Hydrodynamics for a system of inhomogeneous hard rods

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We study a one-dimensional system of **inhomogeneous hard rods** interacting inertially between collisions.

- A rod is a three dimensional point (y, v, r) with a certain position $y \in \mathbb{R}$, traveling speed $v \in \mathbb{R}$ and length $r \in \mathbb{R}_+$
- The state space is $\mathbb{R}^2 \times \mathbb{R}_+$ or $\mathbb{R}^3$ if we allow negative size
- Configuration denoted by $Y \subset \mathbb{R}^3$
- $\mathfrak{Y}$ set of hard rod configurations such that
  - rods do not intersect, i.e. $(y_1, y_1 + r_1) \cap (y_2, y_2 + r_2) = \emptyset$
  - finite number of rods, i.e. $\sharp \{(y, v, r) \in Y \ : \ a \leq y \leq b\} < \infty$

Given the initial condition, hard rods evolve deterministically: what happen when they collide?

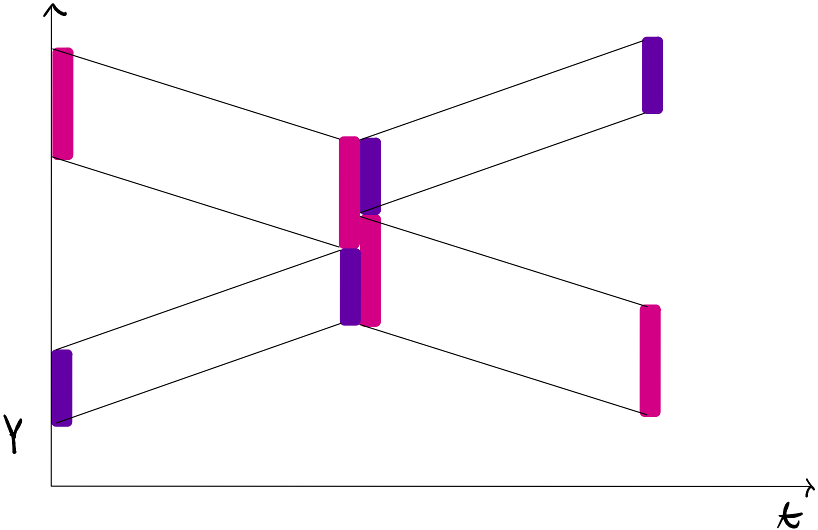
Hard rods evolution

Consider the two rods (y_1, v_1, r_1) and (y_2, v_2, r_2).
If at time t^- they have positions that satisfy $y_2 = y_1 + r_1$ then at time t they exchange their order by a shift in the direction of the other rod.
Namely,

Before collision at time t^-	After collision at time t
$(\boldsymbol{y_1}, v_1, r_1)$ and $(\boldsymbol{y_2}, v_2, r_2)$	$(\boldsymbol{y_1} + \boldsymbol{r_2}, v_1, r_1)$ and $(\boldsymbol{y_2} - \boldsymbol{r_1}, v_2, r_2)$

In other words, two rods next to each other swap their positions and keep their original speeds.

Collision rule



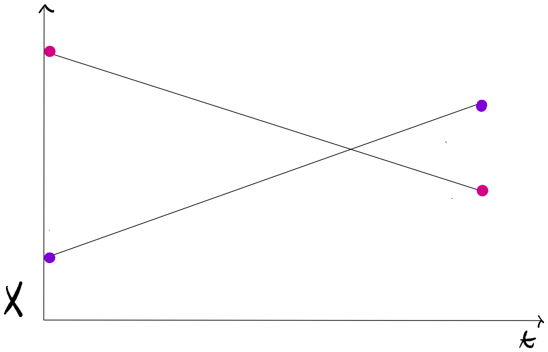
Incomplete backgrounds for hard rods

- Jepsen 1964
- Sinai 1972
- Aizemann Goldstein and Lebowitz 1975
- Boldrighini, Dobrushin and Sukhov 1982
- Spohn 1991
- Boldrighini and Sukhov 1997
- Doyon, Yoshimura and Caux 2017

