$ with the property that if ν_1 and ν_2 are probabilities then

$$|||T_\varepsilon(\nu_1 - \nu_2)|||_\beta \le q_\beta |||\nu_1 - \nu_2|||_\beta \quad \text{with } q_\beta = \mathbb{E}[Z^\beta] < 1$$

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Hence

$$|||m_\varepsilon - \gamma_\varepsilon|||_\beta \le c_\beta |||T_\varepsilon\gamma_\varepsilon - \gamma_\varepsilon|||_\beta \qquad \left(c_\beta = (1 - q_\beta)^{-1}\right)$$

What to do if $\alpha = 0$? (Sketch of proof)

$|||\cdot|||_0$ is well defined, actually for m_1 and m_2 a probabilities

$$|||m_1 - m_2|||_0 = \int_0^\infty |G_{m_1}(t) - G_{m_2}(t)| \frac{\mathrm{d}t}{t}$$

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Problem is: the contractive constant $q_0 = \mathbb{E}[Z^0] = 1$ and $c_0 = \infty$!

Change of variables (and perspective): work with $X_j := \log \sigma_j \in \mathbb{R}$, so $X_{j+1} = \log Z_j + f_\varepsilon(X_j)$ with

$$f_\varepsilon : x \mapsto x + \log Z + \log \left(\frac{1 + \varepsilon e^{-x}}{1 + \varepsilon e^x} \right)$$

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New Markov process on $\mathbb{R}$: $X_{j+1} = \log Z_j + f_\varepsilon(X_j)$ with

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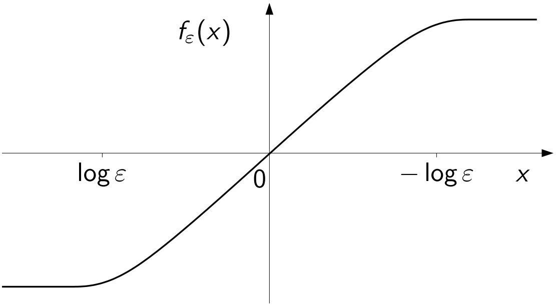

So (X_j) is a walk with centered increments on which a strong repulsion acts when it attempts leaving $[\log \varepsilon, -\log \varepsilon]$.

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First approximation

$$\gamma_\varepsilon(x) \stackrel{?}{=} \frac{1}{2\log(1/\varepsilon)} \mathbf{1}_{[\log\varepsilon, \log(1/\varepsilon)]}(x)$$

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Recenter the process on $\log \varepsilon$ and on $-\log \varepsilon$ (qualitatively symmetric problems): the $\varepsilon \searrow 0$ Markov chain is a well known walk with one barrier key tool in the analysis of the Sinai RWRE [Kesten, Kozlov, Spitzer,. . .] (much studied also as critical case of random affine iterations [Babillot, Bougerol, Elie, Brofferio, Buraczewski, Damek]). The one barrier walk is a null recurrent processes.

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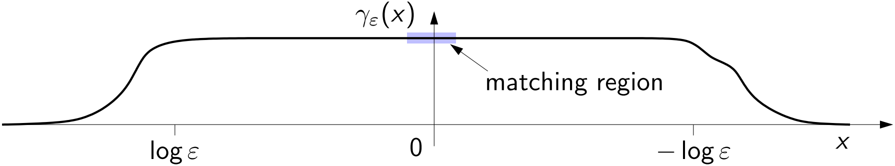

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We still need to circumvent the lack of contraction.

Recover a *micro-contraction* by exploiting the structure of the (X_j) process at $\varepsilon > 0$: we show that for $c > 2$

$$|||m_\varepsilon - \gamma_\varepsilon|||_0 \leq (\log(1/\varepsilon))^c |||T_\varepsilon \gamma_\varepsilon - \gamma_\varepsilon|||_0 = O\left((\log(1/\varepsilon))^c \varepsilon^a\right)$$

which largely suffices.

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$$\mathcal{L}(\varepsilon) = \frac{\kappa_1}{\log(1/\varepsilon)} \sum_{j=0}^{\infty} \left(\frac{-\kappa_2}{\log(1/\varepsilon)} \right)^j + O\left(\varepsilon^a\right)$$

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Conclusions and perspectives

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- Lots of room in the last estimate... and the result itself

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  is not the minimal result one is after.
- Expected (?) that

$$\mathcal{L}(\varepsilon) \sim \frac{\kappa_1}{\log(1/\varepsilon)}$$

  holds under much weaker conditions (e.g., support of Z spans $(0, \infty)$?) However our tools really do not get there: difficulties both in building γ_ε and showing that it is close to m_ε.