Tessellations

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Random tessellations and cellular structures occur in many domains of application, such as astrophysics, ecology, telecommunications, biochemistry and naturally cellular biology (see (Stoyan, Kendall and Mecke, 1987) or (Okabe, Boots, Sugihara and Chiu, 2000) for complete surveys). The theoretical study of these objects was initiated in the second half of the twentieth century by D. G. Kendall, J. L. Meijering, E. N. Gilbert and R. E. Miles notably. Two isotropic and stationary models have emerged as the most basic and useful: the Poisson hyperplane tessellation and the Poisson–Voronoi tessellation. Since then, a large majority of questions raised about random tessellations have concerned statistics of the population of cells ("how many cells are triangles in the plane?", "how many cells have a volume greater than one?") or properties of a specific cell (typically the one containing the origin). Two types of results are presented below: exact distributional calculations and asymptotic estimations.

In a first part, we describe the two basic constructions of random tessellations (i.e. by throwing random hyperplanes or by constructing Voronoi cells around random nuclei) and we introduce the fundamental notion of typical cell of a stationary tessellation. The second part is devoted to the presentation of exact distributional results on basic geometrical characteristics (number of hyperfaces, typical k-face, etc.). The following part concerns asymptotic properties of the cells. It concentrates in particular on the well-known D. G. Kendall's conjecture which states that large planar cells in a Poisson line tessellation are close to the circular shape. In the last part, we present some recent models of iterated tessellations which appear naturally in applied fields (study of crack structures, telecommunications).

Intentionally, this chapter does not contain an exhaustive presentation of all the models of random tessellations existing in the literature (in particular, dynamical constructions such as Johnson–Mehl tessellations will be omitted). The aim of the text below is to provide a significative view of recent selected methods and results on a few specific models.

0.1 Definitions and basic properties of random tessellations

0.1.1 *Introduction*

Let $\mathcal{T} = \{C_i\}_{i\geq 1}$ be a locally finite collection of closed sets of $\mathbb{R}^d$, $d \geq 1$. The family $\mathcal{T}$ is said to be a *tessellation* of $\mathbb{R}^d$ if C_i and C_j have disjoint interiors for $i \neq j$ and $\bigcup_{i\geq 1} C_i = \mathbb{R}^d$. The sets C_i, $i \geq 1$, are called the *cells* of the tessellation $\mathcal{T}$. In this chapter, we will consider the particular case of a *convex tessellation* where each cell is a convex polyhedron.

We endow the set $\mathbb{T}$ of all convex tessellations of $\mathbb{R}^d$ with the σ-algebra generated by sets of the form

$$\{\mathcal{T} = \{C_i\}_{i\geq 1} : \ [\cup_{i\geq 1}\partial C_i] \cap K \neq \emptyset\}$$

where K is any compact subset of $\mathbb{R}^d$.

A *random convex tessellation* is then defined as a random variable with values in $\mathbb{T}$, see (Stoyan, Kendall and Mecke, 1987). In the following, we will focus on two fundamental examples: the hyperplane tessellation and the Voronoi tessellation.

Let X be a point process in $\mathbb{R}^d$ which does not contain the origin almost surely. For every $x \in X \setminus \{0\}$, we denote by H_x the affine hyperplane orthogonal to x and containing x, i.e.

$$H_x = \{y \in \mathbb{R}^d : \langle y - x, x \rangle = 0\}, \tag{0.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. The *hyperplane tessellation* induced by X is the convex tessellation constituted with the closure of each connex component of $\mathbb{R}^d \setminus \cup_{x \in X} H_x$.

Let X be a point process in $\mathbb{R}^d$. For every $x \in X$, we define the cell $C(x)$ associated with x as

$$C(x) = \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - x'\| \text{ for all } x' \in X\}.$$

The *Voronoi tessellation* induced by X is the tessellation $\{C(x) : x \in X\}$. The points in X are called the *nuclei* of the tessellation.

