Precise formulas for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional Poisson-Voronoi tessellation and a Poisson line process. *

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Abstract

In this paper, we give an explicit integral expression for the joint distribution of the number and the respective positions of the sides of the typical cell $\mathcal{C}$ of a two-dimensional Poisson-Voronoi tessellation. We deduce from it precise formulas for the distributions of the principal geometric charateristics of $\mathcal{C}$ (area, perimeter, area of the fundamental domain). We also adapt the method to the Crofton cell and the empirical (or typical) cell of a Poisson line process.

1 Introduction and principal results.

1.1 The typical cell of a two-dimensional Poisson-Voronoi tessellation.

Consider Φ a homogeneous Poisson point process in $\mathbb{R}^2$, with the two-dimensional Lebesgue measure V_2 for intensity measure. The set of cells

$$C(x) = \{y \in \mathbb{R}^2; ||y - x|| \le ||y - x'||, x' \in \Phi\}, \quad x \in \Phi,$$

(which are almost surely bounded polygons) is the well-known *Poisson-Voronoi tessellation* of $\mathbb{R}^2$. Introduced by Meijering [13] and Gilbert [6] as a model of crystal aggregates, it provides now models for many natural phenomena such as thermal conductivity [9], telecommunications [3], astrophysics [22] and ecology [19]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [20] and Okabe et al. [18].

In order to describe the statistical properties of the tessellation, the notion of *typical cell* $\mathcal{C}$ in the Palm sense is commonly used [17]. Consider the space $\mathcal{K}$ of convex compact sets of $\mathbb{R}^2$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set $B \subset \mathbb{R}^2$ such that $0 < V_2(B) < +\infty$. The typical cell $\mathcal{C}$ is defined by means of the identity [17]:

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_2(B)}\mathbf{E}\sum_{x \in B\cap\Phi} h(C(x) - x),$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the space of bounded measurable functions.

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Consider now the cell

$$C(0)=\{y\in\mathbb{R}^2;||y||\leq||y-x||,x\in\Phi\}$$

obtained when the origin is added to the point process Φ. It is well known [17] that $C(0)$ and $\mathcal{C}$ are equal in law. From now on, we will use $C(0)$ as a realization of the typical cell $\mathcal{C}$. We will call a point y of Φ a *neighbor* of the origin if the bisecting line of the segment $[0,y]$ intersects the boundary of $C(0)$. Let us denote by $N_0(\mathcal{C})$ the number of sides (or equivalently vertices) of the typical cell $\mathcal{C}$. In [5], we provided an integral formula for the distribution function of $N_0(\mathcal{C})$. We extend the method to obtain the joint distribution of the respective positions of the k lines bounding $C(0)$ conditionally to the event $\{N_0(C(0))=k\}$, $k\geq 3$.

Theorem 1 *(i) For every $k\geq 3$, we have*

$$\mathbf{P}\{N_0(\mathcal{C})=k\}=\frac{(2\pi)^k}{k!}\int d\sigma_k(\delta_1,\cdots,\delta_k)\int\prod_{i=1}^{k}e^{-H(\delta_i,p_i,p_{i+1})}\mathbf{1}_B(p_{i-1},p_i,p_{i+1},\delta_{i-1},\delta_i)p_idp_i, \quad (1)$$

where σ_k is the (normalized) uniform measure on the simplex

$$\mathcal{S}_k=\{(\delta_1,\cdots,\delta_k)\in[0,2\pi];\sum_{i=1}^{k}\delta_i=2\pi\}, \quad (2)$$

with

$$B=\{(p,q,r,\alpha,\beta)\in(\mathbb{R}_+)^3\times(0,\pi)^2;p\sin(\beta)+r\sin(\alpha)\geq q\sin(\alpha+\beta)\}, \quad (3)$$

with for every $\delta\in(0,\pi)$, $p,q\geq 0$,

$$H(\delta,p,q)=\frac{1}{2\sin^2(\delta)}\left\{(p^2+q^2-2pq\cos(\delta))\frac{\delta}{2}+pq\sin(\delta)-\frac{p^2}{4}\sin(2\delta)-\frac{q^2}{4}\sin(2\delta)\right\}, \quad (4)$$

and with the conventions $p_0=p_k$, $p_{k+1}=p_1$, and $\delta_0=\delta_k$;
(ii) conditionally to $\{N_0(C(0))=k\}$, let us denote by $(P_1,\Theta_1),\cdots,(P_k,\Theta_k)$ the polar coordinates of the consecutive neighbors of the origin in the trigonometric order.