Ideal gas evolution

A **particle** of ideal gas is a three dimensional point (x, v, r).
We denote $X \subset \mathbb{R}^3$ a free gas configuration.
The dynamics is described by the operator T_t

$$T_t(X) := \left\{(x + vt, v, r) \in \mathbb{R}^3 : \ (x, v, r) \in X\right\}$$



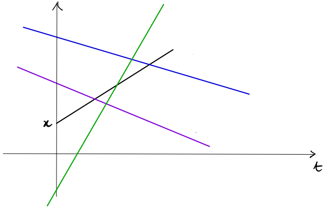
Length flow and mass of X

The **mass** and the **signed mass** between $a, b \in \mathbb{R}$ are

$$m(X) := \sum_{(x,v,r)\in X} r \qquad m_a^b(X) := \begin{cases} m\left((x,v,r) \in X\right) & \text{if } a \le x < b \\ -m\left((x,v,r) \in X\right) & \text{if } b \le x < a \\ 0 \text{ if } a = b \end{cases}$$

The **length flow** is

$$\begin{aligned} \mathrm{j}(x,v,t) &:= m(\text{particles with velocity} < v) - m(\text{particles with velocity} > v) \\ &= \mathrm{j}^+(x,v,t) - \mathrm{j}^-(x,v,t) \end{aligned}$$



$\mathrm{j} = r + r - r$

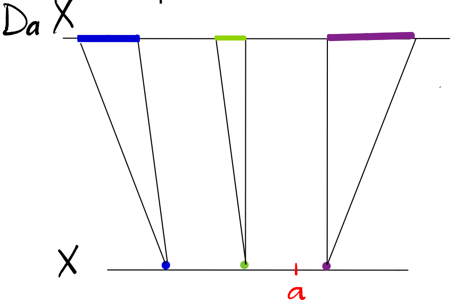
We will consider configurations X with finite flows:

$$\mathfrak{X} := \{X \subset \mathbb{R}^3 : \mathrm{j}^+(x,v,t) < \infty, \quad \mathrm{j}^-(x,v,t) < \infty, \text{ for } x,v,t \in \mathbb{R}\}$$

The key ingredients to show the results is to describe the **hard rods evolution** via the **free gas evolution**. This is done using two maps.

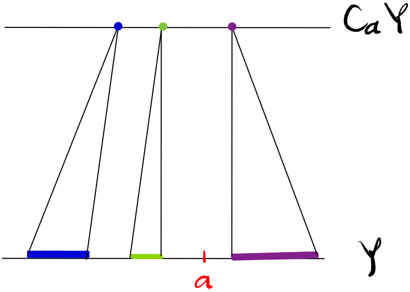
Dilation map

D_a describes the dilation of a free gas configuration X around a point $a \in \mathbb{R}$



Contraction map

C_a describes the contraction of a hard rod configuration Y to a point $a \in \mathbb{R}$



The dilation and contraction maps are one the inverse of the other.

The dilation and the contraction maps for configurations

Consider the hard rod configuration space with no rod containing $a \in \mathbb{R}$: $\mathfrak{Y}_a := \{Y \in \mathfrak{Y} : a \notin (y, y+r, \ (y,v,r) \in Y\}$

- The **dilation map** for the configuration X is defined as

$$\begin{aligned} D_a : \mathfrak{X} &\longrightarrow \mathfrak{Y}_a \\ X &\longrightarrow D_a(X) := \{(D_a(x), v, r) \ \ (x,v,r) \in X\} \end{aligned}$$

  where $D_a(x) := x + m_a^x(X)$

- The **contraction map** for the configuration Y is defined as

$$\begin{aligned} C_a : \mathfrak{Y}_a &\longrightarrow \mathfrak{X} \\ Y &\longrightarrow C_a(Y) := \{(C_a(y), v, r) \ \ (y,v,r) \in Y\} \end{aligned}$$

