The multicolour East model

Rencontres de Probabilités 2022

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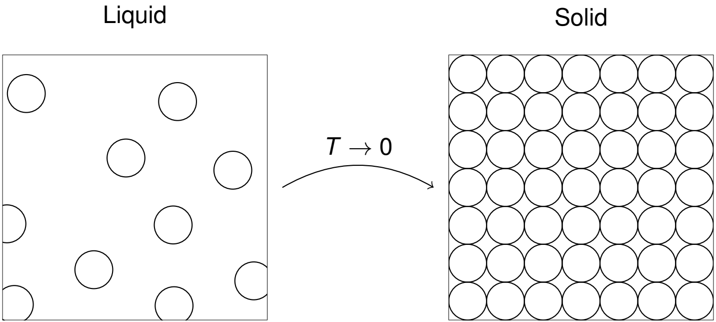
Motivation



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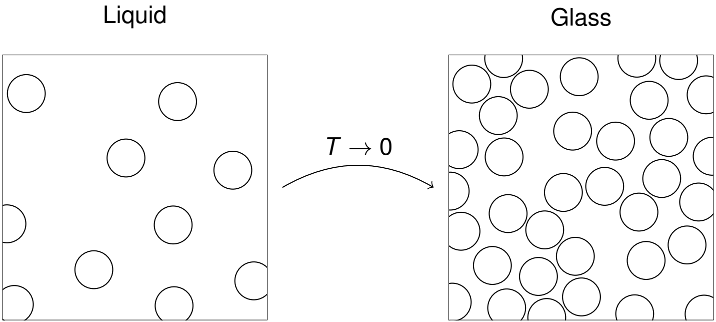

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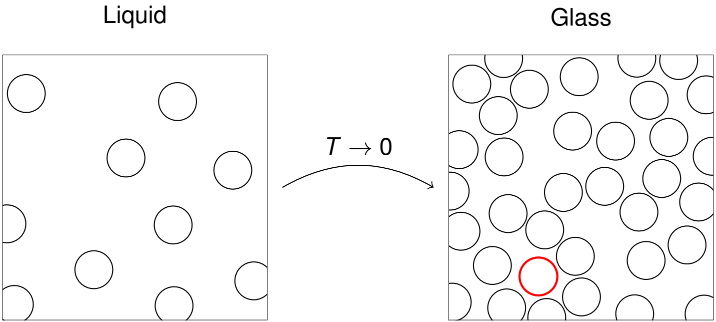

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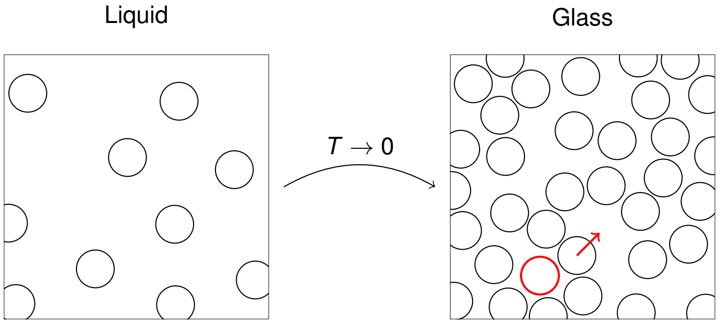

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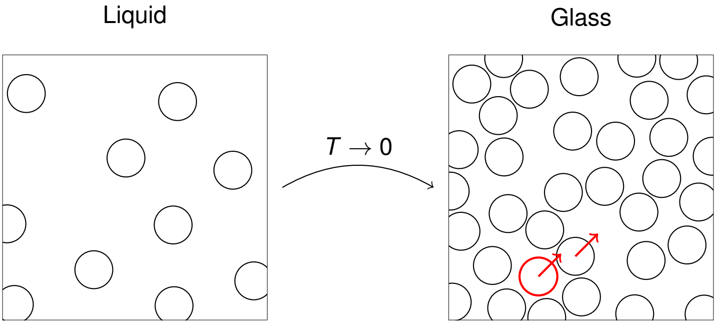
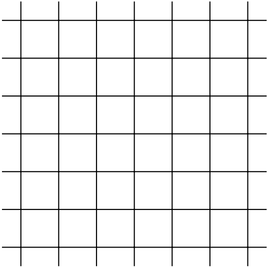


Two-dimensional East model

- State space $\mathbb{Z}^2$.
- Two states, parameter $q \in (0, 1)$.