In particular, if X is a stationary point process in $\mathbb{R}^d$, the associated Voronoi tessellation is stationary (invariant under any translation). In the particular case where X is a homogeneous Poisson point process, it is stationary and isotropic (invariant under any rotation): we speak of a *Poisson–Voronoi tessellation* (Okabe, Boots, Sugihara and Chiu, 2000; Møller, 1994).

Let us consider the measure Θ_0 on $\mathbb{R}^d$ such that its density with respect to the Lebesgue measure is $\|\cdot\|^{-(d-1)}$. If X is a Poisson point process with intensity measure Θ_0 (up to a multiplicative constant), the associated hyperplane tessellation is isotropic and stationary. We speak of a (stationary) *Poisson hyperplane tessellation* (Gilbert, 1962; Miles, 1964*a*; Miles, 1964*b*; Miles, 1969; Miles, 1971).

There are obviously many other types of tessellations which are of great interest and could not be discussed here, for instance Johnson–Mehl tessellations (Møller, 1992) or Laguerre tessellations (Lautensack, 2007; Lautensack and Zuyev, 2008). Though the only Voronoi tessellation that will receive attention in the rest of the chapter is the (homogeneous) Poisson–Voronoi tessellation, it should be noted that Voronoi tessellations generated by other types of more complicated point processes have also been investigated and have led to significative results, see e.g. (Heinrich, 1998; Błaszczyszyn and Schott, 2003; Heinrich and Muche, 2008).

0.1.2 *Zero-cell, ergodic means and typical cell*

One of the fundamental questions raised in the study of random tessellations is to find a way to isolate a particular cell which will be a good descriptor of the collection of all cells, i.e. to define a uniform sample among all the cells.

0.1.2.1 *Zero-cell.*

A first idea is to fix a point in $\mathbb{R}^d$ and consider the cell containing that point.

If there is almost surely a unique cell containing the origin 0, it is called the *zero-cell* of the tessellation and will be denoted by C_0. It is correctly defined for a Poisson–Voronoi tessellation or for a hyperplane tessellation where the associated point process has an intensity measure which does not charge the origin. In particular, the *zero-cell* of the stationary Poisson hyperplane tessellation is called the *Crofton cell*.

Let us remark that C_0 is not a "mean cell", in the sense that it does not have the mean characteristics of the whole population of cells. It is in particular bigger than the typical cell defined below, see the ergodic convergence (0.2) applied to $f = \lambda_d$.

0.1.2.2 *Ergodic means.* It is therefore intuitive to consider a finite set of cells included in a non-empty compact set W and to calculate the mean over these cells of a real-valued, bounded, measurable and invariant-under-translations function f defined on the family $\mathcal{K}$ of convex and compact sets in $\mathbb{R}^d$. If the tessellation is stationary and ergodic (as in both Poisson–Voronoi and Poisson hyperplane cases), Wiener's ergodic theorem (Wiener, 1939) and a proper treatment of "edge regions" ensure that this mean converges when the size of W goes to infinity (Cowan, 1978; Cowan, 1980).

Let N_R be the number of cells of a Poisson–Voronoi or stationary Poisson hyperplane tessellation included in RW for every $R > 0$. Then for any bounded, measurable and invariant-under-translations function $f : \mathcal{K} \to \mathbb{R}$,

$$\lim_{R\to\infty} \frac{1}{N_R} \sum_{i=1}^{N_R} f(C_i) = \frac{1}{\mathbf{E}(\lambda_d(C_0)^{-1})} \mathbf{E}\left(\frac{f(C_0)}{\lambda_d(C_0)}\right), \tag{0.2}$$

where λ_d is the d-dimensional Lebesgue measure. The *typical cell* $\mathscr{C}$ is then defined as a random variable which takes values in the set $\mathcal{K}$ and has a density with respect to the distribution of C_0 equal to $(1/\lambda_d)$ up to a multiplicative constant.