The joint distribution of the vector

$$(P_1,\cdots,P_k,\Theta_2-\Theta_1,\cdots,\Theta_k-\Theta_{k-1},2\pi+\Theta_1-\Theta_k)$$

then has a density with respect to the measure

$$d\nu_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)=dp_1\cdots dp_kd\sigma_k(\delta_1,\cdots,\delta_k), \quad (5)$$

and its density φ_k is given by the following equality for every $p_1,\cdots,p_k\geq 0$, $(\delta_1,\cdots,\delta_k)\in\mathcal{S}_k$,

$$\varphi_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)=\frac{1}{\mathbf{P}\{N_0(\mathcal{C})=k\}}\frac{(2\pi)^k}{k!}\prod_{i=1}^{k}p_ie^{-H(\delta_i,p_i,p_{i+1})}\mathbf{1}_B(p_{i-1},p_i,p_{i+1},\delta_{i-1},\delta_i).$$

A table of numerical values for the distribution function of $N_0(\mathcal{C})$ has already been provided (see [5], table 1).

Let us denote by $\mathcal{F}(C(0))$ the fundamental domain associated to $C(0)$, i.e.

$$\mathcal{F}(C(0))=\cup_{x\in C(0)}D(x,||x||),$$

where $D(y,r)$ is the disk centered at $y\in\mathbb{R}^2$ and of radius $r\geq 0$.

Theorem 1 provides an easy way to obtain the distribution of the area of $\mathcal{F}(C(0))$ conditionally to $\{N_0(C(0))=k\}$, $k\geq 3$, and explicit integral formulas for the distribution of the area $V_2(\mathcal{C})$ and the perimeter $V_1(\mathcal{C})$ of $\mathcal{C}$.

Corollary 1 *Conditionally to the event* $\{N_0(\mathcal{C})=k\}$, $k\geq 3$,
(i) the area $V_2(\mathcal{F}(C(0)))$ *is Gamma distributed of parameters* $(k,1)$;
(ii) the distribution of $V_2(\mathcal{C})$ *is given by the following equality for every* $t\geq 0$:

$$\mathbf{P}\{V_2(\mathcal{C})\geq t|N_0(\mathcal{C})=k\}=\int(\mathbf{1}_{C_t}\cdot\varphi_k)(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)d\nu_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k),$$

where

$$C_t=\{(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)\in(\mathbb{R}_+)^k\times(0,\pi)^k;\\ \frac{1}{8}\sum_{i=1}^{k}\frac{1}{\sin(\delta_{i-1})\sin(\delta_i)}p_i(p_{i-1}\sin(\delta_i)+p_{i+1}\sin(\delta_{i-1})-p_i\sin(\delta_{i-1}+\delta_i))\geq t\};\ (6)$$

(iii) the distribution of $V_1(\mathcal{C})$ *is given by the following equality for every* $t\geq 0$:

$$\mathbf{P}\{V_1(\mathcal{C})\geq t|N_0(\mathcal{C})=k\}=\int(\mathbf{1}_{E_t}\cdot\varphi_k)(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)d\nu_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k),$$

where

$$E_t=\{(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)\in(\mathbb{R}_+)^k\times(0,\pi)^k;\\ \frac{1}{2}\sum_{i=1}^{k}\frac{1}{\sin(\delta_{i-1})\sin(\delta_i)}(p_{i-1}\sin(\delta_i)+p_{i+1}\sin(\delta_{i-1})-p_i\sin(\delta_{i-1}+\delta_i))\geq t\}.$$

Remark 1 The point (i) was already obtained by Zuyev [23] with a different method based on Russo's formula. The result can be easily extended to a d-dimensional Poisson-Voronoi tessellation, $d\geq 3$, in the following way: condtionally to the event $\{\text{number of hyperfaces of } C(0)=k\}$, $k\geq d+1$, the Lebesgue measure of the fundamental domain of $C(0)$ is Gamma distributed of parameters $(k,1)$.

