The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *

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Abstract

Denote by R_m (respectively R_M) the radius of the largest (respectively smallest) disk centered at a typical particle of the two-dimensional Poisson-Voronoi tessellation and included within (respectively containing) the polygonal cell associated with that particle. In this article, we obtain the joint distribution of R_m and R_M. This result is derived from the covering properties of the circle due to Stevens, Siegel and Holst. The same method works for studying the Crofton cell associated to the Poisson line process in the plane. The computation of the conditional probabilities $\mathbf{P}\{R_M \geq r+s | R_m = r\}$ reveals the circular property of the Poisson-Voronoi typical cells (as well as the Crofton cells) having a "large" in-disk.

Introduction and presentation of results.

Consider $\Phi = \{x_n; n \geq 1\}$ a homogeneous Poisson point process in $\mathbb{R}^2$, with the 2-dimensional Lebesgue measure V_2 for intensity measure. The set of cells

$$C(x) = \{y \in \mathbb{R}^d; ||y-x|| \leq ||y-x'||, x' \in \Phi\}, \quad x \in \Phi,$$

(which are almost surely bounded polygons) is the well-known *Poisson-Voronoi tessellation* of $\mathbb{R}^2$. Introduced by Meijering [12] and Gilbert [4] as a model of crystal aggregates, it provides now models for many natural phenomena such as thermal conductivity [11], telecommunications [1], astrophysics [26] and ecology [20]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [25] and Okabe et al. [18].

In order to describe the statistical properties of the tessellation, the notion of *typical cell* $\mathcal{C}$ in the Palm sense is commonly used [16]. Consider the space $\mathcal{K}$ of convex compact sets of $\mathbb{R}^2$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set $B \subset \mathbb{R}^2$ such that $0 < V_2(B) < +\infty$. The typical cell $\mathcal{C}$ is defined by means of the identity [16]:

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_2(B)}\mathbf{E}\sum_{x \in B \cap \Phi} h(C(x)-x),$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the space of bounded measurable functions.

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Consider now the cell

$$C(0) = \{y \in \mathbb{R}^2; ||y|| \le ||y - x||, x \in \Phi\}$$

obtained when the origin is added to the point process Φ. It is well known [16] that $C(0)$ and $\mathcal{C}$ are equal in law. From now on, we will use $C(0)$ as a realization of the typical cell $\mathcal{C}$.

The explicit distributions of the main geometrical characteristics of the typical cell are mainly unknown. For example, we do not have any precise idea of the asymptotic behaviour of the distribution function of the area of $\mathcal{C}$ and the best estimation up to now was obtained by Gilbert [4] in 1961 (see also [18]):

$$e^{-4t} \le \mathbf{P}\{V_2(\mathcal{C}) \ge t\} \le \frac{t-1}{e^{t-1}-1}, \quad t > 0. \tag{1}$$

Nevertheless, the law of the radius R_m of the largest ball centered at the origin and contained in $C(0)$ can be obtained easily. Indeed:

$$\begin{aligned} \mathbf{P}\{R_m \ge r\} &= \mathbf{P}\{D(r) \subset C(0)\} \\ &= \mathbf{P}\{\Phi \cap D(2r) = \emptyset\} = e^{-4\pi r^2}, \quad r > 0, \end{aligned}$$

where $D(r)$, $r > 0$, denotes the closed disk centered at the origin of radius r.

It is more difficult to determine the law of the radius R_M of the smallest disk centered at the origin containing $C(0)$. This problem was investigated by Foss and Zuyev [3] in the framework of a mathematical modelization of a telecommunications network. They obtained the following upper bound:

$$\mathbf{P}\{R_M \ge r\} \le 7e^{-\mu r^2}, \quad r > 0, \tag{2}$$

where $\mu = 2(\sin(\pi/14)\cos(5\pi/14) + \pi/7) \approx 1.09$.

In this work, we obtain the exact distribution of R_M.

Theorem 1 *The law of R_M is given by the following equality*

$$\mathbf{P}\{R_M \ge r\} = e^{-4\pi r^2}\left(1 - \sum_{k\ge1} \frac{(-4\pi r^2)^k}{k!}\xi_k\right), \quad r > 0, \tag{3}$$

with

$$\xi_k = \int \left[\prod_{i=1}^k F(u_i)\right] e^{4\pi r^2 \sum_{i=1}^k \int_0^{u_i} F(t)dt} d\sigma_k(u), \quad k \ge 1,$$

and

$$F(t) = \begin{cases} \sin^2(\pi t) & \text{if } 0 \le t \le 1/2 \\ 1 & \text{if } t \ge 1/2, \end{cases} \tag{4}$$

where σ_k denotes the (normalized) area measure of the simplex

$$\{u = (u_1, \cdots, u_k) \in [0,1]^k; \sum_{i=1}^k u_i = 1\}.$$

The proof of Theorem 1 relies on the observation that $\mathbf{P}\{R_M \geq r\}$ can be expressed in terms of probabilities of coverage of a circle by random independent and identically distributed arcs. More precisely, for any probability measure ν on $[0,1]$, let us denote by $P(\nu,n)$ the probability of the coverage of a circle of circumference one by n open random arcs $\mathcal{A}_i$, $1 \leq i \leq n$ such that:

(i) the lengths $0 \leq L_i \leq 1$, $1 \leq i \leq n$, of the arcs are independent and identically distributed random variables of law ν;

(ii) The centers C_i, $1 \leq i \leq n$, of these arcs are independent and uniformly distributed (on the unit circle) random variables;

(iii) The sequences $\{L_i; i \geq 1\}$ and $\{C_i; i \geq 1\}$ are independent.