The use of this kind of convergence as an approximation of the typical cell requires the existence of central limit theorems: in the two-dimensional Poisson–Voronoi case, it was proved by Avram and Bertsimas (1993) when f is the perimeter of a polygon, through a stabilisation-type method. Afterwards, Paroux (1998) obtained with the method of moments a similar result for the Poisson line tessellation in the plane and for several functionals among which the perimeter and the number of vertices. More recently, Heinrich, Schmidt and Schmidt (2005) used Hoeffding's decomposition of U-statistics to derive multivariate central limit theorems for a d-dimensional Poisson hyperplane tessellation ($d \geq 2$) and for the number and volume of k-faces ($0 \leq k \leq (d-1)$) of the tessellation.

0.1.2.3 *Typical cell and Palm measure.* Defining the typical cell through ergodic means may not be the most convenient way to study its specific properties. Stationarity allows us to deduce an equivalent definition through the use of

Palm measures (Mecke, 1967; Neveu, 1977) (see also [**CHAPTER Classical stochastic geometry**]).

Indeed, let us suppose that we can assign to any cell C in the tessellation a unique centroid $z(C)$ such that $z(C+x) = z(C) + x$ and the point process Y of all these centroids is stationary (of intensity γ). For instance, in the Poisson–Voronoi case, $z(C)$ can be the nucleus associated with C. In the general case of a stationary tessellation, we can take the center of mass, or the center of the largest ball included in the cell or equivalently the lowest point of the cell (with respect to one of the coordinates).

The typical cell $\mathscr{C}$ is then equivalently defined as a random variable such that for every bounded, measurable and invariant-under-translations function $f : \mathcal{K} \to \mathbb{R}$ and every $B \in \mathcal{B}$ such that $0 < \lambda_d(B) < \infty$

$$\mathbf{E}(f(\mathscr{C})) = \frac{1}{\gamma \lambda_d(B)} \mathbf{E}\left\{ \sum_{\{C : z(C) \in B\}} f(C - z(C)) \right\}. \tag{0.3}$$

That definition (Mecke, 1967; Mecke, 1975; Møller, 1986) does not depend on the chosen centroid process and the method is equivalent to the ergodic procedure, since $\mathbf{E}(f(\mathscr{C}))$ is precisely the limit of the ergodic means, i.e.

$$\mathbf{E}(f(\mathscr{C})) = \frac{1}{\mathbf{E}(\lambda_d(C_0)^{-1})} \mathbf{E}\left(\frac{f(C_0)}{\lambda_d(C_0)} \right). \tag{0.4}$$

This last equality (Møller, 1989) can also be seen as a generalization of a formula which can be found in (Gilbert, 1962), page 964, and would here correspond to the choice $f = {\lambda_d}^2$.

In simple words, $\mathscr{C}$ is the cell containing the origin 0 when the point process of centroids is conditioned on containing 0. It is also the cell "seen from a typical centroid".

In the same way, it is possible to define the *typical k-face* $\mathscr{C}_k$, $0 \leq k \leq (d-1)$ by associating with any k-face of the tessellation a precise centroid and by using the underlying Palm probability measure $\widetilde{P}_k^0$, see e.g. (Møller, 1989; Møller, 1994).

Finally, a different approach consists in defining a typical point on a k-face by considering the stationary random measure $M_k = \sum_{\{F \ k\text{-face}\}} \mathcal{H}_k(F \cap \cdot)$ and its associated Palm probability measure P_k^0 on the set of locally finite subsets of $\mathbb{R}^d$:

$$\int f(\omega) \mathrm{d}P_k^0(\omega) = \frac{1}{\gamma_k \lambda_d(B)} \mathbf{E}\left\{ \int_B f(\{z(C) - x, C \text{ cell}\}) \mathrm{d}M_k(x) \right\}, \tag{0.5}$$

where f is a measurable bounded function defined on locally finite subsets of $\mathbb{R}^d$ and γ_k is defined as the multiplicative constant such that the deterministic measure $\mathbf{E}(M_k)$ which is invariant under translations is equal to $\gamma_k \lambda_d$, see e.g. (Baumstark and Last, 2007). We say that P_k^0 is the distribution of the point

process of the centroids of the cells "seen from a typical point on a k-face". We denote by $C_{0,k}$ the k-face containing the origin under P_k^0.