1.2 The Crofton cell of a Poisson line process.

Let us now consider Φ' a Poisson point process in $\mathbb{R}^2$ of intensity measure

$$\mu(A)=\int_0^{+\infty}\int_0^{2\pi}\mathbf{1}_A(r,u)d\theta dr,\quad A\in\mathcal{B}(\mathbb{R}^2).$$

Let us consider for all $x\in\mathbb{R}^2$, $H(x)=\{y\in\mathbb{R}^2;(y-x)\cdot x=0\}$, ($x\cdot y$ being the usual scalar product). Then the set $\mathcal{H}=\{H(x);x\in\Phi\}$ is called a *Poisson line process* and divides the plane into convex polygons that constitute the so-called *two-dimensional Poissonian tessellation.* This tessellation is isotropic, i.e. invariant in law by any isometric transformation of the Euclidean space.

This random object was used for the first time by S. A. Goudsmit [8] and by R. E. Miles ([14], [15] and [16]). In particular, it provides a model for the fibrous structure of sheets of paper.

The origin is almost surely included in a unique cell C_0', called the *Crofton cell.* As in Theorem 1, we can get the joint distribution of the number of sides $N_0(C_0')$ of C_0' and the respective positions of its bounding lines.

Theorem 2 *(i) For every $k \geq 3$, we have*

$$\mathbf{P}\{N_0(C_0') = k\} = \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \int \prod_{i=1}^{k} e^{-p_i\left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right)} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) dp_i; \quad (7)$$

(ii) conditionally to $\{N_0(C_0') = k\}$, let us denote by $(P_1', \Theta_1'), \cdots, (P_k', \Theta_k')$ the polar coordinates of the projections of the origin on the consecutive lines bounding C_0' in the trigonometric order.

The joint distribution of the vector

$$(P_1', \cdots, P_k', \Theta_2' - \Theta_1', \cdots, \Theta_k' - \Theta_{k-1}', 2\pi + \Theta_1' - \Theta_k')$$

then has a density with respect to the measure ν_k (defined by (5)) and its density φ_k' is given by the following equality for every $p_1, \cdots, p_k \geq 0$, $(\delta_1, \cdots, \delta_k) \in \mathcal{S}_k$,

$$\varphi_k'(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) = \frac{1}{\mathbf{P}\{N_0(C_0') = k\}} \frac{(2\pi)^k}{k!} \prod_{i=1}^{k} e^{-p_i\left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right)} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i).$$

As for the Voronoi case, the point (i) of Theorem 2 provides numerical values estimated by a Monte-Carlo procedure which are listed in Table 1.

We deduce from Theorem 2 the joint distributions of the couples $(N_0(C_0'), V_1(C_0'))$ and $(N_0(C_0'), V_2(C_0'))$.

Corollary 2 *Conditionally to the event $\{N_0(C_0') = k\}$, $k \geq 3$,*
(i) the perimeter $V_1(C_0')$ is Gamma distributed of parameters $(k, 1)$;
(ii) the distribution of $V_2(C_0')$ is given by the following equality for every $t \geq 0$:

$$\mathbf{P}\{V_2(C_0') \geq t | N_0(C_0') = k\} = \int (\mathbf{1}_{C_{t/4}} \cdot \varphi_k')(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) d\nu_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k),$$

where the set $C_{t/4}$ is defined by the equality (6).

Remark 2 The point (i) was already obtained by G. Matheron (see [10], p.177). It can be extended to any d-dimensional Poissonian tessellation, $d \geq 3$, in the following way: conditionally to the event {number of hyperfaces of $C_0' = k$}, $k \geq d+1$, the mean width of C_0' is Gamma distributed of parameters $\left(k, \frac{\Gamma(d/2)}{\pi^{d/2}}\right)$.