We show that

Theorem 2 *For all $r \geq 0$,*

$$\mathbf{P}\{R_M \geq r\} = \sum_{n \geq 0} e^{-4\pi r^2} \frac{(4\pi r^2)^n}{n!} (1 - P(\nu_0, n)), \tag{5}$$

where $\nu_0(dt) = \pi \sin(2\pi t) \mathbf{1}_{[0,1/2]}(t) dt$.

The probabilities $P(\nu,n)$ (see formula (18)) were explicitely calculated by Siegel and Holst [23]. By inserting their expressions in (5), we obtain Theorem 1.

Using *Matlab*, we obtain precise estimates for $P(\nu_0,n)$, $n \geq 0$ that we insert in (5). It provides us numerical values for the distribution function of R_M that are listed in Table 1.

Besides, we deduce from Theorem 2 theoretical lower and upper bounds for $\mathbf{P}\{R_M \geq r\}$ that improve significantly the latest result (2) due to Foss and Zuyev:

Theorem 3 *For all $r > 0$, we have*

$$\begin{aligned} 2\pi r^2 e^{-\pi r^2} \left(1 + \frac{1}{2\pi r^2} e^{-\pi r^2}\right) \quad &\leq \quad \mathbf{P}\{R_M \geq r\} \\ &\leq 2\pi r^2 e^{-\pi r^2} \left(2 - 2\pi r^2 e^{-\pi r^2} + \frac{\pi^2 r^4}{3} e^{-2\pi r^2} + \frac{1}{2\pi r^2} e^{-3\pi r^2}\right). \end{aligned}$$

In particular, for $r \geq \alpha \approx 0.337$,

$$2\pi r^2 e^{-\pi r^2} \leq \mathbf{P}\{R_M \geq r\} \leq 4\pi r^2 e^{-\pi r^2}. \tag{6}$$

These estimations, particularly essential when r is large, are difficult to obtain. In order to do it, we use a conjecture of Siegel [22] which we deduce from a non-trivial result proved by Huffer and Shepp [9].

Let us notice that Theorem 3 provides an upper bound for the distribution function of the area of $\mathcal{C}$ which is better than Gilbert's one (1) for $0 < t \leq t^* \approx 1.043$ and worse for $t \geq t^*$.

By the same method we obtain the conditional distributions

$$\mathbf{P}\{R_M \geq t | R_m = r\}, \quad r \geq 0, t > 0$$

as well as the corresponding asymptotic estimations.

r	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\mathbf{P}\{R_M \geq r\}$	1	0.999	0.995	0.983	0.946	0.874	0.758	0.604	0.441
r	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\mathbf{P}\{R_M \geq r\}$	0.292	0.177	0.098	0.050	0.023	0.010	0.004	0.001	0

Table 1: Numerical values for $\mathbf{P}\{R_M \geq r\}$.

Theorem 4 *For all $r, s > 0$,*

$$\mathbf{P}\{R_M \geq r+s | R_m = r\} = e^{-4\pi(s^2+2rs)a_{r,s}} + e^{-4\pi(s^2+2rs)} \left(1 + \sum_{k\geq 1} (-1)^k \frac{(4\pi(s^2+2rs))^k}{k!} \xi_k(r,s) \right), \tag{7}$$

where, for all $k \geq 1$,

$$\begin{aligned} \xi_k(r,s) \quad = & \int \mathbf{1}_{\{u_1 \geq l_{r,s}\}} \left[\prod_{i=2}^{k+1} F_{r,s}(u_i) \right] e^{4\pi(s^2+2rs)\sum_{i=1}^{k+1} \int_0^{u_i} F_{r,s}(t)dt} d\sigma_{k+1}(u) \\ & - \int \left[\prod_{i=1}^{k} F_{r,s}(u_i) \right] e^{4\pi(s^2+2rs)\sum_{i=1}^{k} \int_0^{u_i} F_{r,s}(t)dt} \left[\sum_{i=1}^{k} (u_i - l_{r,s})_+ \right] d\sigma_k(u), \end{aligned}$$

with

$$l_{r,s} = \arccos(r/(r+s))/\pi, \tag{8}$$

$$F_{r,s}(t) = \nu_{r,s}([0,t]) = \begin{cases} \frac{(r+s)^2}{2rs+s^2} \sin^2(\pi t) & \text{if } 0 \leq t \leq l_{r,s} \\ 1 & \text{if } t \geq l_{r,s}, \end{cases} \tag{9}$$

and

$$\begin{aligned} a_{r,s} \quad &= \int_0^1 t \, d\nu_{r,s}(t) \\ &= \frac{1}{2\sqrt{2\pi}} \sqrt{\frac{r}{s}} (1 + \frac{s}{2r})^{-1/2} + \frac{1}{\pi} \arccos(1 - \frac{s}{s+r})(1 + \frac{s}{2r})^{-1}(-\frac{r}{4s} + \frac{1}{2} + \frac{s}{4r}). \end{aligned} \tag{10}$$

Theorem 5 *For all $0 < c < 8/(3\sqrt{2})$ and all fixed $-1 < \alpha < 1/3$,*

$$\mathbf{P}\{R_M \geq r + \frac{1}{r^\alpha} | R_m = r\} = O(e^{-cr^{\frac{1}{2}(1-3\alpha)}}), \quad \text{when } r \to +\infty. \tag{11}$$

The asymptotic result (11) follows from (7) and an inequality proved by Shepp (see [21]). It means that the boundary of the cells such that the in-disk (centered at the nucleus associated to the cell) has a "large radius" r, is included in the annulus $A(r, r+1/r^\alpha)$ (with probability close to one). We observe, expressed in a different form, the circular property of the large cells of the two-dimensional Poisson-Voronoi tessellation that we already noticed in [7].