The two Palm procedures (seeing the tessellation from the centroid of a k-face or from a typical point of a k-face) are related in the same way as $\mathscr{C}$ and C_0 in equation (0.4). Indeed, Neveu's exchange formula (Neveu, 1977) provides the relation

$$\mathbf{E}(f(\mathscr{C}_k)) = \frac{1}{\mathbf{E}(\mathcal{H}_k(C_{0,k})^{-1})}\mathbf{E}\left(\frac{f(C_{0,k})}{\mathcal{H}_k(C_{0,k})}\right) \tag{0.6}$$

for any bounded, measurable and invariant-under-translations $f : \mathcal{K} \to \mathbb{R}$, see Baumstark and Last (2007). As a matter of fact, (0.6) is generally applied to functions f which are invariant-under-rotations as well.

0.1.2.4 *Realizations of the typical cell.* The typical cell is not a particular cell isolated from a realization of the tessellation. Nevertheless, it can be explicitly constructed in both Poisson–Voronoi and Poisson hyperplane cases.

The key result in both cases is the well-known and very useful Slivnyak's formula for Poisson processes (see [**CHAPTER Classical stochastic geometry**]). If X is a Poisson point process in $\mathbb{R}^d$ of intensity measure μ, then for every $n \geq 1$

$$\mathbf{E}\left\{\sum_{\{x_1,\ldots,x_n\}\subset X} f(x_1,\ldots,x_n,X)\right\} = \frac{1}{n!}\int \mathbf{E}\left\{f(x_1,\ldots,x_n,X\cup\{x_1,\ldots,x_n\})\right\}\mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_n), \tag{0.7}$$

where f is a bounded, measurable and invariant-under-permutations function defined on the product $(\mathbb{R}^d)^n \times \mathcal{N}$, $\mathcal{N}$ being the set of locally finite subsets of $\mathbb{R}^d$, see Møller (1994).

If the set of centroids is a homogeneous Poisson point process, a basic use of Slivnyak's formula implies that the associated Palm measure is the distribution of the same process with an extra point at the origin. In particular, the typical cell $\mathscr{C}$ of a Poisson–Voronoi tessellation generated by a homogeneous Poisson point process X is equal in distribution to the zero-cell of a Voronoi tessellation constructed with the new set of nuclei $X \cup \{0\}$. In the Poisson hyperplane tessellation case, a specific choice of a centroid process is required. The result below is a particular example of a possible realization of the typical cell. It is based on (Calka, 2001).

If we take for $z(C)$ the center of the largest ball included in the cell C, Slivnyak's formula allows us to obtain a generalization in any dimension of the construction given by Miles in dimension two (Miles, 1973).

- Let R and $(U_0,\ldots,U_d)$ be independent random variables with values in $\mathbb{R}_+$ and $(\mathbb{S}^{d-1})^{(d+1)}$ respectively such that R is exponentially distributed with mean $1/\omega_d$ and $(U_0,\ldots,U_d)$ has a density with respect to the uniform

measure on $(\mathbb{S}^{d-1})^{(d+1)}$ which is proportional to the volume of the simplex constructed with these $(d+1)$ vectors (multiplied by the indicator function that this simplex contains the origin).

- Let Y be a point process such that conditionally on $\{R = r\}$, $r > 0$, Y is a Poisson point process of intensity measure $\mathbf{1}_{\mathbb{R}^d \setminus B_r(0)}(x)\|x\|^{-(d-1)}dx$.
- Let $\mathscr{C}_1$ be the polyhedron containing the origin obtained as the intersection of the $(d+1)$ half-spaces bounded by the hyperplanes H_{RU_i}, $0 \leq i \leq d$.
- Let $\mathscr{C}_2$ be the zero-cell of the hyperplane tessellation associated with Y.

Then the typical cell of the stationary Poisson hyperplane tessellation is distributed as the intersection $\mathscr{C}_1 \cap \mathscr{C}_2$.