  where $C_a(y) := y - m_a^y(Y)$

Hard rod position vs free gas position

The position of the hard rod associated to the ideal particle (x, v, r) is

$$\mathrm{y}_{v,t}(x) := D_0(x) + vt + j(x, v, t)$$

The hard rod evolution is given by the configuration at time t

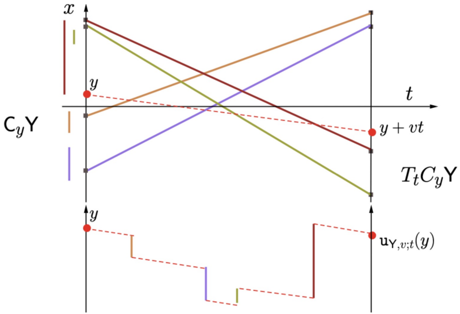
$$U_t Y := \{(\mathrm{y}_{v,t}(x), v, r) : (x, v, r) \in X\}$$

with $U_0 Y = Y$

Hard rods dynamics via a tagged rod

The position at time t of a single hard rod inserted at $t=0$ in y is

$$\mathrm{u}_{v,t}(y) := y + vt + \mathrm{j}(y,v,t)[C_y Y] \qquad \text{for} \qquad Y \in \mathfrak{Y}_y$$



Starting with the configuration Y, the configuration at time t is

$$U_t Y := \{(\mathrm{u}_{v,t}(y), v, r) : (y,v,r) \in Y\}$$

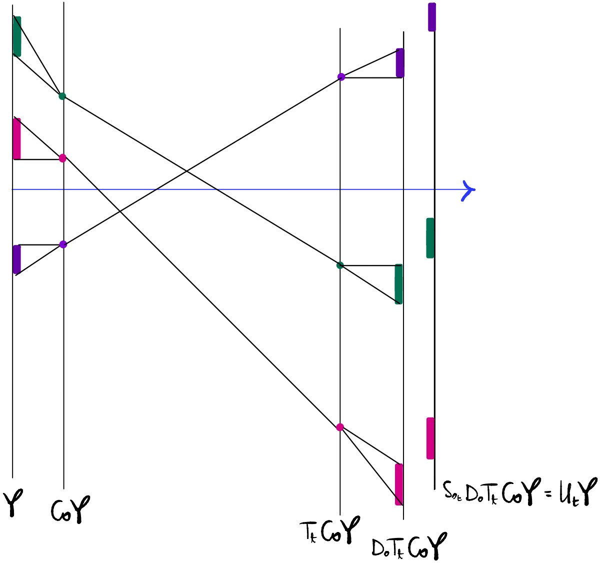
Hard rods dynamics via dilation and contraction

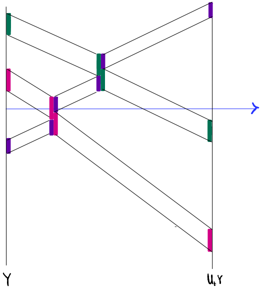
The shift operator is $S_a Y := \{(y + a, v, r) : (y, v, r) \in Y\}$ then the hard rod configuration at time t is

$$U_t Y = S_{o_t} D_0 T_t C_0 Y \qquad \text{for} \qquad Y \in \mathfrak{Y}_0$$

where the point o_t denotes the position at time t of the rod $(0, 0, 0)$ namely,

$$o_t := \mathrm{u}_{0,t}(0) = \mathrm{j}(0, 0, t)[C_0 Y]$$


Y
$C_0 Y$
$T_t C_0 Y$
$D_0 T_t C_0 Y$
$S_{0_t} D_0 T_t C_0 Y = U_t Y$


Y
$U_t Y$

Poisson line process with marks

We interpret the free particle (x, v, r) as the **line** (x, v) **with mark** r.
Let μ be a space locally finite measure on $\mathbb{R}^3$ with the Borel sigma algebra.
We denote by X the Poisson process with mean measure μ and intensity $f(x, v, r)$:

$$\mu(A) = \iiint_A f(x, v, r) dxdvdr$$

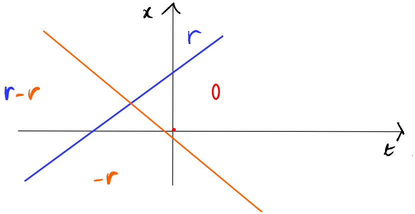
then the configuration at time t, $T_t X$ is also a Poisson process with $\mu T_t^{-1} = \mu T_{-t}$ and with the same distribution of the initial configuration.