Besides, we can adapt the procedure to study the radius R'_M of the smallest disk centered at the origin containing the Crofton cell of the Poisson line process in the plane, of intensity measure

$$\mu(A) = \int_0^{+\infty} \int_0^{2\pi} \mathbf{1}_A(\rho, \theta) d\theta d\rho, \quad A \in \mathcal{B}(\mathbb{R}^2).$$

Theorem 6 *The law of R'_M is given by the equality*

$$\mathbf{P}\{R'_M \geq r\} = e^{-2\pi r}\left(1 - \sum_{k\geq 1} \frac{(-2\pi r)^k}{k!}\zeta_k\right), \quad r > 0, \tag{12}$$

where for any $k \geq 1$,

$$\zeta_k = \int \left[\prod_{i=1}^{k} G(u_i)\right] e^{4\pi r^2 \sum_{i=1}^{k}\int_0^{u_i} G(t)dt} d\sigma_k(u),$$

and

$$G(t) = \begin{cases} 1 - \cos(\pi t) & \text{if } 0 \leq t \leq 1/2 \\ 1 & \text{if } t \geq 1/2. \end{cases} \tag{13}$$

Theorem 7 *We have*

$$\mathbf{P}\{R'_M \geq r\} = \sum_{n\geq 0} e^{-2\pi r}\frac{(2\pi r)^n}{n!}(1 - P(\nu'_0, n)),$$

where $\nu'_0(dt) = \pi \sin(\pi t)\mathbf{1}_{[0,1/2]}(t)dt$.

Theorem 8 *We have*

$$2\pi r e^{-2r}\left(\cos 1 + \frac{e^{-2(\pi\cos 1 - 1)r}}{2\pi r}\right) \leq \mathbf{P}\{R'_M \geq r\}$$
$$\leq 2\pi r e^{-2r}\left(1 - (\pi - 2)re^{-2r} + \frac{2}{3}(\pi - 3)^2 r^2 e^{-4r} + \frac{e^{-2(\pi-1)r}}{2\pi r}\right).$$

Denoting by R'_m the radius of the largest disk centered at the origin and contained in the cell, we have:

Theorem 9 *For all $r, s > 0$,*

$$\mathbf{P}\{R'_M \geq r + s | R'_m = r\} = e^{-2\pi s b_{r,s}} + e^{-2\pi s}\left(1 + \sum_{k\geq 1}(-1)^k \frac{(2\pi s)^k}{k!}\zeta_k(r,s)\right),$$

where for any $k \geq 1$,

$$\begin{aligned}\zeta_k(r,s) &= \int \mathbf{1}_{\{u_1 \geq l_{r,s}\}}\left[\prod_{i=2}^{k+1} G_{r,s}(u_i)\right] e^{2\pi s \sum_{i=1}^{k+1}\int_0^{u_i} G_{r,s}(t)dt} d\sigma_{k+1}(u) \\ &\quad - \int \left[\prod_{i=1}^{k} G_{r,s}(u_i)\right] e^{2\pi s \sum_{i=1}^{k}\int_0^{u_i} G_{r,s}(t)dt}\left[\sum_{i=1}^{k}(u_i - l_{r,s})_+\right] d\sigma_k(u),\end{aligned}$$

with

$$G_{r,s}(t) = \nu'_{r,s}([0,t]) = \begin{cases} \frac{r+s}{s}(1 - \cos(\pi t)) & \text{if } 0 \leq t \leq l_{r,s} \\ 1 & \text{if } t \geq l_{r,s}, \end{cases} \tag{14}$$

and

$$\begin{aligned} b_{r,s} &= \int_0^1 t\, d\nu'_{r,s}(t) \\ &= \frac{\sqrt{2}}{\pi}\sqrt{\frac{r}{s}}(1+\frac{s}{2r})^{1/2} - \frac{r}{\pi s}\arccos(1-\frac{s}{s+r}). \end{aligned} \tag{15}$$

Theorem 10 *For all* $0 < c < 8/(3\sqrt{2})$ *and all fixed* $1/3 < \alpha < 1$*,*

$$\mathbf{P}\{R'_M \geq r + r^\alpha | R'_m = r\} = O(e^{-cr^{\frac{1}{2}(3\alpha-1)}}) \quad \text{when } r \to +\infty.$$

This paper is structured as follows. We prove first Theorem 2 which connects the distribution of R_M to the coverage probabilities. From this we deduce Theorem 1. Then we prove the conjecture of Siegel that the probability of coverage of the circle is an increasing function of the concentration (about the mean) of the distribution of the arc lengths. This result (which seems to be unknown) is derived quite easily from a comparison lemma of Huffer and Shepp. It provides us Theorem 3. Then we apply the same method to determine the conditional distributions $\mathbf{P}\{R_M \geq r+s|R_m = r\}$, $r, s > 0$ (Theorems 4 and 5). Finally we conclude this article by using the same arguments in order to obtain similar results in the case of the Crofton cell of the Poisson line process in the plane (Theorems 6 to 10).