The next part provides some precise results on the distribution of several geometrical characteristics of the typical cell and the typical k-faces.

0.2 Exact distributional results

0.2.1 *Number of hyperfaces and distribution of the cell conditioned on the number of hyperfaces*

In this section, we consider the zero-cell C_0 of a Poisson hyperplane process such that the intensity measure of the underlying point process is (in spherical coordinates)

$$\mathrm{d}\Theta(t,u) = \gamma \mathbf{1}_{\mathbb{R}_+}(t) t^{\alpha-1} \mathrm{d}t \mathrm{d}\sigma_d(u),$$

where σ_d is the uniform measure on $\mathbb{S}^{d-1}$, $\alpha \geq 1$ and $\gamma > 0$.

In particular, if $\alpha = 1$, we obtain the Crofton cell and if $\alpha = d$ and $\gamma = 2^d$, C_0 is distributed as the typical cell of a Poisson–Voronoi tessellation of unit intensity.

Let us denote by N_{d-1} the number of hyperfaces of C_0. In the Poisson–Voronoi case, it can be identified as the number of neighbors of the typical nucleus 0. We explain here how to calculate the probability of having n hyperfaces as a multiple integral of order n. The formula can be made fully explicit in dimension two. The lines below are based on (Calka, 2003*b*; Calka, 2003*a*).

For n points $x_1, \ldots, x_n \in \mathbb{R}^d \setminus \{0\}$, we define $D(x_1, \ldots, x_n)$ as the connected component containing the origin of $\mathbb{R}^d \setminus \cup_{i=1}^n H_{x_i}$, see (0.1) for the definition of H_{x_i}.

In any dimension, Slivnyak's formula (0.7) yields that for any $n \geq (d+1)$,

$$\mathbf{P}\{N_{d-1} = n\} = \frac{1}{n!} \int \mathrm{e}^{-\gamma\Phi(x_1,\ldots,x_n)} \mathbf{1}_{A_n}(x_1, \ldots, x_n) \mathrm{d}\Theta(x_1) \ldots \mathrm{d}\Theta(x_n), \quad (0.8)$$

where $\gamma\Phi(x_1, \ldots, x_n) = \Theta\left(\{x \in \mathbb{R}^d : H_x \cap D(x_1, \ldots, x_n) \neq \emptyset\}\right)$ and A_n is the subset of n-tuples $(x_1, \ldots, x_n)$ such that $D(x_1, \ldots, x_n)$ is a convex polyhedron with n hyperfaces and containing the origin 0.

Let us remark that this formula was heuristically obtained in dimension two and in the Poisson–Voronoi case by Miles and Maillardet (1982). The functional Φ as well as the indicator function $\mathbf{1}_{\mathbf{A_n}}$ need to be made more explicit in function

of $x_1, \dots, x_n$. For any convex subset K of $\mathbb{R}^d$ which contains the origin, we denote by $h(K,u) = \sup\{\langle x,u\rangle : x \in K\}$, $u \in \mathbb{S}^{d-1}$, its support function. Then for any real $\alpha \geq 1$,

$$\Phi(x_1,\dots,x_n) = \frac{1}{\alpha}\int h(D(x_1,\dots,x_n),u)^\alpha \mathrm{d}\sigma_d(u).$$

In the two-dimensional case, if the points $x_1 = (r_1,\theta_1),\dots,x_n = (r_n,\theta_n)$ are sorted by angular coordinate such that $0 \leq \theta_1 < \cdots < \theta_n < 2\pi$, we have for every $u_\theta = (\cos(\theta),\sin(\theta))$ with $\theta_i \leq \theta < \theta_{i+1}$ $(1 \leq i < n)$

$$h(D(x_1,\dots,x_n),u_\theta) = \frac{1}{\sin(\theta_{i+1}-\theta_i)}(\sin(\theta_{i+1}-\theta)r_i + \sin(\theta-\theta_i)r_{i+1})$$

