Hydrodynamics for a system of inhomogeneous hard rods

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We study a one-dimensional system of **inhomogeneous hard rods** interacting inertially between collisions.

- A rod is a three dimensional point \((y, v, r)\) with a certain position \(y \in \mathbb{R}\), traveling speed \(v \in \mathbb{R}\) and length \(r \in \mathbb{R}_+\)
- The state space is \(\mathbb{R}^2 \times \mathbb{R}_+\) or \(\mathbb{R}^3\) if we allow negative size
- Configuration denoted by \(Y \subset \mathbb{R}^3\)
- \(\mathcal{Y}\) set of hard rod configurations such that
  - rods do not intersect, i.e. \((y_1, y_1 + r_1) \cap (y_2, y_2 + r_2) = \emptyset\)
  - finite number of rods, i.e. \(# \{ (y, v, r) \in Y : a \leq y \leq b \} < \infty\)

*Given the initial condition, hard rods evolve deterministically: what happen when they collide?*
Consider the two rods \((y_1, v_1, r_1)\) and \((y_2, v_2, r_2)\). If at time \(t^-\) they have positions that satisfy \(y_2 = y_1 + r_1\) then at time \(t\) they exchange their order by a shift in the direction of the other rod. Namely,

Before collision at time \(t^-\) \((y_1, v_1, r_1)\) and \((y_2, v_2, r_2)\) \n
After collision at time \(t\) \((y_1 + r_2, v_1, r_1)\) and \((y_2 - r_1, v_2, r_2)\)

In other words, two rods next to each other swap their positions and keep their original speeds.
Collision rule
Incomplete backgrounds for hard rods

- Jepsen 1964
- Sinai 1972
- Aizemmann Goldstein and Lebowitz 1975
- Boldrighini, Dobrushin and Sukhov 1982
- Spohn 1991
- Boldrighini and Sukhov 1997
- Doyon, Yoshimura and Caux 2017
Ideal gas evolution

A particle of ideal gas is a three dimensional point \((x, v, r)\). We denote \(X \subset \mathbb{R}^3\) a free gas configuration. The dynamics is described by the operator \(T_t\)

\[
T_t (X) := \{(x + vt, v, r) \in \mathbb{R}^3 : (x, v, r) \in X\}
\]
Length flow and mass of $X$

The **mass** and the **signed mass** between $a, b \in \mathbb{R}$ are

$$m(X) := \sum_{(x, v, r) \in X} r$$

$$m^b_a(X) := \begin{cases} 
  m((x, v, r) \in X) & \text{if } a \leq x < b \\
  -m((x, v, r) \in X) & \text{if } b \leq x < a \\
  0 & \text{if } a = b 
\end{cases}$$

The **length flow** is

$$j(x, v, t) := m(\text{particles with velocity } < v) - m(\text{particles with velocity } > v)$$

$$= j^+(x, v, t) - j^-(x, v, t)$$

We will consider configurations $X$ with finite flows:

$$\mathcal{X} := \left\{ X \subset \mathbb{R}^3 : j^+(x, v, t) < \infty, \quad j^-(x, v, t) < \infty, \quad \text{for } x, v, t \in \mathbb{R} \right\}$$
The key ingredients to show the results is to describe the **hard rods evolution** via the **free gas evolution**. This is done using two maps.

**Dilation map**

\[ D_a \text{ describes the dilation of a free gas configuration } X \text{ around a point } a \in \mathbb{R} \]

**Contraction map**

\[ C_a \text{ describes the contraction of a hard rod configuration } Y \text{ to a point } a \in \mathbb{R} \]

The dilation and contraction maps are one the inverse of the other.
The dilation and the contraction maps for configurations

Consider the hard rod configuration space with no rod containing \( a \in \mathbb{R} \):

\[ \mathcal{Y}_a := \{ Y \in \mathcal{Y} : a \notin (y, y + r, (y, v, r) \in Y \} \]

- The **dilation map** for the configuration \( X \) is defined as

\[ D_a : \mathcal{X} \rightarrow \mathcal{Y}_a \]

\[ X \rightarrow D_a(X) := \{(D_a(x), v, r) \mid (x, v, r) \in X\} \]

where \( D_a(x) := x + m^x_a(X) \)

- The **contraction map** for the configuration \( Y \) is defined as

\[ C_a : \mathcal{Y}_a \rightarrow \mathcal{X} \]

\[ Y \rightarrow C_a(Y) := \{(C_a(y), v, r) \mid (y, v, r) \in Y\} \]

where \( C_a(y) := y - m^y_a(Y) \)
Hard rod position vs free gas position

The position of the hard rod associated to the ideal particle \((x, v, r)\) is

\[
y_{v,t}(x) := D_0(x) + vt + j(x, v, t)
\]