1 Proofs of Theorems 1 and 2.

The probability of coverage of the circle by arcs of constant length equal to $0 \leq a \leq 1$ (corresponding to the choice $\nu = \delta_a$) was obtained by Stevens [23]. A proof of the following theorem can be found in [24].

Theorem 11 (Stevens, 1939) *For all* $n \geq 1$*, we have*

$$P(\delta_a, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1-ka)_+^{n-1}. \tag{16}$$

In particular, we deduce easily from Theorem 11 the following corollary.

Corollary 1 *For all* $p \in [0,1]$ *and all* $n \geq 1$*, we have*

$$P((1-p)\delta_0 + p\delta_a, n) = 1 - (1-p)^n + \sum_{k=1}^{n} (-1)^k \binom{n}{k} p^k (1-ka)_+^{k-1}[1-p+p(1-ka)_+]^{n-k}. \tag{17}$$

The formula (16) was extended to the general case by Siegel and Holst [23] under the form:

Theorem 12 (Siegel, Holst, 1982) *For any probability measure* ν *on* $[0,1]$ *with* F_ν *for distribution function, we have*

$$P(\nu, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int \left[\prod_{i=1}^{k} F_\nu(u_i)\right] \left[\sum_{i=1}^{k} \int_0^{u_i} F_\nu(t)dt\right]^{n-k} d\sigma_k(u), \quad n \in \mathbb{N}^*, \tag{18}$$

where σ_k*,* $k \geq 1$*, is the (normalized) uniform measure of the simplex*

$$\{u = (u_1, \cdots, u_k); \sum_{i=1}^{k} u_i = 1\}$$

.

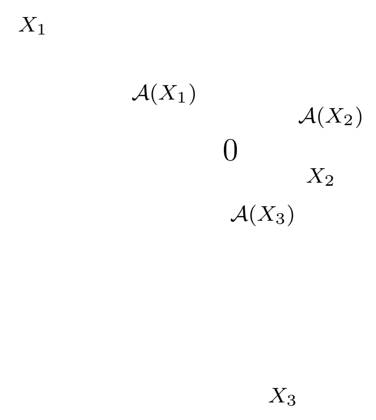

Figure 1: Covering the circle with the arcs $\mathcal{A}(X_i)$, $1 \le i \le n$.

The formula (3) giving the law of R_M is derived directly from (5) and (18). It remains to prove Theorem 2.

Let us fix $r > 0$ and notice first that the convexity of $C(0)$ implies that

$$R_M \ge r \Longleftrightarrow \text{ there exists } x \in C(0) \text{ such that } ||x|| = r.$$

From the definition of the cell $C(0)$, we deduce the identity:

$$\begin{aligned} \{R_M \ge r\} &= \{\exists x \in C(0); ||x|| = r\} \\ &= \{\exists x; ||x|| = r \text{ and } ||x-y|| \ge r \ \ \forall y \in \Phi\} \\ &= \{\exists x; ||x|| = r \text{ and } ||x-y|| \ge r \ \ \forall y \in \Phi \cap D(2r)\}, \end{aligned} \tag{19}$$

Let us define for all $x \in D(2r)$,

$$\mathcal{A}(x) = \{y; ||y|| = r \text{ and } ||y - x|| < r\}.$$

The sets $\mathcal{A}(x)$, $x \in D(2r)$, are open arcs of the circle of radius $r > 0$ (see Figure 2). From (19) we get:

$$\{R_M \ge r\} = \{\exists x; ||x|| = r \text{ and } x \notin \cup_{y \in \Phi \cap D(2r)} \mathcal{A}(y)\}. \tag{20}$$

Besides, let us recall that

$$\Phi \cap D(2r) = \{X_n; 1 \le n \le N\},$$

where:

(i') $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables, taking values in $D(2r)$, of law:

$$X_1(\mathbf{P}) = \frac{1}{4\pi r^2} \mathbf{1}_{D(2r)}(x)dx;$$

(ii') N is a Poisson variable of mean $\mathbf{E}N = 4\pi r^2$ and independent of the sequence $\{X_n; n \geq 1\}$.

Let us note

$$\mathcal{A}_i = \frac{1}{2\pi r}\mathcal{A}(X_i), \quad i \geq 1.$$

Then by an elementary geometrical argument, we deduce from (i') that for all $n \geq 1$, the sequence $\{\mathcal{A}_i; 1 \leq i \leq n\}$, satisfies the conditions (i)-(iii) with

$$L_i = \frac{1}{\pi} \arccos\left(\frac{||X_i||}{2r}\right), \quad i \geq 1,$$

which corresponds to the fact that the law ν_0 of the arc lengths is

$$\nu_0(dt) = \pi \sin(2\pi t) \mathbf{1}_{[0,\frac{1}{2}]}(t)dt.$$

Finally applying the property (ii') we obtain with (20):

$$\begin{aligned}
\mathbf{P}\{R_M \geq r\} &= \mathbf{P}\{N = 0\} + \sum_{n \geq 1} \mathbf{P}\{N = n\} \times \mathbf{P}\{\exists x; ||x|| = r \text{ and } x \notin \cup_{1 \leq i \leq n} \mathcal{A}(X_i)\} \\
&= e^{-4\pi r^2}\left(1 + \sum_{n \geq 1} \frac{(4\pi r^2)^n}{n!}(1 - P(\nu_0, n))\right).
\end{aligned}$$