and the integration of h^α can be carried out for $\alpha = 1$ and $\alpha = 2$. If $\gamma = 1$, we then obtain in the first case the perimeter of the set $D(x_1,\dots,x_n)$ and in the second case the area of the fundamental domain of $D(x_1,\dots,x_n)$ (often called *Voronoi flower*), i.e. the union of the n discs centered at $y_i/2$ and of radius $\|y_i\|/2$, where the y_i, $1 \leq i \leq n$, are the vertices of $D(x_1,\dots,x_n)$. Lastly, we can express the indicator function with the use of polar coordinates: $(x_1,\dots,x_n) \in C_n$ if and only if for every $1 \leq i \leq n$,

$$r_{i-1}\sin(\theta_{i+1}-\theta_i) + r_{i+1}\sin(\theta_i-\theta_{i-1}) > r_i\sin(\theta_{i+1}-\theta_{i-1})$$

with the conventions $x_0 = (r_0,\theta_0) = x_n$ and $x_{n+1} = x_1$.

Consequently, we obtain an explicit formula which provides a way to do numerical calculations or to look for asymptotic estimates and limit shapes for many-sided cells, see e.g. (Hilhorst, 2005; Hilhorst, 2006; Hilhorst and Calka, 2008). Nevertheless it is of little help for estimating the moments, for instance, it is very hard to verify the well-known equality $\mathbf{E}(N_1) = 6$ in the planar Poisson–Voronoi case. In this connection, it should be noted that a calculation of the second moment of the number of vertices of the typical Poisson–Voronoi cell in any dimension has been recently provided in (Heinrich and Muche, 2008) via the use of second-order properties of the point process of nodes of the tessellation, see also (Heinrich, Körner, Mehlhorn and Muche, 1998).

Going back to (0.8), let us add that the functional of $(x_1,\dots,x_n)$ inside the integral is up to a multiplicative constant the density of the distribution of the respective positions of the n hyperplanes which surround the zero-cell conditioned on having n hyperfaces. In particular, it provides an easy way to verify that conditionally on $\{N_{d-1} = n\}$, $n \geq (d+1)$, $\Theta(\{x \in \mathbb{R}^d : H_x \cap C_0 \neq \emptyset\})$ is Gamma-distributed with parameters n and 1, see (Zuyev, 1992; Cowan, Quine and Zuyev, 2003) and also the work by S. Zuyev on "stopping sets" techniques (Zuyev, 1999).

0.2.2 *Typical k-face of a section of a Poisson–Voronoi tessellation*

This section is a quick survey of a number of papers (Møller, 1989; Muche and Stoyan, 1992; Mecke and Muche, 1995; Muche, 1996; Schlather, 2000; Muche,

2005; Baumstark and Last, 2007), which concern exclusively the d-dimensional Poisson–Voronoi tessellation or its sections with deterministic affine subspaces and propose mainly explicit formulas for the distribution of the typical k-face or typical edge.

Let $\mathcal{T}$ be a Voronoi tessellation generated by a set of nuclei distributed as a homogeneous Poisson point process of intensity 1. Miles (1970), Mecke and Muche (1995) and Muche (1996) provided a precise description of the Palm measure P_0^0 as defined in (0.5), i.e. the distribution of the point process of the nuclei as seen from a typical vertex. Baumstark and Last (2007) prove a generalization of their result, i.e. a full characterization of the Palm measure P_k^0 $(0 \leq k \leq d)$ which is explained below.