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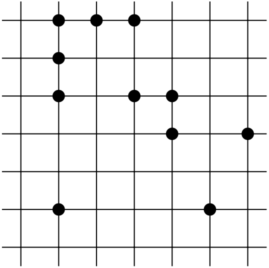

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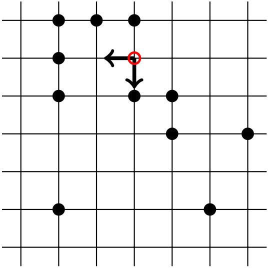

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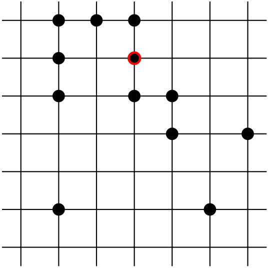
- At legal update sample from $\mu_x = \mathrm{Ber}(p)$, $p = 1 - q$.



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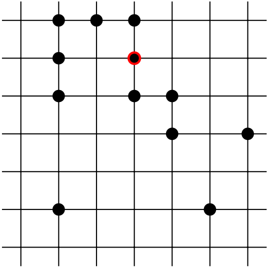
- Process reversible w.r.t. $\mu = \bigotimes_{x \in \mathbb{Z}^d} \mu_x$.

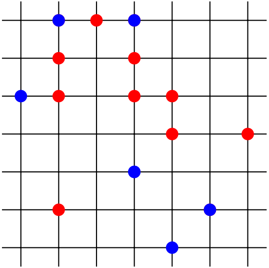


2 colour East model on $\mathbb{Z}^2$

- State space $\mathbb{Z}^2$.

- Three states, parameters

$$q_{\color{blue}\bullet}, q_{\color{red}\bullet}, p := 1 - q_{\color{blue}\bullet} - q_{\color{red}\bullet} \in (0,1)\ .$$



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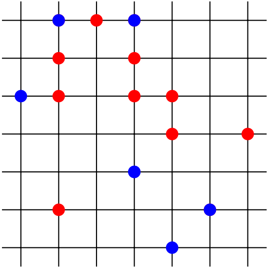

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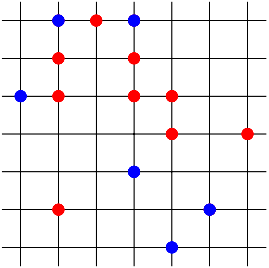

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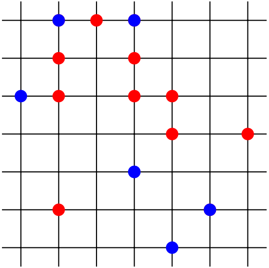

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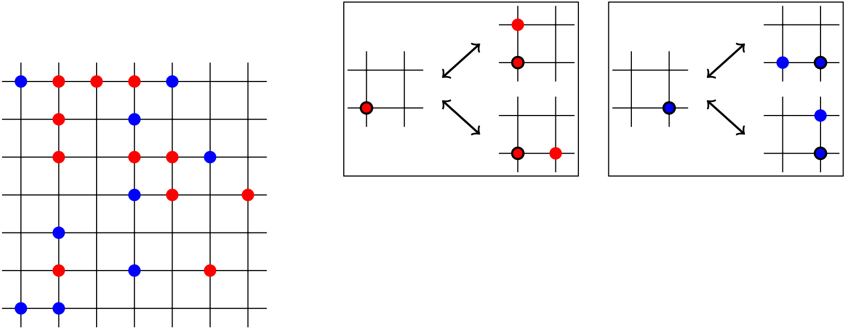

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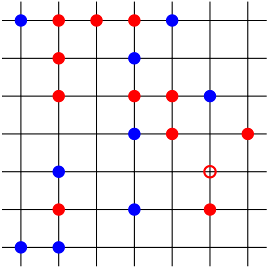