This is the required result.□

2 A result of comparison for probabilities of coverage.

A. F. Siegel introduced [22] the following notion of comparison of probability distributions.

Definition 1 (Siegel, 1978) *Consider ν_1 and ν_2 two probability distributions on $[0,1]$ with common expectation*

$$\int_0^1 t \, d\nu_1(t) = \int_0^1 t \, d\nu_2(t) = e \in [0,1]. \tag{21}$$

ν_1 is said to be more concentrated (about the mean) than ν_2 if

$$\begin{cases} \nu_1([0,t]) \leq \nu_2([0,t]) & \text{for } t < e \\ \nu_1([0,t]) \geq \nu_2([0,t]) & \text{for } t \geq e \end{cases}$$

In particular, if ν_1 is more concentrated (about the mean) than ν_2, then ν_1 is *less dangerous than* ν_2, which is written in modern notation as $\nu_1 \leq_D \nu_2$ (see [17], p. 23).

Using simulation observations, A. F. Siegel conjectured [22] that

Theorem 13 *If ν_1 is more concentrated than ν_2, then we have*

$$P(\nu_1, n) \leq P(\nu_2, n) \quad \forall n \geq 1. \tag{22}$$

We are going to prove Siegel's conjecture. Let us remark first that

$$P(\nu_1,n)=\int \overline{P}(l_1,\cdots,l_n)d\nu_1(l_1)\cdots d\nu_1(l_n), \tag{23}$$

where $\overline{P}(l_1,\cdots,l_n)$, $l_1,\cdots,l_n\in[0,1]$, denotes the probability that the circle of circumference one is covered by n arcs of lengths respectively equal to $l_1,\cdots,l_n$, where the centers of the arcs are independent and uniformly distributed (on the circle) random variables.

Moreover, Huffer and Shepp [9] proved the following non-trivial result.

Theorem 14 (Huffer, Shepp, 1987) *The function*

$$(l_1,\cdots,l_n)\longmapsto \overline{P}(l_1,\cdots,l_n)$$

is convex in each argument when the others are held fixed.

Besides, it is well known (see [17], pages 16-17, 23) that the comparison $\nu_1\le_D \nu_2$ associated to (21) implies $\nu_1\le_{cx}\nu_2$, i.e. comparison in the convex order of distributions, which by definition ensures that for any convex function f on $[0,1]$, we have

$$\int f(t)d\nu_1(t)\le\int f(t)d\nu_2(t). \tag{24}$$

Then applying Theorem 14 and (24), we obtain by successive iterations

$$\begin{aligned}
&\int\left[\int \overline{P}(l_1,l_2,\cdots,l_n)d\nu_1(l_2)\cdots d\nu_1(l_n)\right]d\nu_1(l_1)\\
&\qquad\le\int\left[\int \overline{P}(l_1,l_2,\cdots,l_n)d\nu_1(l_2)\cdots d\nu_1(l_n)\right]d\nu_2(l_1)\\
&\qquad=\int\left[\int \overline{P}(l_1,l_2,\cdots,l_n)d\nu_2(l_1)d\nu_1(l_3)\cdots d\nu_1(l_n)\right]d\nu_1(l_2)\\
&\qquad\le\int\left[\int \overline{P}(l_1,l_2,\cdots,l_n)d\nu_2(l_1)d\nu_1(l_3)\cdots d\nu_1(l_n)\right]d\nu_2(l_2)\\
&\qquad\vdots\\
&\qquad\le\int \overline{P}(l_1,l_2,\cdots,l_n)d\nu_2(l_1)\cdots d\nu_2(l_n).
\end{aligned}$$

Consequently, using (23), we get

$$P(\nu_1,n)\le P(\nu_2,n),\quad n\ge 1.\square$$

3 Proof of Theorem 3.

The upper and lower bounds on the distribution function of R_M given by Theorem 3 may be obtained easily from the preceding comparison result. Actually, we have clearly

$$\int t\,d\nu_0(t)=\pi\int_0^{1/2}t\sin(2\pi t)dt=\frac{1}{4},$$

and besides $\nu_0([0,1/4])=1/2$. So

Lemma 1 *(i) The measure $\delta_{1/4}$ is more concentrated than ν_0;*

(ii) The measure ν_0 is more concentrated than $\frac{1}{2}(\delta_0+\delta_{1/2})$.

Moreover, Stevens's formula (Theorem 11) and Corollary 1 provide the following expressions.

Lemma 2 *We have for all $n \geq 1$,*

$$\begin{aligned} 1 - P(\delta_{1/4}, n) &= n\left(\frac{3}{4}\right)^{n-1} - \binom{n}{2}\left(\frac{1}{2}\right)^{n-1} + \binom{n}{3}\left(\frac{1}{4}\right)^{n-1} & (25)\\ 1 - P(1/2(\delta_0+\delta_{1/2}), n) &= 2^{-n} + \frac{n}{2}\left(\frac{3}{4}\right)^{n-1}. & (26) \end{aligned}$$

Consequently, it suffices to apply Theorems 2 and 13 as well as Lemmas 1 and 2.□

4 Proofs of Theorems 4 and 5.

Proof of Theorem 4. Notice first the following identity

$$\{\Phi | R_m = r\} \stackrel{law}{=} \Phi_r \cup \{X_0\},$$

where

(i) Φ_r is a Poisson planar point process of intensity measure $\mathbf{1}_{D(2r)^c}dx$

(ii) X_0 is a random variable uniformly distributed on the circle centered at the origin of radius $2r$, and independent of Φ_r.