- The random set P_k^0 contains no point (from the homogeneous Poisson point process) apart from the origin 0 in a random ball $B_{R_k}(0)$ such that R_k is Gamma-distributed variable with parameters $(d-k+k/d)$ and κ_d.
- Conditioned on $\{R_k = r\}$, $r > 0$, $P_k^0 \cap \{x \in \mathbb{R}^d : \|x\| > r\}$ is distributed as a homogeneous Poisson point process of intensity 1.
- The intersection $P_k^0 \cap (R_k \mathbb{S}^{d-1})$ contains exactly $(d+1-k)$ points from the homogeneous Poisson point process which are independent with $(R_k, \{x \in P_k^0 : \|x\| > r\})$ and distributed as follows: we denote by Z_k and R'_k the center and the radius of the unique $(d-1-k)$-dimensional sphere containing these $(d+1-k)$ nuclei. The $(d+1-k)$ neighbors of the origin are then distributed as $\sqrt{R_k^2 - R_k'^2}U + R'_k U_i$, $0 \leq i \leq (d-k)$ (up to a special orthogonal transformation), where
  - we have $U \in \mathbb{S}^{d-1}$ and $(U_0, \ldots, U_{d-k}) \in (\mathbb{S}^{d-1-k})^{(d-k+1)}$,
  - the $(d-k+1)$-tuple $(U_0, \ldots, U_{d-k})$ has a density (with respect to the uniform measure on $(\mathbb{S}^{d-1-k})^{(d-k+1)}$) which is proportional to the $(d-k)$-dimensional Hausdorff measure raised to the power $(k+1)$ of the simplex spanned by the $(d-k+1)$ vectors,
  - conditioned on $(U_0, \ldots, U_{d-k})$, the direction U is uniformly distributed on $\mathbb{S}^{d-1} \cap \{U_0, \ldots, U_d\}^{\perp}$,
  - the quantity $R_k'^2/R_k^2$ is independent with the vector of directions $(U, U_0, \ldots, U_{d-k})$ and is Beta-distributed with parameters $d(d-k)/2$ and $k/2$.

The main tools for proving this decomposition are Slivnyak's formula (0.7) and the Blaschke–Petkantschin change of variables formula, see Satz 7.2.1 of (Schneider and Weil, 2000). It can be thought as a generalization of the previously known distributional results about the Poisson–Delaunay typical cell which would correspond here to the case $k = 0$, see e.g. (Miles, 1970; Møller, 1994). Let us remark finally that this results have been recently extended to Laguerre tessellations by Lautensack (2007).

This description combined with the relation expressed by (0.6) between the Palm measure P_k^0 and the typical k-face $\mathscr{C}_k$ implies some precise distributional results on $\mathscr{C}_k$. If the centroid of a k-face is chosen as the equidistant point from

the $(d-k+1)$ neighbors of the k-face and in the affine subspace generated by the neighbors, it is possible to calculate explicitly the joint distribution of the vector constituted with $\mathcal{H}_k(\mathscr{C}_k)$, the distance ρ_k from the centroid 0 of the typical k-face to the $(d-k+1)$ neighbors of the k-face and the $(d-k+1)$ unit-vectors which determine the directions from the centroid to the neighbors. In particular, these directions are shown to be independent with $(\mathcal{H}_k(\mathcal{C}_k), \rho_k)$ and have the same distribution as $(U_0, \ldots, U_{d-k})$ in the construction above.

If $k = 1$, the calculation can be simplified and provides the same formulas as in (Muche, 2005), which will be given afterwards. In that paper, L. Muche investigates more generally the typical edge of a section of a Poisson–Voronoi tessellation of intensity 1 by a deterministic s-dimensional affine subspace where $1 \leq s \leq d$ and $d \geq 2$. This work unifies and extends previous efforts due to Brakke (1985), Møller (1989), Muche and Stoyan (1992), and Schlather (2000).

Exploiting the fact that the typical edge is equal in distribution to a randomly chosen edge emanating from the typical vertex, L. Muche makes explicit the joint distribution of the vector constituted with the length L of the typical edge and the two adjacent angles $B_1 = \angle(v_1, v_2, x)$ and $B_2 = \angle(v_2, v_1, x)$ where v_1 and v_2 are the vertices of the edge and x is one of the s neighbors of the edge. The density of (L, B_1, B_2) is up to a multiplicative constant equal to

$$f(l, \beta_1, \beta_2) = \frac{l^{d(s+2)-s-1}(\sin(\beta_1)\sin(\beta_2))^d}{\sin^{ds+2d-s}(\beta_1+\beta_2)} f_B(\beta_1) f_B(\beta_2) \exp\left(-\nu(l, \beta_1, \beta_2)\right),$$

where

- the quantity $\nu(l, \beta_1, \beta_2)$ is the d-dimensional Lebesgue measure of the union of two balls such that their centers v_1 and v_2 are at distance l and the angles $\angle(v_2, v_1, x)$ and $\angle(v_1, v_2, x)$ are equal to β_1 and β_2, x being any point at the intersection of the boundaries of the balls.