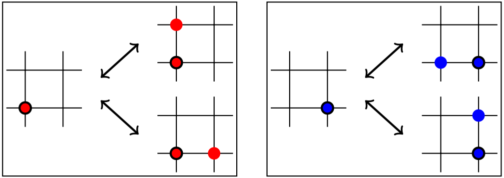

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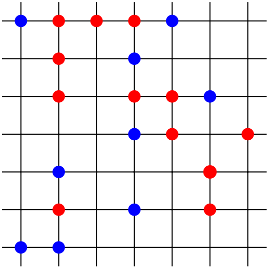

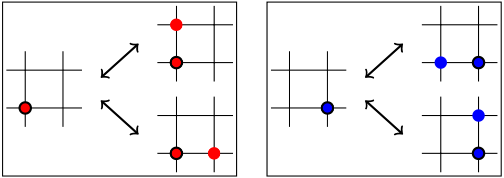

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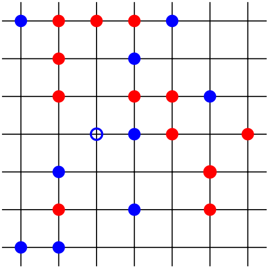

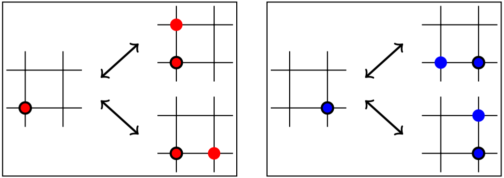

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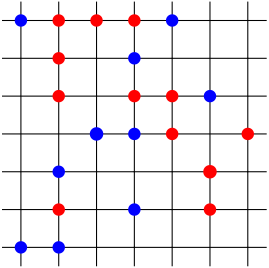

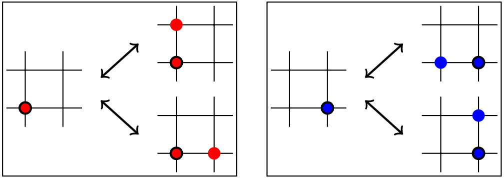

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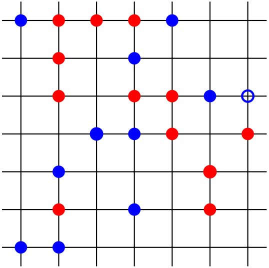

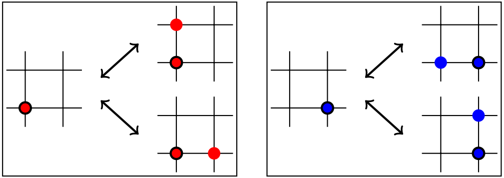

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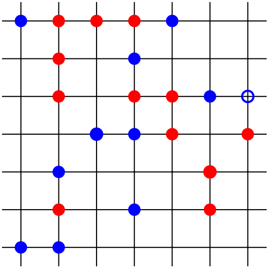

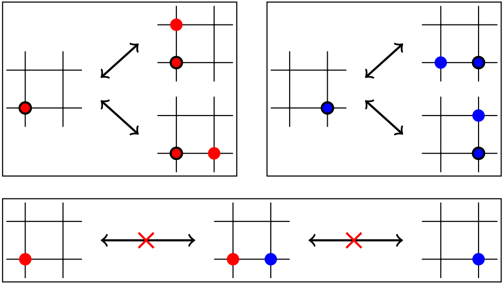

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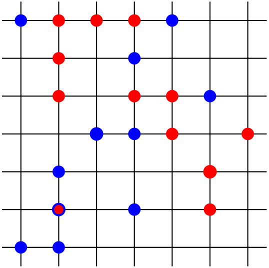

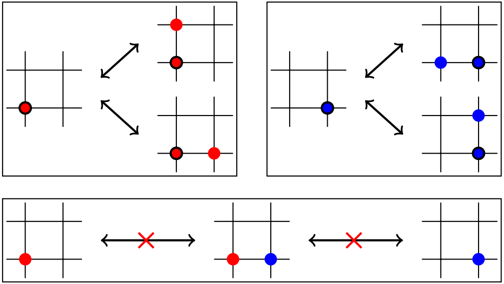

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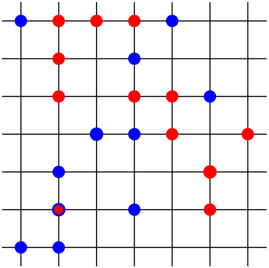