So we can apply word for word the arguments described in the proof of Theorem 1 by replacing Φ by the point process $\Phi_r \cup \{X_0\}$. We obtain

$$\begin{aligned} \mathbf{P}\{R_M \geq r+s | R_m = r\} &= \mathbf{P}\{\exists x; ||x|| = r+s \text{ and } x \notin \cup_{y\in\Phi_r\cup\{X_0\}}\mathcal{A}(y)\}\\ &= \sum_{n\geq 0}\mathbf{P}\{N=n\}\times\mathbf{P}\{\exists x; ||x||=r+s \text{ and } x\notin \cup_{0\leq i\leq n}\mathcal{A}(X_i)\}, \quad (27) \end{aligned}$$

where

(i) $\{X_n, n\geq 1\}$ is a sequence of random variables independent and identically distributed of law

$$X_1(\mathbf{P}) = \frac{1}{4\pi(s^2+2rs)}\mathbf{1}_{D(2(r+s))\setminus D(2r)}(x)dx,$$

(ii) N is a Poisson variable of mean $\mathbf{E}N = 4\pi(s^2+2rs)$ and independent of the sequence $\{X_n; n\geq 1\}$.

The arcs

$$\mathcal{A}_i = \frac{1}{2\pi(r+s)}\mathcal{A}(X_i), \quad i \geq 0,$$

are independent. The arc $\mathcal{A}_0$ is of constant length equal to

$$L_0 = l_{r,s} = \arccos(r/(r+s))/\pi.$$

The arcs $\mathcal{A}_i$, $i \geq 1$, are of length L_i, $i \geq 1$, having the distribution

$$\nu_{r,s}(dt) = \frac{\pi(r+s)^2}{2rs+s^2} \sin(2\pi t) \mathbf{1}_{[0,l_{r,s}]}(t) dt.$$

The corresponding probability of coverage is not contained in the framework of Siegel and Holst's formula. Nevertheless by adapting the proof of [23] to the case where one of the arcs has a constant length and the others have i.i.d. lengths, it is not too difficult to obtain the formula

$$\begin{aligned}
&\mathbf{P}\left\{\exists x; ||x|| = 1 \text{ and } x \notin \cup_{0\leq i\leq n}\mathcal{A}_i\right\} \\
&= (1-a_{r,s})^n + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \left\{ \int \mathbf{1}_{\{u_1 \geq l_{r,s}\}} \left[\prod_{i=2}^{k+1} F_{r,s}(u_i)\right] \left[\sum_{i=1}^{k+1} \int_0^{u_i} F_{r,s}(t)dt\right]^{n-k} d\sigma_{k+1}(u) \right. \\
&\qquad \left. - \int \left[\prod_{i=1}^{k} F_{r,s}(u_i)\right] \left[\sum_{i=1}^{k} \int_0^{u_i} F_{r,s}(t)dt\right]^{n-k} \left[\sum_{i=1}^{k} (u_i - l_{r,s})_+\right] d\sigma_k(u) \right\}, \quad (28)
\end{aligned}$$

where

$$a_{r,s} = \mathbf{E}L_1 = \int t d\nu_{r,s}(t).$$

This last equality associated to (27) provides us (7).□

Proof of Theorem 5. Fix $r, s > 0$. Remark that

$$L_i \leq l_{r,s}, \quad \text{a.s. } i \geq 1.$$

Consequently, we obtain with Theorem 13 that

$$\begin{aligned}
\mathbf{P}\{\exists x; ||x|| = 1 \text{ and } x \notin \cup_{i=0}^{n}\mathcal{A}_i\} &\leq \mathbf{P}\{\exists x; ||x|| = 1 \text{ and } x \notin \cup_{i=1}^{n+1}\mathcal{A}_i\} \\
&= 1 - P(\nu_{r,s}, n+1) \\
&\leq 1 - P(\delta_{a_{r,s}}, n+1), \quad n \geq 0, \qquad (29)
\end{aligned}$$

where $a_{r,s} = \int t d\nu_{r,s}(t)$ denotes the expectation of $\nu_{r,s}$ given by the formula (10).

Besides, Shepp [21] proved by using a stopping-time argument that

Lemma 3 (Shepp, 1972) *If $0 \leq a \leq 1/4$, then we have*

$$1 - P(\delta_a, n) \leq \frac{2(1-a)^{2n}}{\int_0^a (1-a-t)^n dt + (\frac{1}{4} - a)(1-2a)^n}, \quad n \geq 1.$$

With the choice $r/s \geq \cos(\pi/12)/(1-\cos(\pi/12))$, we have

$$a_{r,s} = \mathbf{E}L_1 \leq l_{r,s} = \frac{1}{\pi} \arccos(\frac{r}{r+s}) \leq \frac{1}{12},$$

so in particular

$$\frac{1}{4} - \frac{1}{n+1} - (1 - \frac{2}{n+1})a_{r,s} \geq 0, \quad \forall n \geq 4.$$

Then by using Lemma 3, we obtain for all $n \geq 1$,

$$\begin{aligned} 1 - P(\delta_{a_{r,s}}, n) &\leq \frac{2(1-a_{r,s})^{2n}}{\int_0^{a_{r,s}}(1-a_{r,s}-t)^n dt + (\frac{1}{4}-a_{r,s})(1-2a_{r,s})^n} \\ &= \frac{2(1-a_{r,s})^{2n}}{\frac{1}{n+1}(1-a_{r,s})^{n+1} + (\frac{1}{4}-\frac{1}{n+1}-(1-\frac{2}{n+1})a_{r,s})(1-2a_{r,s})^n} \\ &\leq \begin{cases} 2(n+1)(1-a_{r,s})^{n-1} & \text{for } n \geq 4 \\ 1 \leq 2(n+1)(1-a_{r,s})^{n-1} & \text{for } n \leq 3. \end{cases} \end{aligned} \tag{30}$$