1.3 The typical cell of a Poisson line process.

The notion of *typical* (or *empirical*) cell $\mathcal{C}'$ for the Poisson tessellation was first introduced by Miles [14], [15] through the convergence of ergodic means and has been reinterpreted since by means of a Palm measure (see [11], [12] and [4]). The typical cell $\mathcal{C}'$ is connected in law to the Crofton cell by the following equality (see for example [4]):

$$\mathbf{E}h(\mathcal{C}') = \frac{1}{\mathbf{E}(1/V_2(C_0'))} \mathbf{E}\left(\frac{h(C_0')}{V_2(C_0')}\right), \quad (8)$$

k	3	4	5	6	7	8	9
$\mathbf{P}\{N_0(C_0') = k\}$	0.0767	0.3013	0.3415	0.1905	0.0682	0.0155	0.0052

Table 1: Numerical values for $\mathbf{P}\{N_0(C_0') = k\}$.

for all measurable and bounded function $h : \mathcal{K} \longrightarrow \mathbf{R}$ which is invariant by translation. Besides, it is well known [20] that

$$\mathbf{E}\{V_2(\mathcal{C}')\} = \left[\mathbf{E}\left(\frac{1}{V_2(C_0')}\right)\right]^{-1} = \frac{1}{\pi}. \tag{9}$$

Since Corollary 2 provides the joint distribution of the couple $(N_0(C_0'), V_2(C_0'))$, we can deduce from the equality (8) the law of the number of sides $N_0(\mathcal{C}')$ and also generalize all the results obtained for the Crofton cell.

Theorem 3 *(i) For every $k \geq 3$, we have*

$$\begin{aligned}\mathbf{P}\{N_0(\mathcal{C}') = k\} &= \frac{(2\pi)^k}{\pi \cdot k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \\ & \int \frac{\prod_{i=1}^k e^{-p_i\left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right)} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i)}{W_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k)} dp_1 \cdots dp_k, \end{aligned} \tag{10}$$

where

$$W_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) = \frac{1}{2}\sum_{i=1}^k \frac{1}{\sin(\delta_{i-1})\sin(\delta_i)} p_i(p_{i-1}\sin(\delta_i) + p_{i+1}\sin(\delta_{i-1}) - p_i \sin(\delta_{i-1} + \delta_i));$$

(ii) Let

$$(Q_1, \cdots, Q_k, \Sigma_1, \cdots, \Sigma_k) \in (\mathbb{R}_+)^k \times \mathcal{S}_k$$

be a random vector which has a density ψ_k with respect to the measure ν_k (given by (5)) satisfying the following equality for every $p_1, \cdots, p_k \geq 0$, $(\delta_1, \cdots, \delta_k) \in \mathcal{S}_k$,

$$\psi_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) = a_k \cdot \frac{\varphi_k'(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k)}{W_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k)}.$$

where $a_k = (\mathbf{P}\{N_0(C_0') = k\}/(\pi \mathbf{P}\{N_0(\mathcal{C}') = k\}))$.

Let us consider a random angle Θ independent of the preceding vector and uniformly distributed on the circle. We denote by $X_1, X_2, \cdots, X_k$ the points of the plane of respective polar coordinates (Q_1, Θ), $(Q_2, \Theta + \Sigma_1), \cdots, (Q_k, \Theta + \Sigma_1 + \cdots + \Sigma_{k-1})$. The typical cell $\mathcal{C}'$ then is equal in law to the convex polygon bounded by the lines $H(X_1), \cdots, H(X_k)$.

Numerical values for the distribution function of $N_0(\mathcal{C}')$ using the point (i) and a Monte-Carlo method are listed in Table 2. Let us remark that Miles [14] obtained that $\mathbf{P}\{N_0(\mathcal{C}') = 3\} = 2 - \frac{\pi^2}{6}$ and Tanner [21] get the exact value for $\mathbf{P}\{N_0(\mathcal{C}') = 4\}$.

As for the Crofton cell, we deduce from the preceding theorem a corollary about the joint distributions of the number of sides and the perimeter $V_1(\mathcal{C}')$ (resp. the area $V_2(\mathcal{C}')$) of the typical cell.

k	3	4	5	6	7	8	9
$\mathbf{P}\{N_0(\mathcal{C}')=k\}$	0.3554	0.3815	0.1873	0.0596	0.0129	0.0023	0.0004

Table 2: Numerical values for $\mathbf{P}\{N_0(\mathcal{C}')=k\}$.