The hard rod evolution is given by the configuration at time \(t\)

\[
U_t Y := \{(y_{v,t}(x), v, r) : (x, v, r) \in X\}
\]

with \(U_0 Y = Y\)
Hard rods dynamics via a tagged rod

The position at time $t$ of a single hard rod inserted at $t = 0$ in $y$ is

$$u_{v,t}(y) := y + vt + j(y, v, t)[C_y Y] \quad \text{for} \quad Y \in \mathcal{Y}_y$$

Starting with the configuration $Y$, the configuration at time $t$ is

$$U_t Y := \{(u_{v,t}(y), v, r) : (y, v, r) \in Y\}$$
Hard rods dynamics via dilation and contraction

The shift operator is \( S_a Y := \{(y + a, v, r) : (y, v, r) \in Y\} \) then the hard rod configuration at time \( t \) is

\[
U_t Y = S_{o_t} D_0 T_t C_0 Y \quad \text{for} \quad Y \in \mathcal{Y}_0
\]

where the point \( o_t \) denotes the position at time \( t \) of the rod \((0, 0, 0)\) namely,

\[
o_t := u_{0,t}(0) = j(0, 0, t)[C_0 Y]
\]
\[
\begin{align*}
I & \quad Y \\
U & = E - D \\
T_t C_0 Y & = U_t Y \\
S_{0c} D_{0t} C_0 Y & = U_t Y
\end{align*}
\]
We interpret the free particle \((x, v, r)\) as the \textbf{line} \((x, v)\) \textbf{with mark} \(r\). Let \(\mu\) be a space locally finite measure on \(\mathbb{R}^3\) with the Borel sigma algebra. We denote by \(X\) the Poisson process with mean measure \(\mu\) and intensity \(f(x, v, r)\):

\[
\mu(A) = \iiint_A f(x, v, r) \, dx \, dv \, dr
\]

then the configuration at time \(t\), \(T_t X\) is also a Poisson process with \(\mu T_t^{-1} = \mu T_{-t}\) and with the same distribution of the initial configuration.

\[X^\epsilon := \text{rescaled Poisson process with intensity } \epsilon^{-1} f.\]
Chentsov Lantuéjoul field induced by the marked lined

Starting from the marked line \((x, v, r)\) we construct the surface
\[
H_{(x,v,r)} : \mathbb{R}^2 \to \mathbb{R}
\]

\[
a \mapsto H_{(x,v,r)}(a) := \begin{cases} 
0 & \text{if } (x, v) \notin \overline{oa} \\
+ r & \text{if } (x, v) \in \overline{oa}_+ \\
- r & \text{if } (x, v) \in \overline{oa}_- 
\end{cases}
\]

Given a line configuration \(X \in \mathcal{X}\), define the CL filed as
\[
H(a) := \sum_{(x,v,r) \in X} H_{(x,v,r)}(a) \quad \text{for} \quad a \in \mathbb{R}^2
\]
LLN for Chentsov Lantuéjoul field

The rescaled Chentsov Lantuéjoul field associated to $X^\epsilon$ is

$$H^\epsilon(a) := \epsilon \sum_{(x,v,r) \in X^\epsilon} H_{(x,v,r)}(a) \quad \text{for} \quad a \in \mathbb{R}^2$$

Then

$$\lim_{\epsilon \to 0} H^\epsilon(a) = \mu_1(\overline{oa_-}) - \mu_1(\overline{oa_+})$$

since from Campbell’s theorem

$$\mathbb{E}\left[ \sum_{(x,v,r) \in X_i} r \mathbb{1}_{\{(x,v) \in \overline{oa_-}\}} \right] = \iint \int r \mathbb{1}_{\{(x,v) \in \overline{oa_-}\}} \mu(dx, dv, dr)$$

and $\mu_1(dx, dv, dr) := r \mu(dx, dv, dr)$. 
**LLN for empirical length measure**

The empirical length measure for the hard rod process is

\[
K_t^\varepsilon \varphi := \varepsilon \sum_{(y,v,r) \in U_t Y^\varepsilon} r \varphi(y,v,r)
\]