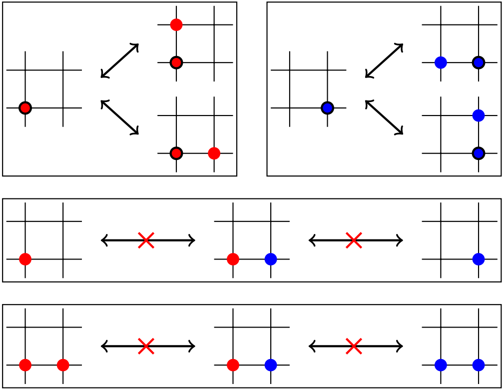

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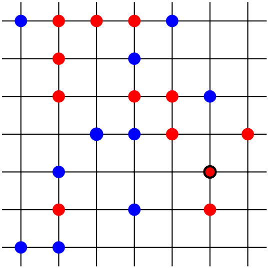

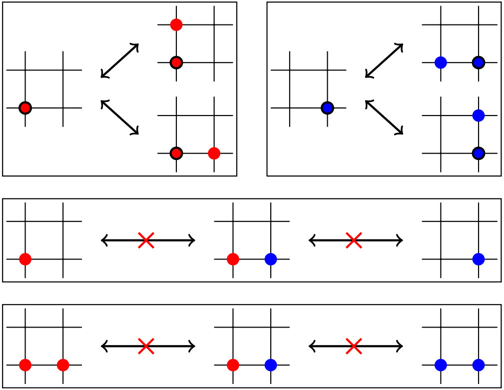

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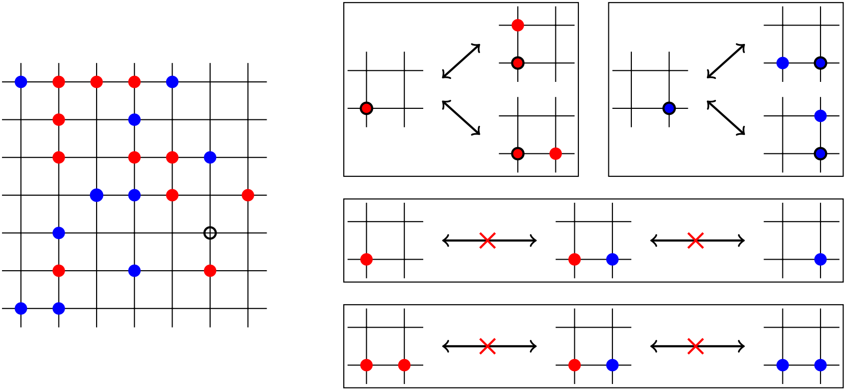
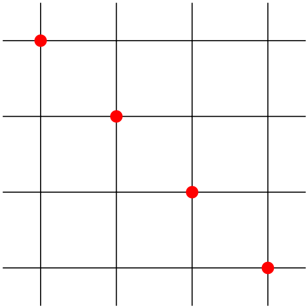


Blocking dynamics



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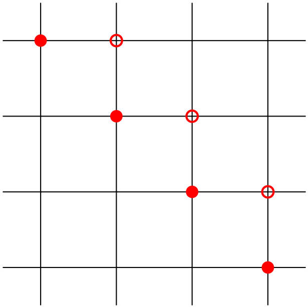

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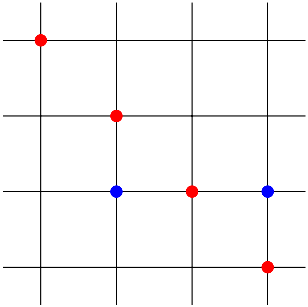


- Diagonal cannot remove itself.
- Despite closeness, no relaxation.