$$X^\epsilon := \text{rescaled Poisson process with intensity } \epsilon^{-1} f.$$

Chentsov Lantuéjoul field induced by the marked lined

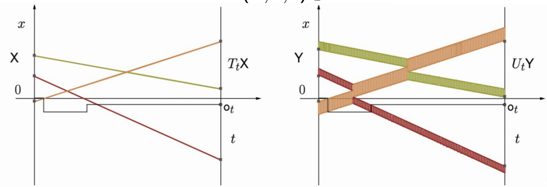
Starting from the marked line (x, v, r) we construct the surface $\mathtt{H}_{(x,v,r)} : \mathbb{R}^2 \to \mathbb{R}$

$$a \to \mathtt{H}_{(x,v,r)}(a) := \begin{cases} 0 & if\,(x,v) \notin \overline{oa} \\ +r & if\,(x,v) \in \overline{oa}_+ \\ -r & if\,(x,v) \in \overline{oa}_- \end{cases}$$



Given a line configuration $X \in \mathfrak{X}$, define the CL filed as

$$\mathtt{H}(a) := \sum_{(x,v,r)\in X} \mathtt{H}_{(x,v,r)}(a) \qquad \text{for} \quad a \in \mathbb{R}^2$$



LLN for Chentsov Lantuéjoul field

The rescaled Chentsov Lantuéjoul field associated to X^{ϵ} is

$$\mathtt{H}^{\epsilon}(a) := \epsilon \sum_{(x,v,r)\in X^{\epsilon}} \mathtt{H}_{(x,v,r)}(a) \qquad \text{for} \quad a \in \mathbb{R}^2$$

Then

$$\lim_{\epsilon\to 0} \mathtt{H}^{\epsilon}(a) = \mu_1(\overline{oa}_-) - \mu_1(\overline{oa}_+)$$

since from Campbell's theorem

$$\mathbb{E}\Big[\sum_{(x,v,r)\in X_i} r\mathbb{1}\{(x,v)\in \overline{oa}_-\}\Big] = \iiint r\mathbb{1}\{(x,v)\in \overline{oa}_-\}\mu(dx,dv,dr)$$

and $\mu_1(dx,dv,dr) := r\mu(dx,dv,dr)$.

LLN for empirical length measure

The empirical length measure for the hard rod process is

$$\mathtt{K}_t^\epsilon \varphi := \epsilon \sum_{(y,v,r)\in U_t Y^\epsilon} r\varphi(y,v,r)$$

For $t = 0$ assume that

$$\lim_{\epsilon\to 0} \mathtt{K}_0^\epsilon \varphi = k_0\varphi := \iiint \varphi(y,v,r) r g(y,v,r) dydvdr$$

then for all $t \in \mathbb{R}$

$$\lim_{\epsilon\to 0} \mathtt{K}_t^\epsilon \varphi = k_t\varphi := \iiint \varphi(y,v,r) r g_t(y,v,r) dydvdr$$

where g_t can be characterized.

