Critical asymptotic behaviour in the SIR model

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SIR model

Kermack and McKendrick (1927)

Susceptible class S = healthy individuals that might be infected
Infected class I = infected individuals that can spread the disease to susceptibles
Recovered class R = individuals that have recovered after having been infected and have became immune

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$$\begin{cases} \frac{d}{dt}S = -\lambda SI \\ \frac{d}{dt}I = \lambda SI - \mu I \\ \frac{d}{dt}R = \mu I \end{cases} \qquad \lambda, \mu > 0 \tag{1}$$

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Kurtz (1971):
Kurtz proved that (1) arises as the hydrodynamic limit of a stochastic epidemic model in the mean field regime.

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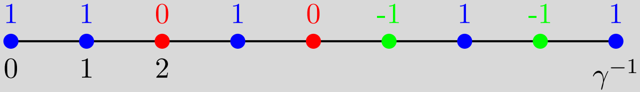

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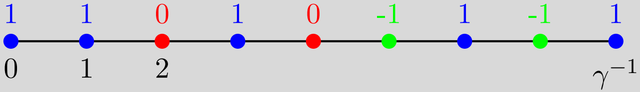

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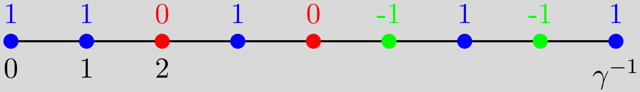

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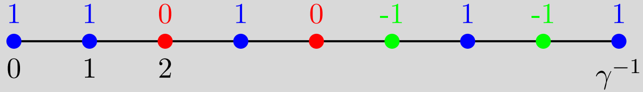

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The rates:

For every $\eta \in \Omega_\gamma,\ x \in \mathbb{T}_\gamma$,

$$\eta(x) : 0 \to 1 \qquad c_0(x,\eta) := \gamma \sum_{y \in \mathbb{T}_\gamma} \beta J(\gamma x, \gamma y)\, \mathbb{I}_{\{\eta(y)=1\}}$$

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▷ Our model is set in a local mean field regime

Empirical measures

We consider the empirical measures associated to susceptible and infected individuals

$$\pi_t^{\gamma,i}(dr) := \gamma \sum_{x \in \mathbb{T}_\gamma} \mathbb{I}_{\{\eta_t(x)=i\}} \delta_{\{\gamma x\}}(dr) \in \mathcal{M}^+(\mathbb{T}), \qquad i \in \{0,1\}$$

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For every $G \in C(\mathbb{T}, \mathbb{R})$ we denote by

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Initial distribution

Given $\rho_0 : \mathbb{T} \to [0,1]$ a density profile, we suppose that,

$$\left\langle \pi_0^{\gamma,0}, G \right\rangle \xrightarrow[\gamma\to 0]{P} \int_{\mathbb{T}} \rho_0(r) G(r)\, dr$$

$$\left\langle \pi_0^{\gamma,1}, G \right\rangle \xrightarrow[\gamma\to 0]{P} \int_{\mathbb{T}} (1-\rho_0(r))\, G(r)\, dr$$

for every $G \in C(\mathbb{T},\mathbb{R})$.

Hydrodynamic limit

Theorem

For every $T > 0$, $G \in C(\mathbb{T}, \mathbb{R})$ *and* $i \in \{0, 1\}$ *we have that*

$$\sup_{t \in [0,T]} \left| \left\langle \pi_t^{\gamma,i}, G \right\rangle - \int_{\mathbb{T}} u_i(t,r)\, G(r)\, dr \right| \xrightarrow[\gamma \to 0]{P} 0$$

where $u_i : [0,T] \times \mathbb{T} \to [0,1]$ *satisfies*

$$\begin{cases} \dfrac{\partial}{\partial t} u_0(t,r) = -\beta (J * u_1)(t,r) u_0(t,r) \\ \dfrac{\partial}{\partial t} u_1(t,r) = \beta (J * u_1)(t,r) u_0(t,r) - u_1(t,r) \\ u_0(0,r) = \rho_0(r), u_1(0,r) = 1 - \rho_0(r) \end{cases}$$

where $(J * u_1)(t,r) := \int_{\mathbb{T}} J(r,r') u_1(t,r') dr'$.