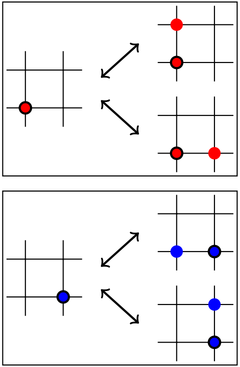
Ergodicity

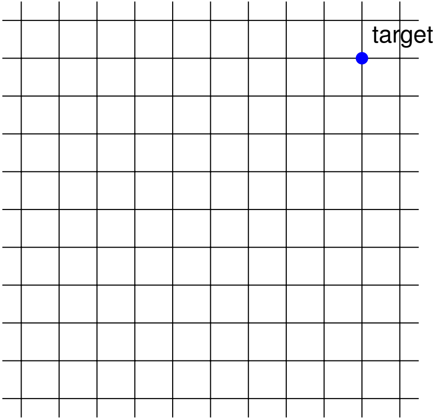
Theorem (Y.C.'22)

The two-colour East model has positive spectral gap for any $\{q_h\}_{h\in\{\bullet,\bullet\}}$ such that $\min q_h > 0$.

Positive spectral gap proof

Show that starting from μ any vacancy can a.s. be removed





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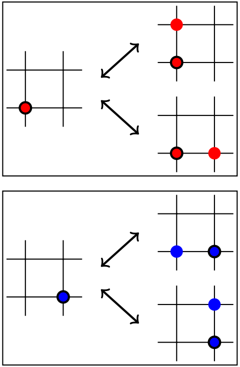

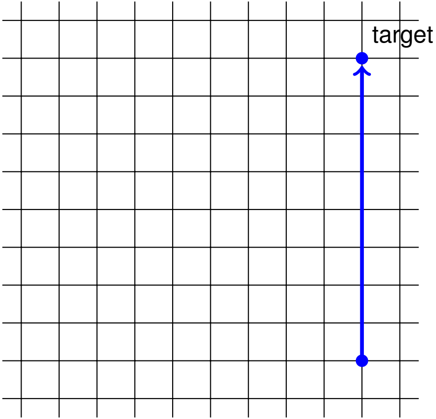

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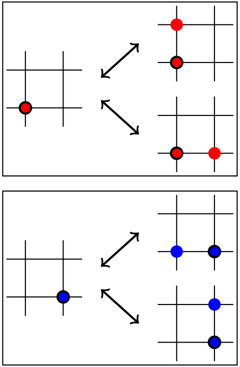

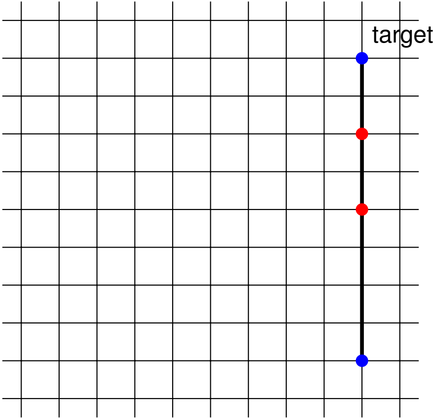

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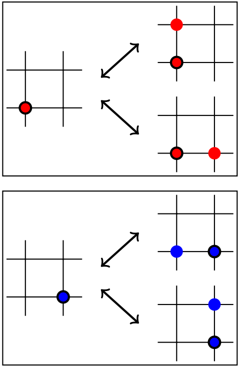

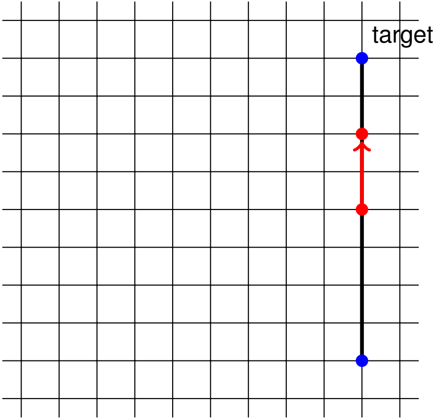

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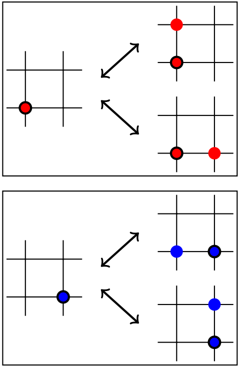

