The time constant for Bernoulli percolation is Lipschitz continuous strictly above p_c

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Percolation

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- Graph $(\mathbb{Z}^d, \mathbb{E}^d)$, $d \geq 2$.
- $(B(e))_{e \in \mathbb{E}^d}$: i.i.d. family of Bernoulli random variable of parameter $p \in [0, 1]$.
- $B(e) = 1 \implies e$ is open.
- $B(e) = 0 \implies e$ is closed.

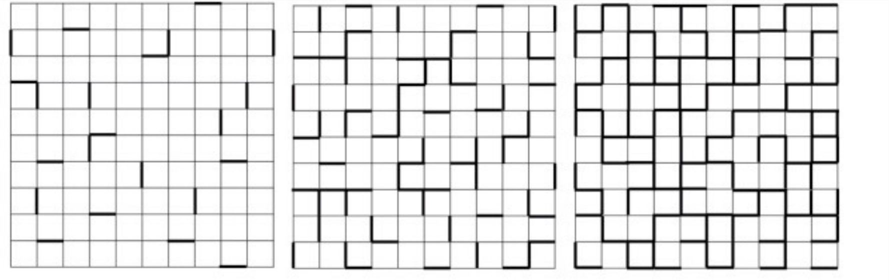

Figure 1: Simulation of percolation for parameters $p = 0.1$; 0.3 and 0.6

Percolation probability

- Random graph $\mathcal{G}_p = (\mathbb{Z}^d, \{e \in \mathbb{E}^d : B(e) = 1\})$.
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- $p \mapsto \theta(p)$ is nondecreasing

Phase transition

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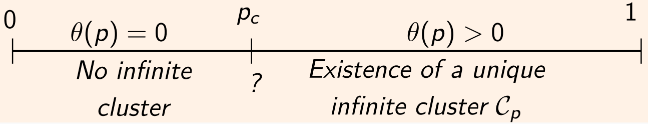
Phase transition at $p_c \in]0,1[$:

Theorem (Broadbendt-Hammersley 57-59,...)



Time constant

Graph distance

We are interested in the random metric induced by $\mathcal{G}_p$ when $p > p_c$. We define for x and y in $\mathbb{Z}^d$

$$\mathcal{D}_p(x,y) = \inf\{|\gamma| : \gamma \text{ path that joins } x \text{ and } y \text{ in } \mathcal{G}_p\}$$

with the convention that $\mathcal{D}_p(x,y) = \infty$ if x and y are not in the same connected component in $\mathcal{G}_p$.

First passage percolation : Definition of the time constant for the graph distance

Theorem (Kingman 73-75, Cerf-Théret 14)

For $p > p_c$, for any $x \in \mathbb{Z}^d$, there exists $\mu_p(x) > 0$ such that

$$\lim_{n\to\infty} \frac{\mathcal{D}_p(\widetilde{0}, \widetilde{nx})}{n} = \mu_p(x) \text{ almost surely and in } L^1$$

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Regularity of μ_p in p ?

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Theorem (D. 18)

Let $p_0 > p_c$, there exists a positive constant C (depending on p_0) such that

$$\forall p, q \in [p_0, 1] \qquad \sup_{\|x\|=1} |\mu_p(x) - \mu_q(x)| \leq C|q-p| \log |q-p| \,.$$

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$$\mathbb{P}(e \text{ is } p\text{-closed}|\, e \text{ is } q\text{-open}) = \mathbb{P}(U(e) \geq p \,|\, U(e) \leq q) = \frac{q-p}{q}$$

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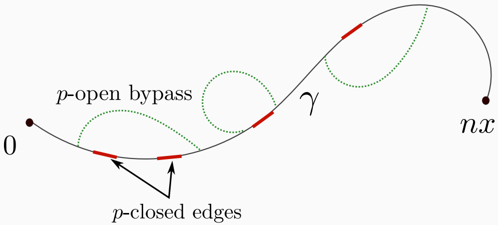

Figure 2: Build a p-open path upon a q-open path for $q > p > p_c$

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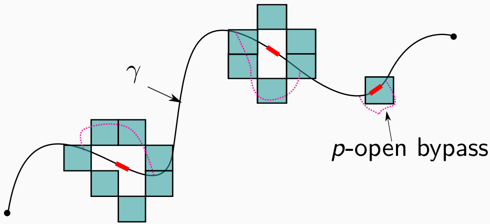
$$\mathcal{D}_p(0, nx) \leq |\gamma'| \leq |\gamma| + |\gamma' \setminus \gamma| = \mathcal{D}_q(0, nx) + |\gamma' \setminus \gamma|$$

If we prove that $|\gamma' \setminus \gamma| \leq C_0|q-p|n$ then

$$\mu_p \leq \mu_q + C_0|q-p|.$$

First approach: renormalization



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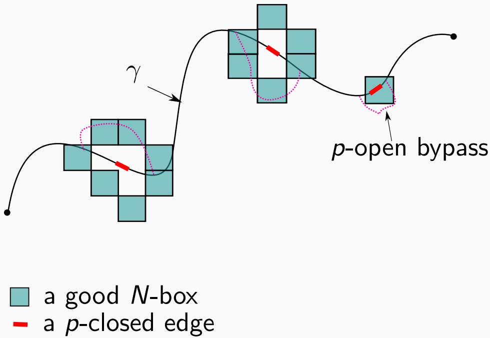

Divide the lattice into boxes of mesoscopic size N. A good box is a box that has good connectivity property. Being a good box is something very likely for N large.Two cases :

1. Bad edge in good box
2. Bad edge in bad box

A different approach

Let $q > p > p_c$. γ is the q-geodesic between 0 and nx. We don't reveal which edges need to be bypassed. For each $e \in \gamma$, we define $c(e)$ the cost to bypass e such that:

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We have

$$\mathcal{D}_p(0, nx) \leq |\gamma'| \leq |\gamma| + |\gamma' \setminus \gamma| \leq \mathcal{D}_q(0, nx) + \sum_{e \in \gamma} \mathbb{1}_{e \text{ is } p\text{-closed}} c(e)$$

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To build $c(e)$ we need a multiscale renormalisation.

Thank you for your attention !