Asymptotic behaviour of the hydrodynamic limit

We have the following convergence

$$\left(u_0\left(t,r\right),u_1\left(t,r\right)\right)\xrightarrow[t\to+\infty]{}\left(\rho_\infty\left(r\right),0\right)$$

where

$$\rho_\infty\left(r\right)=\rho_0\left(r\right)e^{-\beta J*(1-\rho_\infty)(r)} \tag{2}$$

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Observe that from the statistics of the final survivors

- if we know β and J then we can deduce location and density of the initial infectors as $\rho_0\left(r\right)=\rho_\infty\left(r\right)e^{\beta J*(1-\rho_\infty)(r)}$,
- if we know J and that $\rho_0(r)=1$ then we can compute

$$\beta=-\left(J*\left(1-\rho_\infty\right)\left(r\right)\right)^{-1}\log\rho_\infty\left(r\right) \tag{3}$$

Question: Is it possible to exchange the order of the limits in t and γ?

Mean field regime

Assumption: $J \equiv 1$

The main observables of the system are now the total densities of susceptible and infected individuals

$$x^{\gamma}(t) = \gamma \sum_{x \in \mathbb{T}_{\gamma}} \mathbb{I}_{\{\eta_t(x)=0\}} = \left\langle \pi_t^{\gamma,0}, 1 \right\rangle,$$

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Initial distribution

Given $\rho_0 \in [0,1]$ we assume that

$$x^{\gamma}(0) \xrightarrow[\gamma \to 0]{P} \rho_0, \qquad y^{\gamma}(0) \xrightarrow[\gamma \to 0]{P} 1 - \rho_0$$

Hydrodynamic limit in the mean field regime

For finite time ranges $(x^{\gamma}(t), y^{\gamma}(t)) \xrightarrow[\gamma\to 0]{P} (x(t), y(t))$ which satisfy

$$\begin{cases} \frac{d}{dt}x(t) = -\beta x(t)y(t) \\ \frac{d}{dt}y(t) = \beta x(t)y(t) - y(t) \\ x(0) = \rho_0,\ y(0) = 1 - \rho_0 \end{cases} \tag{4}$$

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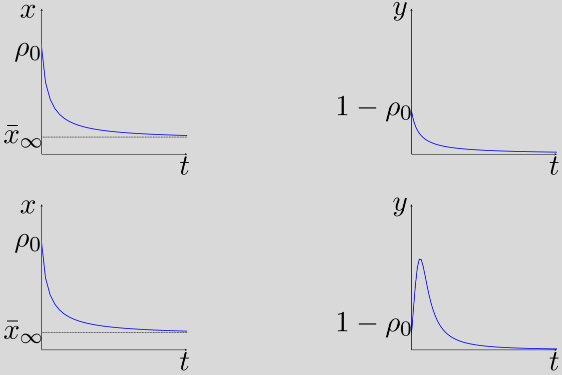
Trivial solution of (4):

- If $\rho_0 = 1 \Rightarrow (1, 0)$, unstable equilibrium
- If $\rho_0 = 0 \Rightarrow (0, e^{-t})$

Qualitative behaviour of the macroscopic solution in the mean field regime, $\rho_0 \neq 0, 1$

$\rho_0 \leq \frac{1}{\beta}$

$\rho_0 > \frac{1}{\beta}$



where $\bar{x}_\infty$ is the smallest solution of

$$x = 1 + \frac{1}{\beta}\log x - \frac{1}{\beta}\log \rho_0 \tag{5}$$

Long time behaviour of the microscopic model

We prove the convergence

$$\left(x^{\gamma}\left(t\right), y^{\gamma}\left(t\right)\right) \xrightarrow[t\to\infty]{} \left(x^{\gamma}\left(\infty\right), 0\right) \quad \text{a.s.}$$

where $x^{\gamma}\left(\infty\right)$ is a random variable in $[0, 1]$.

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- Question: How does $x^{\gamma}\left(\infty\right)$ behave as $\gamma\to 0$?

Theorem

Let $\rho_0\in(0,1)$ and $\beta>0$, then $x^{\gamma}\left(\infty\right)\xrightarrow[\gamma\to 0]{P}\bar{x}_{\infty}$

Criticality at $\beta = 1$

Theorem

1. *If* $\beta < 1$ *and* $\rho_0 = 1$, *then* $x^\gamma(\infty) \xrightarrow[\gamma \to 0]{P} 1$
2. *if* $\beta > 1$, $x^\gamma(0) = 1 - \gamma^\alpha$, $y^\gamma(0) = \gamma^\alpha$ *and* $\alpha \in (0, \frac{1}{2})$, *then*

$$x^\gamma(\infty) \xrightarrow[\gamma \to 0]{P} \hat{x}_\infty < 1$$

where $\hat{x}_\infty$ *is the smallest solution of*

$$x = 1 + \frac{1}{\beta} \log x \tag{6}$$

Thank you for your attention!