For \( t = 0 \) assume that

\[
\lim_{\varepsilon \to 0} K_0^\varepsilon \varphi = k_0 \varphi := \int \int \int \varphi(y,v,r)rg(y,v,r)dydvdr
\]

then for all \( t \in \mathbb{R} \)

\[
\lim_{\varepsilon \to 0} K_t^\varepsilon \varphi = k_t \varphi := \int \int \int \varphi(y,v,r)rg_t(y,v,r)dydvdr
\]

where \( g_t \) can be characterized.
Macroscopic evolution

For a system of inhomogeneous hard rods, the equation satisfied by the hard rod evolution $g_t$ is described by the hydrodynamic equation, i.e. $g_t := \mathcal{U}_t g$ is the unique solution of the Cauchy problem:

$$\begin{cases}
\partial_t g_t(y, v, r) + \partial_y (g_t(y, v, r)v^{\text{eff}}(y, v, t)) = 0 \\
g_0(y, v, r) = g(y, v, r)
\end{cases}$$

where

$$v^{\text{eff}}(y, v, t) = v + \frac{\iint r(v - w)g_t(y, w, r)dwdr}{1 - \iint rg_t(y, w, r)dwdr}$$
Macroscopic evolution

Let $f$ be the density of $\mu$ such that the corresponding mass and momentum functions are

$$
\sigma_f(x) := \int \int rf(x, v, r) dv dr \quad \zeta_f(x) := \int \int vrf(x, v, r) dv dr
$$

The macroscopic counterpart of contraction, dilation, free time evolution and shift operators are

$$
\mathcal{D}_{f,a}(b) := b + \int_a^b \sigma_f(x) dx \\
\mathcal{D}_a f(y, v, r) := \frac{f(\mathcal{D}_{f,a}^{-1}(y), v, r)}{1 + \sigma_f(\mathcal{D}_{f,a}^{-1}(y))} \\
S_a f(x, v, r) := f(x - a, v, r)
$$

$$
\mathcal{C}_{g,a}(b) := b - \int_a^b \sigma_g(y) dy \\
\mathcal{C}_a g(y, v, r) := \frac{g(\mathcal{C}_{g,a}^{-1}(x), v, r)}{1 - \sigma_g(\mathcal{C}_{g,a}^{-1}(x))} \\
\mathcal{T}_t f(x, v, r) := (x - vt, v, r)
$$
The hard rod evolution of $g$ as seen from the origin is

$$\mathcal{U}_t : g \rightarrow \mathcal{U}_t g := S_{ot} D_0 \mathcal{T}_t C_0 g$$

An alternative formulation of the density evolution formula is

$$\mathcal{U}_t g = g(u_{v,t}^{-1}(y), v, r) \frac{d}{dy} u_{v,t}^{-1}(y)$$

where $u_{v,t}(y) := y + vt + j_{\mathcal{C}_y g}(y, v, t)$
Hydrodynamics for the tagged rod

Recall that $u_{v,t}(y)[Y]$ is the position of a tagged rod initially in $y$ for the configuration $Y \in \mathcal{Y}$. Let $u^\varepsilon_{v,t}(y) := \varepsilon u_{v,t}(y)[Y^\varepsilon]$ the rescaled position in the configuration $Y^\varepsilon$, then a.s.

$$\lim_{\varepsilon \to 0} u^\varepsilon_{v,t}(y) = u_{v,t}(y)$$

where

$$\begin{align*}
\partial_t u_{v,t}(y) &= \nu^{\mathrm{eff}}(u_{v,t}(y), \nu, t) \\
u_{v,0}(y) &= y
\end{align*}$$

Follows from the fact that we can write $\partial_t u_{v,t}^{-1}(y) = -\partial_t u_{v,t}(\hat{q})$
Collision theorem

The effective velocity can be written in terms of mass and momentum as

$$v^{\text{eff}}(y, v, t) = \frac{v - \zeta_{gt}(y)}{1 - \sigma_{gt}(y)}$$

and in particular

$$v^{\text{eff}}(y, v, t) - v^{\text{eff}}(y, w, t) = \frac{v - w}{1 - \sigma_{gt}(y)}.$$

Moreover $v^{\text{eff}}$ satisfies the following

$$v^{\text{eff}}(y, v, t) = v + \iint \Phi(v, w, r) \mid v^{\text{eff}}(y, v, t) - v^{\text{eff}}(y, w, t) \mid g_{t}(y, w, r) dw dr$$

where the collision rule is given by

$$\Phi(v, w, r) = \begin{cases} +r & \text{if } v > w \\ -r & \text{if } v < w \end{cases}$$
Macroscopic evolution

Next

- Stochastic redistribution of length after collision
- Particles with acceleration
- External force in the system
- Fluctuations
- Large deviation
- Box Ball System
- Other models with similar framework? KdV soliton gas, Lieb-Liniger

THANKS FOR THE ATTENTION