Macroscopic evolution

For a system of inhomogeneous hard rods, the equation satisfied by the hard rod evolution g_t is described by the hydrodynamic equation, i.e. $g_t := \mathscr{U}_t g$ is the unique solution of the Cauchy problem:

$$\begin{cases} \partial_t g_t(y,v,r) + \partial_y \left(g_t(y,v,r) v^{eff}(y,v,t)\right) = 0 \\ g_0(y,v,r) = g(y,v,r) \end{cases}$$

where

$$v^{eff}(y,v,t) = v + \frac{\iint r(v-w) g_t(y,w,r) dwdr}{1 - \iint r g_t(y,w,r) dwdr}$$

Macroscopic evolution

Let f be the density of μ such that the corresponding mass and momentum functions are

$$\sigma_f(x) := \iint rf(x,v,r)dvdr \qquad \zeta_f(x) := \iint vrf(x,v,r)dvdr$$

The macroscopic counterpart of contraction, dilation, free time evolution and shift operators are

$$\mathscr{D}_{f,a}(b) := b + \int_a^b \sigma_f(x)dx \qquad \mathscr{C}_{g,a}(b) := b - \int_a^b \sigma_g(y)dy$$

$$\mathscr{D}_a f(y,v,r) := \frac{f(\mathscr{D}_{f,a}^{-1}(y),v,r)}{1+\sigma_f(\mathscr{D}_{f,a}^{-1}(y))} \qquad \mathscr{C}_a g(y,v,r) := \frac{g(\mathscr{C}_{g,a}^{-1}(x),v,r)}{1-\sigma_g(\mathscr{G}_{g,a}^{-1}(x))}$$

$$S_a f(x,v,r) := f(x-a,v,r) \qquad \mathscr{T}_t f(x,v,r) := (x-vt,v,r)$$

Macroscopic dynamics

The hard rod evolution of g as seen from the origin is

$$\mathscr{U}_t : \ g \to \mathscr{U}_t g := S_{o_t} \mathscr{D}_0 \mathscr{T}_t \mathscr{C}_0 g$$

An alternative formulation of the density evolution formula is

$$\mathscr{U}_t g = g(u_{v,t}^{-1}(y), v, r) \frac{d}{dy} u_{v,t}^{-1}(y)$$

where $u_{v,t}(y) := y + vt + j_{\mathscr{C}_y g}(y, v, t)$

Hydrodynamics for the tagged rod

Recall that $\mathtt{u}_{v,t}(y)[Y]$ is the position of a tagged rod initially in y for the configuration $Y \in \mathfrak{Y}_y$. Let $\mathtt{u}^{\varepsilon}_{v,t}(y) := \varepsilon \mathtt{u}_{v,t}(y)[Y^{\varepsilon}]$ the rescaled position in the configuration Y^{ϵ}, then a.s.

$$\lim_{\epsilon \to 0} \mathtt{u}^{\varepsilon}_{v,t}(y) = u_{v,t}(y)$$

where

$$\begin{cases} \partial_t u_{v,t}(y) = v^{eff}(u_{v,t}(y), v, t) \\ \mathtt{u}_{v,0}(y) = y \end{cases}$$

Follows from the fact that we can write $\partial_t u^{-1}_{v,t}(y) = -\partial_t u_{v,t}(\hat{q})$

Collision theorem

The **effective velocity** can be written in terms of mass and momentum as

$$v^{eff}(y, v, t) = \frac{v - \zeta_{g_t}(y)}{1 - \sigma_{g_t}(y)}$$

and in particular $v^{eff}(y, v, t) - v^{eff}(y, w, t) = \dfrac{v - w}{1 - \sigma_{g_t}(y)}$.

Moreover v^{eff} satisfies the following

$$v^{eff}(y, v, t) = v + \iint \Phi(v, w, r) \mid v^{eff}(y, v, t) - v^{eff}(y, w, t) \mid g_t(y, w, r) dwdr$$

where the collision rule is given by

$$\Phi(v, w, r) = \begin{cases} +r & \text{if} \quad v > w \\ -r & \text{if} \quad v < w \end{cases}$$

Next

- Stochastic redistribution of length after collision
- Particles with acceleration
- External force in the system
- Fluctuations
- Large deviation
- Box Ball System
- Other models with similar framework? KdV soliton gas, Lieb-Liniger

THANKS FOR THE ATTENTION