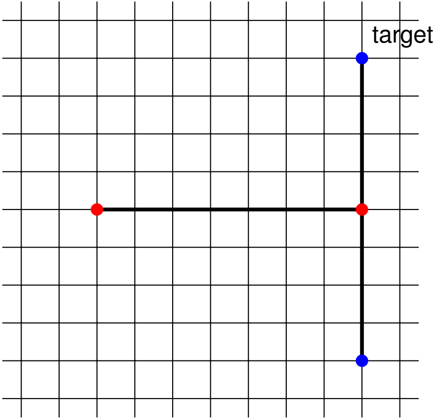

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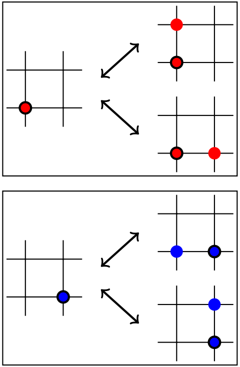

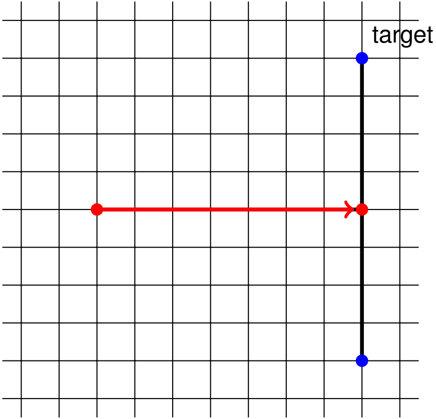

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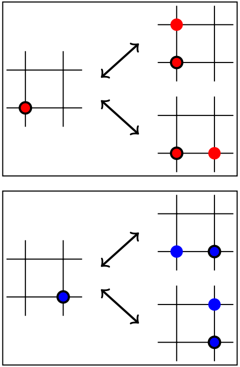

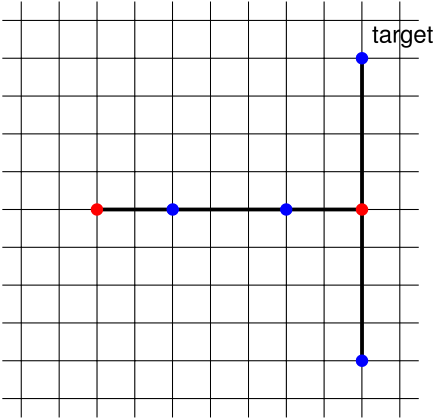

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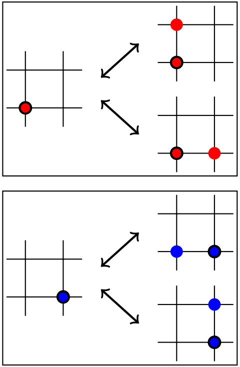

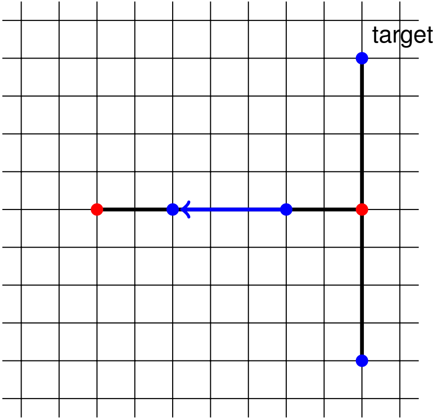

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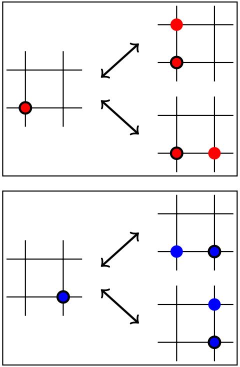