Corollary 3 *Conditionally to the event* $\{N_0(\mathcal{C}')=k\}$, $k\ge 3$,
(i) the perimeter $V_1(\mathcal{C}')$ *is Gamma distributed of parameters* $(k-2,1)$;
(ii) the distribution of $V_2(\mathcal{C}')$ *is given by the following equality for every* $k\ge 3$, $t\ge 0$:

$$\mathbf{P}\{V_2(\mathcal{C}')\ge t|N_0(\mathcal{C}')=k\}=\int(\mathbf{1}_{C_{t/4}}\cdot\psi_k)(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)d\nu_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k),$$

where the set $C_{t/4}$ *is defined by the equality (6).*

Remark 3 The point (i) was already obtained by R. E. Miles [14]. It can be extended to any d-dimensional Poissonian tessellation in the following way: conditionally to the event {number of hyperfaces of $\mathcal{C}'=k$}, $k\ge d+1$, the mean width of $\mathcal{C}'$ is Gamma distributed of parameters $\left(k-d,\frac{\Gamma(d/2)}{\pi^{d/2}}\right)$.

Remark 4 Comparing the points (i) of Corollaries 1 and 3, we notice that the area of the fundamental domain of $C(0)$ plays the same role for the Poisson-Voronoi case as the perimeter of $\mathcal{C}'$ for the Poisson line process. This analogy may be explained as follows: for every fixed measure in $\mathbb{R}^2$, the set of the lines $H(x)$, $x\in\mathbb{R}^2$, induces a pseudo-metric in the plane in the sense of R. V. Ambartzumian [1], [2]. The quantity $V_2(\mathcal{F}(C(0)))$ (resp. $V_1(\mathcal{C}')$) then is proportional to the perimeter of the typical cell with respect to the pseudo-metric associated to the intensity measure of the Poisson point process Φ (resp. Φ').

In the paper, we first prove the resuls relative to the Poisson-Voronoi tessellation and secondly the analogous facts for the Crofton cell of a Poisson line process. Let us remark that Theorem 3 and Corollary 3 are direct consequences of Theorem 2 and Corollary 2 combined with (8) and (9).

2 Proofs of Theorem 1 and Corollary 1.

We use the same technique as in [5] based on Slivnyak's formula (see e.g. [17]).

For every $x\in\mathbb{R}^2$, let us denote by $L(x)$ (respectively $\mathcal{D}(x)$) the bisecting line of the segment $[0,x]$ (respectively the half-plane containing 0 delimited by $L(x)$).

We then define for all $k\ge 3$, and $x_1,\cdots,x_k\in\mathbb{R}^2$, the domain

$$\mathcal{D}(x_1,\cdots,x_k)=\cap_{i=1}^k\mathcal{D}(x_k).$$

Besides, we consider the set of $(\mathbb{R}^2)^k$

$$A_k=\{(x_1,\cdots,x_k)\in(\mathbb{R}^2)^k;\mathcal{D}(x_1,\cdots,x_k)\text{ is a convex polygon with }k\text{ sides}\},\tag{11}$$

and for every $(x_1,\cdots,x_k)\in A_k$, the Lebesgue measure of the fundamental domain of $\mathcal{D}(x_1,\cdots,x_k)$, i.e.

$$V(x_1,\cdots,x_k)=V_2[\mathcal{F}(\mathcal{D}(x_1,\cdots,x_k))]=V_2\left[\cup_{x\in\mathcal{D}(x_1,\cdots,x_k)}D(x,||x||)\right].$$

Let $\mathcal{N}_0$ be the set of all neighbors of the origin.

Proposition 1 *For every $k \geq 3$ and every bounded and measurable function $h : \mathbb{R}^k \longrightarrow \mathbb{R}$ invariant by permutation, we have*

$$\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}} h(\mathcal{N}_0)\right\} = \frac{1}{k!}\int h(x_1,\cdots,x_k)\exp\{-V(x_1,\cdots,x_k)\}\mathbf{1}_{A_k}(x_1,\cdots,x_k)dx_1\cdots dx_k. \tag{12}$$

Proof. Let us decompose Ω over all possibilities for the set $\mathcal{N}_0$.