Consequently, the identity (27) and the inequalities (29) and (30) imply that

$$\mathbf{P}\{R_M \geq r+s | R_m = r\} \leq (8\pi(s^2+2rs)+4)e^{-4\pi(s^2+2rs)a_{r,s}},$$

and with the choice $s = 1/r^\alpha$, $r^{1+\alpha} \geq \cos(\pi/12)/(1-\cos(\pi/12))$,

$$\mathbf{P}\{R_M \geq r + \frac{1}{r^\alpha} | R_m = r\} \leq (8\pi(r^{-2\alpha}+2r^{1-\alpha})+4)e^{-4\pi(r^{-2\alpha}+2r^{1-\alpha})a_{r,1/r^\alpha}}. \tag{31}$$

It remains to study the behaviour of $a_{r,1/r^\alpha}$, where $-1 < \alpha < 1/3$ is fixed, and r goes to infinity. We have

$$\begin{aligned} a_{r,1/r^\alpha} &= \frac{1}{2\sqrt{2}\pi} r^{\frac{1+\alpha}{2}}(1+\frac{1}{2r^{1+\alpha}})^{-1/2} \\ &\quad + \frac{1}{\pi}\arccos(1-\frac{1}{1+r^{1+\alpha}})(1+\frac{1}{2r^{1+\alpha}})^{-1}(-\frac{r^{1+\alpha}}{4}+\frac{1}{2}+\frac{1}{4r^{1+\alpha}}) \\ &\sim \frac{1}{3\sqrt{2}\pi}\frac{1}{r^{\frac{1+\alpha}{2}}} \quad \text{when } r \to +\infty. \end{aligned}$$

This asymptotic result associated to (31) implies that for all $0 < c < 8/(3\sqrt{2})$ and all $-1 < \alpha < 1/3$,

$$\mathbf{P}\{R_M \geq r + \frac{1}{r^\alpha} | R_m = r\} = O(e^{-cr^{\frac{1}{2}(1-3\alpha)}}),$$

which provides the result of Theorem 5.□

5 The case of the Crofton cell of a Poisson line process.

We are going to adapt our method to the study of the smallest disk centered at the origin and containing the Crofton cell of a Poisson line process in the plane. Let us recall first the required definitions.

Let Ψ a Poisson point process in $\mathbb{R}^2$, of intensity measure

$$\mu(A) = \int_0^{+\infty}\int_0^{2\pi} \mathbf{1}_A(\rho,\theta)d\theta d\rho, \quad A \in \mathcal{B}(\mathbb{R}^2). \tag{32}$$

For all $x \in \mathbb{R}^2$, let us consider

$$H(x) = \{y \in \mathbb{R}^2; (y-x)\cdot x = 0\},$$

the polar line associated to x ($x \cdot y$ being the usual scalar product). Then the set $\mathcal{H} = \{H(x); x \in \Phi\}$ divides the space into convex polyhedra that constitute the so-called *two-dimensional Poissonian tessellation.*

In particular, this tessellation is isotropic, that means it is invariant by isometric transformations of $\mathbb{R}^2$. The first results on this geometrical object date from the beginning of the forties. There are due to Goudsmit [8] and to Miles [13], [14] and [15]. Some recent contributions can be found in [2], [6], [10], and [19].

Let us denote by C_0 the cell of the tessellation containing the origin. It can be proved [5] that the cell C_0 (known as *Crofton cell*) is almost surely well defined. Denote respectively by R'_m and R'_M the radii of the largest disk centered at the origin included in C_0 and the smallest disk centered at the origin containing C_0.

Proofs of Theorems 6 and 7. We use the same method of proof as for Theorems 1 and 2. The definition of C_0 provides us the following identity.

$$\{R'_M \geq r\} = \{\exists x; ||x|| = r/2 \text{ and } ||x - y|| \geq r/2 \;\; \forall y \in \Psi \cap D(r)\}. \tag{33}$$

Moreover, classically

$$\Psi \cap D(r) = \{X_n; 1 \leq n \leq N\}, \tag{34}$$

where:

(i) $\{X_n; n \geq 1\}$ is a sequence of i.i.d. random variables taking their values in $D(r)$ of law:

$$X_1(\mathbf{P}) = \frac{1}{2\pi r} \mathbf{1}_{D(r)}(x) d\mu(x);$$

(ii) N is a Poisson variable of mean $\mathbf{E}N = 2\pi r$ and independent of the sequence $\{X_n; n \geq 1\}$.