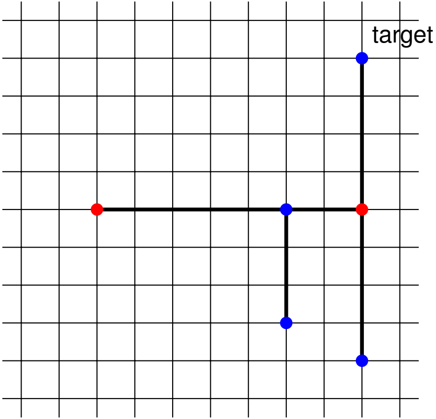

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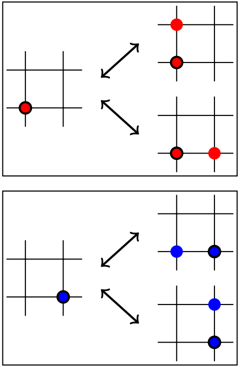
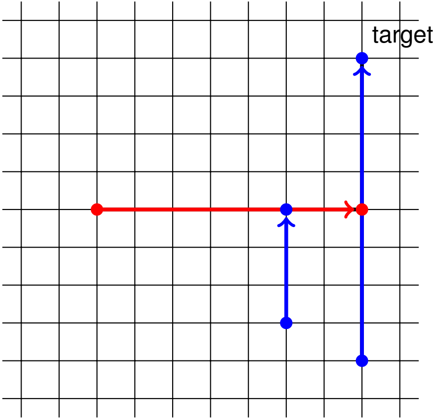




- Target can a.s. be removed if correct colour is met at some point, follows by Borel-Cantelli if $q_{\color{blue}\bullet}, q_{\color{red}\bullet} > 0$.

Ergodicity results

On $\mathbb{Z}^d$ we can add up to 2^d colours.

Theorem (Y.C. '22)

The multicolour East model on $\mathbb{Z}^d$

- *with 2^d colours is not ergodic.*
- *has positive spectral gap if*
  - *all colours share a propagation direction (max colours 2^{d-1}).*
  - *there is a central colour that shares $d-1$ propagation direction with all other colours (max colours $d+1$).*

- For $d = 2$ completely characterized ergodicity landscape.
- For $d > 2$ large gaps.

Spectral gap bounds

For simplicity: Only two-colour East model

Assume w.l.o.g. that $q_{\color{blue}\bullet} < q_{\color{red}\bullet}$ and let $\theta_q := \log_2(1/q)$.

Theorem (Y.C. '22)

Fix $\Delta > 0$. If $p > \Delta$ we have

$$\lim_{q_{\color{blue}\bullet} \to 0} \frac{\gamma(\textit{2-colour})}{\gamma_{2D-East}(q_{\color{blue}\bullet})} = 1$$

If either:

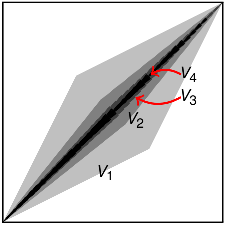
- $\lim_{q_{\color{blue}\bullet} \to 0} q_{\color{red}\bullet} \theta_{\color{blue}\bullet}^3 = 0$, *i.e. "there is no frequent colour".*
- $\lim_{q_{\color{blue}\bullet} \to 0} q_{\color{red}\bullet} \theta_{\color{blue}\bullet}^3 / \log_2(\theta_{\color{blue}\bullet}) = \infty$, *i.e. "there is a frequent colour".*

Bounding λ_D: Finding spectral gap minimizing $V \subset \Lambda$

Proposition (Y.C., F. Martinelli '22)

We find V subset of a square Λ containing both the lower left and top right corner s.t.

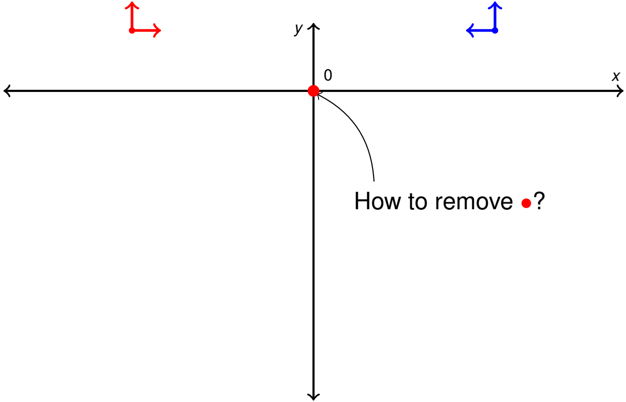
$$\lim_{q \to 0} \frac{\gamma_{\min}(V)}{\gamma_{2D-East}(q)} = 1.$$



- Generalizes to d dimensions.
- Previously only known on boxes with maximal boundary conditions.