$$\begin{aligned}
&\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}} h(\mathcal{N}_0)\right\} \\
&\quad = \mathbf{E}\left\{\sum_{\{x_1,\cdots,x_k\}\subset\Phi} h(x_1,\cdots,x_k)\mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{1}_{\{\mathcal{D}(x_1,\cdots,x_k)=C(0)\}}\right\} \\
&\quad = \mathbf{E}\left\{\sum_{\{x_1,\cdots,x_k\}\subset\Phi} h(x_1,\cdots,x_k)\mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{1}_{\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\setminus\{x_1,\cdots,x_k\}\}}\right\}.
\end{aligned}$$

Using Slivnyak's formula [17], we obtain

$$\begin{aligned}
&\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}} h(\mathcal{N}_0)\right\} \\
&= \frac{1}{k!}\int h(x_1,\cdots,x_k)\mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{E}\left(\mathbf{1}_{\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\}}\right)dx_1\cdots dx_k \\
&= \frac{1}{k!}\int h(x_1,\cdots,x_k)\mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{P}\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\}dx_1\cdots dx_k. \quad (13)
\end{aligned}$$

We can easily verify that for any $z \in \mathbb{R}^2$,

$$L(z)\cap\mathcal{D}(x_1,\cdots,x_k)\neq\emptyset \Longleftrightarrow z\in\cup_{x\in\mathcal{D}(x_1\cdots,x_k)}D(x,||x||),$$

From this remark and the Poissonian property of Φ, we get

$$\begin{aligned}
\mathbf{P}\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\} &= \mathbf{P}\{\Phi\cap\left[\cup_{x\in\mathcal{D}(x_1,\cdots,x_k)}D(x,||x||)\right]=\emptyset\} \\
&= e^{-V(x_1,\cdots,x_k)}.
\end{aligned} \tag{14}$$

Inserting the equality (14) in (13), we deduce Proposition 1.

□

We already expressed the set A_k analytically and calculated the area $V(x_1,\cdots,x_k)$ in function of the polar coordinates of $x_1,\cdots,x_k$ (see [5], lemmas 1 and 2). Let us denote by

$$(p_1,\theta_1),\cdots,(p_k,\theta_k)\in\mathbb{R}_+\times[0,2\pi),$$

the respective polar coordinates of $x_1,\cdots,x_k\in\mathbb{R}^2$. Supposing that $\theta_1,\cdots,\theta_k$ are in growing order, we define $\delta_i=\theta_{i+1}-\theta_i$, $1\leq i\leq(k-1)$, and $\delta_k=2\pi+\theta_1-\theta_k$. We then have the two following results:

$$\mathbf{1}_{A_k}(x_1,\cdots,x_k) = \prod_{i=1}^{k}\mathbf{1}_B(p_{i-1},p_i,p_{i+1},\delta_{i-1},\delta_i), \tag{15}$$

where the set B is defined by (3), and for every $(x_1, \cdots, x_k) \in A_k$,

$$V(x_1, \cdots, x_k) = \sum_{i=1}^{k} \frac{1}{2\sin^2(\delta_i)} \left\{ (p_i^2 + p_{i+1}^2 - 2p_ip_{i+1}\cos(\delta_i))\frac{\delta_i}{2} + p_ip_{i+1}\sin(\delta_i) - \frac{p_i^2}{4}\sin(2\delta_i) - \frac{p_{i+1}^2}{4}\sin(2\delta_i) \right\}. \quad (16)$$

Proof of Theorem 1. Using polar coordinates in the integral of the equality (12), we obtain for every $k \geq 3$,

$$\begin{aligned}
&\mathbf{E}\{\mathbf{1}_{\{N_0(\mathcal{C})=k\}} h(\mathcal{N}_0)\} \\
&= \frac{1}{k!} \int e^{-V(p_1u_{\theta_1}, \cdots, p_ku_{\theta_k})} (h \cdot \mathbf{1}_{A_k})(p_1u_{\theta_1}, \cdots, p_ku_{\theta_k}) \prod_{i=1}^{k} \mathbf{1}_{\{p_i \geq 0\}} \mathbf{1}_{\{0 \leq \theta_i \leq 2\pi\}} p_i dp_i d\theta_i \\
&= \int e^{-V(p_1u_{\theta_1}, \cdots, p_ku_{\theta_k})} (h \cdot \mathbf{1}_{A_k})(p_1u_{\theta_1}, \cdots, p_ku_{\theta_k}) \mathbf{1}_{\{0 \leq \theta_1 \leq \cdots \leq \theta_k \leq 2\pi\}} \prod_{i=1}^{k} \mathbf{1}_{\{p_i \geq 0\}} p_i dp_i d\theta_i, \quad (17)
\end{aligned}$$