Then by using (33) and (34), we obtain that

$$\mathbf{P}\{R'_M \geq r\} = \mathbf{P}\{N = 0\} + \sum_{n \geq 1} \mathbf{P}\{N = n\} \times \mathbf{P}\{\exists x; ||x|| = r/2 \text{ and } x \notin \cup_{i=1}^n \mathcal{A}(X_i)\}.$$

It can be easily verified that the arcs $\mathcal{A}(X_i)$ are i.i.d. of respective lengths $\pi r L_i$ where

$$L_i = \frac{1}{\pi} \arccos\left(\frac{||X_i||}{r}\right), \quad i \geq 1,$$

has the distribution

$$\nu'_0(dt) = \pi \sin(\pi t) \mathbf{1}_{[0,1/2]}(t) dt.$$

We conclude as in the proof of Theorems 1 and 2.□

Proof of Theorem 8. It consists in noticing that the law associated to the distribution function G is less concentrated about the mean than $\delta_{1/\pi}$ and more concentrated than

$$(1 - \cos 1)\delta_0 + \cos 1 \delta_{1/(\pi \cos 1)}.$$

Besides, by applying Stevens's formula (16) and its corollary (17), we obtain

$$1 - P(\delta_{1/\pi}, n) = n\left(1 - \frac{1}{\pi}\right)^{n-1} - \binom{n}{2}\left(1 - \frac{2}{\pi}\right)^{n-1} + \binom{n}{3}\left(1 - \frac{3}{\pi}\right)^{n-1},$$

and on the other hand,

$$1 - P((1 - \cos 1)\delta_0 + \cos 1 \delta_{1/(\pi \cos 1)}, n) = (1 - \cos 1)^n + n \cos 1 \left(1 - \frac{1}{\pi}\right)^{n-1}.$$

Then it remains to use Theorems 7 and 13.□

Proof of Theorem 9. The proof is based on the same arguments as for Theorem 4. We remark first the following identity

$$\{\Psi | R'_m = r\} \stackrel{law}{=} \Psi_r \cup \{X_0\},$$

where

(i) Ψ_r is a Poisson planar point process of intensity measure $\mathbf{1}_{D(r)^c} d\mu$

(ii) X_0 is a random variable uniformly distributed on the circle centered at the origin of radius r and independent of Φ_r.

So we see that the procedure described in the proof of Theorem 4 can be applied. We obtain that

$$\begin{aligned}\mathbf{P}\{R'_M \geq r+s | R'_m = r\} &= \mathbf{P}\{\exists x; ||x|| = r+s \text{ and } x \notin \cup_{y\in\Psi_r\cup\{X_0\}} \mathcal{A}(y)\} \\ &= \sum_{n\geq 0} \mathbf{P}\{N=n\} \times \mathbf{P}\{\exists x; ||x|| = r+s \text{ and } x \notin \cup_{0\leq i\leq n} \mathcal{A}(X_i)\},\end{aligned}$$

where

(i) $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables of law

$$X_1(\mathbf{P}) = \frac{1}{2\pi s} \mathbf{1}_{D(r+s)\setminus D(r)}(x) d\mu(x),$$

(ii) N is a Poisson variable of mean $\mathbf{E}N = 2\pi s$ and independent of the sequence $\{X_n; n \geq 1\}$.

The arcs

$$\mathcal{A}_i = \frac{1}{\pi(r+s)} \mathcal{A}(X_i), \quad i \geq 0,$$

are independent. The arc $\mathcal{A}_0$ is of constant length equal to

$$L_0 = l_{r,s} = \arccos(r/(r+s))/\pi.$$

The arcs $\mathcal{A}_i$, $i \geq 1$, are of lengths L_i, $i \geq 1$, of law

$$\nu'_{r,s}(dt) = \frac{\pi(r+s)}{s} \sin(\pi t) \mathbf{1}_{[0,l_{r,s}]}(t) dt.$$

We conclude then as for Theorem 4 by using (28).□

Proof of Theorem 10. As for Theorem 5, we notice first that

$$\mathbf{P}\{\exists x; ||x|| = r/2 \text{ and } x \notin \cup_{i=0}^n \mathcal{A}(X_i)\} \leq 1 - P(\delta_{b_{r,s}}, n+1), \quad n \geq 0.$$

Besides, using Lemma 3 for $r/s \geq \cos(\pi/12)/(1-\cos(\pi/12))$, we obtain that

$$1 - P(\delta_{b_{r,s}}, n) \leq 2(n+1)(1-b_{r,s})^{n-1}, \quad n \geq 1,$$

so

$$\mathbf{P}\{R'_M \geq r+s | R'_m = r\} \leq (4\pi s + 4) e^{-2\pi s b_{r,s}},$$

and with the choice $s = r^\alpha$, for $r^{1-\alpha} \geq \cos(\pi/12)/(1-\cos(\pi/12))$,

$$\mathbf{P}\{R'_M \geq r + r^\alpha | R'_m = r\} \leq (4\pi r^\alpha + 4)e^{-2\pi r^\alpha b_{r,s}}. \tag{35}$$

It remains to study the behaviour of b_{r,r^α}, $\alpha \in (1/3, 1)$, when r goes to infinity. We have

$$\begin{aligned} b_{r,r^\alpha} &= \frac{\sqrt{2} r^{\frac{1-\alpha}{2}}}{\pi}(1 + \frac{1}{2r^{1-\alpha}})^{1/2} - \frac{r^{1-\alpha}}{\pi}\arccos(1 - \frac{1}{1+r^{1-\alpha}}) \\ &\sim \frac{4}{3\sqrt{2}\pi}\frac{1}{r^{\frac{1-\alpha}{2}}} \quad \text{when } r \to +\infty. \end{aligned} \tag{36}$$

Consequently, we obtain, considering (35) and (36) that for all $0 \leq c < 8/(3\sqrt{2})$,

$$\mathbf{P}\{R'_M \geq r + r^\alpha | R'_m = r\} = O(e^{-cr^{\frac{1}{2}(3\alpha-1)}}).\square$$

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