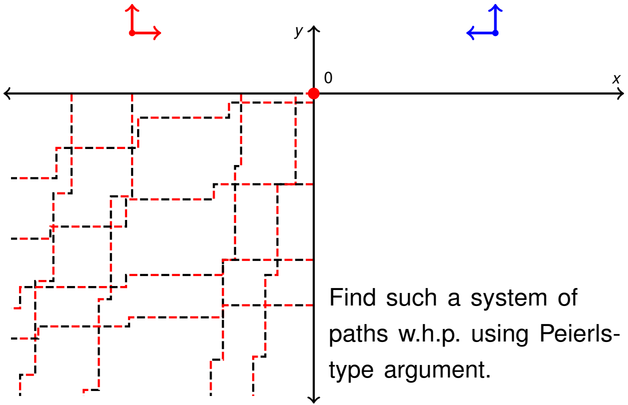
No frequent colour case

$$\lim_{q_{\bullet} \to 0} q_{\bullet} \theta_{\bullet}^3 = 0$$



No frequent colour case

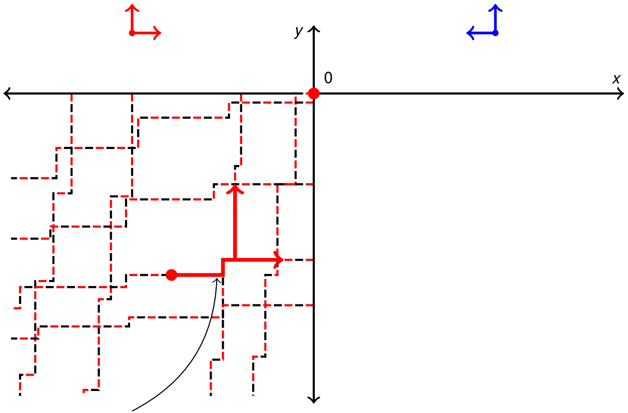
$\lim_{q_\bullet \to 0} q_\bullet \theta_\bullet^3 = 0$



Find such a system of paths w.h.p. using Peierls-type argument.

No frequent colour case

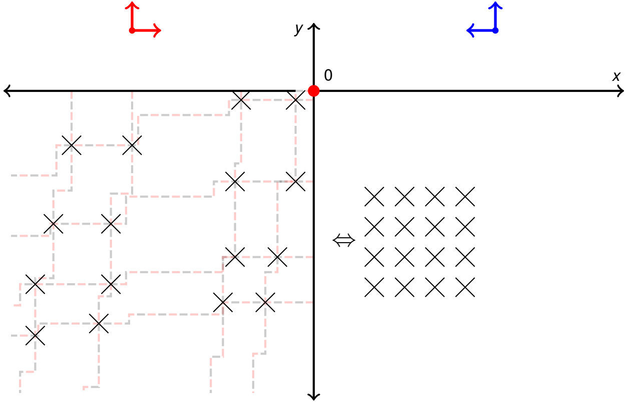
$\lim_{q_{\color{blue}\bullet} \to 0} q_{\color{red}\bullet} \theta_{\color{blue}\bullet}^3 = 0$



- can propagate on paths

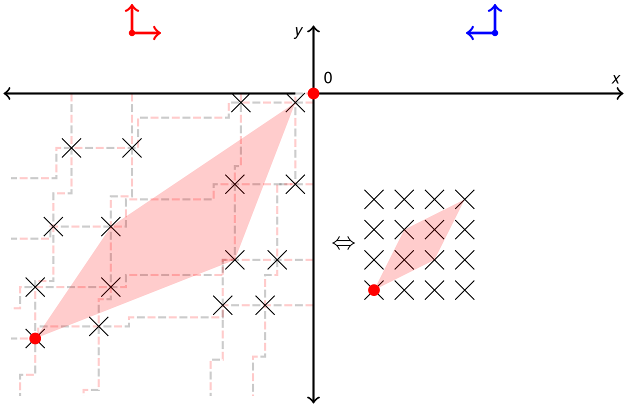
No frequent colour case

$\lim_{q_{\bullet} \to 0} q_{\bullet} \theta^3_{\bullet} = 0$



No frequent colour case

$\lim_{q_{\bullet}\to 0} q_{\bullet}\theta_{\bullet}^3 = 0$



Intersection points isomorphic to box in $\mathbb{Z}^2$

$$\Longrightarrow \lim_{q_{\bullet}\to 0} \frac{\gamma(\text{2-colour})}{\gamma_2(q_{\bullet})} = 1$$

Thank you for listening.