where u_θ, $0 \leq \theta \leq 2\pi$, denotes the unit vector in the plane of rectangular coordinates $(\cos\theta, \sin\theta)$. Let us suppose that h is invariant under rotation, i.e. for all $\theta \in [0, 2\pi]$,

$$h(p_1u_{\theta+\theta_1}, \cdots, p_ku_{\theta+\theta_k}) = h(p_1u_{\theta_1}, \cdots, p_ku_{\theta_k}).$$

Inserting then the results (15) and (16) in (17), we deduce that

$$\begin{aligned}
&\mathbf{E}\{\mathbf{1}_{\{N_0(\mathcal{C})=k\}} h(\mathcal{N}_0)\} \\
&= \int \left[\int h(p_1u_0, p_2u_{\delta_1}, \cdots, p_ku_{\delta_1+\cdots+\delta_{k-1}}) \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) p_i dp_i \right] \\
&\qquad \mathbf{1}_{\{\delta_1+\cdots+\delta_{k-1} \leq 2\pi\}} \delta_k d\delta_1 \cdots d\delta_{k-1} \\
&= \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \int h(p_1u_0, p_2u_{\delta_1}, \cdots, p_ku_{\delta_1+\cdots+\delta_{k-1}}) \\
&\qquad \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) p_i dp_i, \quad (18)
\end{aligned}$$

where the function H is defined by the equality (4).

This last equality provides us the point (ii) of Theorem 1 and replacing h by $\mathbf{1}$, we obtain the point (i).

□

Proof of Corollary 1. Let us first notice that for every $(x_1, \cdots, x_k) \in A_k$,

$$V_2(\mathcal{D}(x_1, \cdots, x_k)) = \frac{1}{8} \sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1})\sin(\delta_i)} p_i(p_{i-1}\sin(\delta_i) + p_{i+1}\sin(\delta_{i-1}) - p_i\sin(\delta_{i-1} + \delta_i)), \quad (19)$$

and

$$\begin{aligned}
V_1(\mathcal{D}(x_1, \cdots, x_k)) &= \frac{1}{2} \sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1})\sin(\delta_i)} (p_{i-1}\sin(\delta_i) + p_{i+1}\sin(\delta_{i-1}) - p_i\sin(\delta_{i-1} + \delta_i)) \quad (20) \\
&= \frac{1}{2} \sum_{i=1}^{k} p_i \left(\frac{1 - \cos(\delta_i)}{\sin(\delta_i)} + \frac{1 - \cos(\delta_{i-1})}{\sin(\delta_{i-1})} \right). \quad (21)
\end{aligned}$$

The point (ii) (resp. (iii)) then is easily obtained by applying the equality (18) to

$$h(x_1,\cdots,x_k)=\mathbf{1}_{\{V_2(\mathcal{D}(x_1,\cdots,x_k))\geq t\}}$$

(resp. $h(x_1,\cdots,x_k)=\mathbf{1}_{\{V_1(\mathcal{D}(x_1,\cdots,x_k))\geq t\}}$). As for point (i), let us apply the equality (12) to

$$h(x_1,\cdots,x_k)=e^{-\lambda V(x_1,\cdots,x_k)},\quad \lambda\geq 0.$$

Let us notice that if $\mathcal{N}_0=\{x_1,\cdots,x_k\}$, we have $V(x_1,\cdots,x_k)=V_2(\mathcal{F}(C(0)))$.

Consequently, we obtain

$$\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}}e^{-\lambda V_2(\mathcal{F}(C(0)))}\right\}=\frac{1}{k!}\int e^{-(\lambda+1)V(x_1,\cdots,x_k)}\mathbf{1}_{A_k}(x_1,\cdots,x_k)dx_1\cdots dx_k.$$

We take the change of variables $x_i'=\sqrt{\lambda+1}x_i$, $1\leq i\leq k$, to deduce that

$$\begin{aligned}\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}}e^{-\lambda V_2(\mathcal{F}(C(0)))}\right\} &= \frac{1}{(\lambda+1)^k}\cdot\frac{1}{k!}\int e^{-V(x_1,\cdots,x_k)}\mathbf{1}_{A_k}(x_1,\cdots,x_k)dx_1\cdots dx_k\\ &= \mathbf{P}\{N_0(C(0)=k\}\frac{1}{(\lambda+1)^k}.\end{aligned}$$

So conditionally to the event $\{N_0(C(0))=k\}$, the Laplace transform of the distribution of $V_2(\mathcal{F}(C(0)))$ is exactly $(\lambda+1)^{-k}$, $\lambda\geq 0$, i.e. $V_2(\mathcal{F}(C(0)))$ is Gamma distributed with parameters $(k,1)$.

□

3 Proofs of Theorem 2 and Corollary 2.

For all $x\in\mathbb{R}^2$, let us define $\mathcal{D}'(x)$ as the half-plane containing the origin delimited by the line $H(x)$. We then denote for every $x_1,\cdots,x_k\in\mathbb{R}^2$,

$$\mathcal{D}'(x_1,\cdots,x_k) = \mathcal{D}'(x_1)\cap\cdots\cap\mathcal{D}'(x_k)=\mathcal{D}(2x_1,\cdots,2x_k).$$

Let $\mathcal{N}_0'$ be the (random) set of all points $x\in\Phi'$ such that $H(x)$ intersects the boundary of the Crofton cell C_0'.

Proposition 2 *For every $k\geq 3$ and every bounded and measurable function $h:\mathbb{R}^k\longrightarrow\mathbb{R}$ invariant by permutation, we have*

$$\mathbf{E}\left\{\mathbf{1}_{\{N_0(C_0')=k\}}h(\mathcal{N}_0')\right\}=\frac{1}{k!}\int(h\cdot\mathbf{1}_{A_k})(x_1,\cdots,x_k)\exp\{-V_1(\mathcal{D}'(x_1,\cdots,x_k))\}dx_1\cdots dx_k. \quad (22)$$

Proof. As for Proposition 1, we apply Slivnyak's formula to obtain

$$\begin{aligned}&\mathbf{E}\left\{\mathbf{1}_{\{N_0(C_0')=k\}}h(\mathcal{N}_0')\right\}\\ &=\frac{1}{k!}\int h(x_1,\cdots,x_k)\mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{P}\{H(y)\cap\mathcal{D}'(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi'\}dx_1\cdots dx_k\end{aligned} \quad (23)$$

We can easily verify (see e.g. [7]) that

$$\begin{aligned}\mathbf{P}\{H(y)\cap\mathcal{D}'(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi'\} &= \mathbf{P}\{\mathcal{D}'(x_1,\cdots,x_k)\subset C_0'\}\\ &= \exp\{-V_1(\mathcal{D}'(x_1,\cdots,x_k))\}.\end{aligned} \quad (24)$$

Inserting the equality (24) in (23), we deduce Proposition 2.

□

Proofs of Theorem 2 and Corollary 2. Let us recall that

$$V_1(\mathcal{D}'(x_1,\cdots,x_k)) = \sum_{i=1}^{k} p_i \left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})} \right), \tag{25}$$

and

$$V_2(\mathcal{D}'(x_1,\cdots,x_k)) = \frac{1}{2}\sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1})\sin(\delta_i)} p_i(p_{i-1}\sin(\delta_i) + p_{i+1}\sin(\delta_{i-1}) - p_i\sin(\delta_{i-1}+\delta_i)). \tag{26}$$

It then suffices to insert in (22) the results (15) and (25) to obtain the two points of Theorem 2.

The proof of Corollary 2 is also analogous to the Voronoi case. In particular, point (i) is deduced from a calculation of the Laplace transform of the distribution of the perimeter of C_0' conditioned by the event $\{N_0(C_0') = k\}$, $k \geq 3$:

$$\mathbf{E}\left\{ \mathbf{1}_{\{N_0(C_0')=k\}} e^{-\lambda V_1(C_0')} \right\} = \mathbf{P}\{N_0(C_0') = k\} \cdot \frac{1}{(\lambda+1)^k}, \quad \lambda \geq 0.$$

□

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