- $f_B(\beta) = \dfrac{(d-1)s\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{ds+d-s}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{ds+d-s+1}{2}\right)} \sin^{ds-s-1}(\beta) \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} b_i(\beta),$

  for every $0 \leq \beta \leq \pi$ with

$$b_0 = \begin{cases} \frac{1}{\pi}[(\pi-\beta)\cos(\beta) + \sin(\beta)] & \text{if d even} \\ \cos^2(\beta/2) & \text{else} \end{cases},$$

$$b_i = \begin{cases} -\frac{\Gamma(i)}{4\sqrt{\pi}\Gamma(i+\frac{3}{2})} \sin^{2i+1}(\beta) & \text{if d even} \\ -\frac{\Gamma(i-\frac{1}{2})}{4\sqrt{\pi}\Gamma(i+1)} \sin^{2i}(\beta) & \text{else} \end{cases}, \quad 1 \leq i \leq \lfloor (d-1)/2 \rfloor.$$

Let us remark that the explicit calculation of all the moments of the variables can be deduced from the formulas above. Additional calculations were added in (Muche, 2005) about the relative positions of the s neighbors. It concerns

in particular the distance from any neighbor to the affine subspace generated by the typical edge and the conditional distribution of the length of the edge conditionally on the fact that the projection of any neighbor on the spanned affine subspace is inside the edge or not.

0.2.3 *The circumscribed radius*

This section provides a different kind of information on the zero-cell C_0 defined in section 0.2.1. Indeed, we are now interested in putting in an optimal way the boundary of C_0 between two spheres centered at 0. The main result is that in any dimension, the joint distribution of the radii of the two spheres can be expressed in terms of covering probabilities of the unit-sphere by random caps and in dimension two, it can be explicitly calculated. The following ideas are basic generalizations of the results contained in (Calka, 2002).

We introduce the quantity $R_m = \sup\{r > 0 : B_r(0) \subset C_0\}$, i.e. the radius of the largest disk centered at 0 and contained in the cell. Clearly, we have

$$\mathbf{P}\{R_m \geq r\} = \exp\left(-\frac{\gamma\omega_d}{\alpha}r^\alpha\right), \quad r > 0,$$

and the distribution of the hyperplane process conditioned on $\{R_m \geq r\}$, $r > 0$, is a new hyperplane process of intensity measure $\mathbf{1}_{\mathbb{R}^d \setminus B_r(0)} d\Theta$.

In order to have a more precise idea of the shape of C_0, it is relevant to consider the radius of the circumscribed ball centered at the origin, i.e. $R_M = \inf\{r > 0 : B_r(0) \supset C_0\}$. It can be shown that the distribution of R_M is related to the covering probability of $\mathbb{S}^{d-1}$ by a Poissonian number of independent random circular caps such that their centers on $\mathbb{S}^{d-1}$ are uniformly distributed and their angular radii (divided by π) have the distribution given below:.

$$d\nu(\theta) = \alpha\pi \sin(\pi\theta)\cos^\alpha(\pi\theta)\mathbf{1}_{[0,1/2]}(\theta)d\theta.$$

Indeed, having $R_M \geq r$ for a fixed $r > 0$ means that there is a non-empty portion of the sphere $r\mathbb{S}^{d-1}$ which is not covered by the intersection with the hyperplanes of the Poisson hyperplane process. To be more precise, let us denote by $P(\nu, n)$, for every $n \geq 0$, the covering probability of $\mathbb{S}^{d-1}$ by n independent random circular caps which are isotropic and of angular radius distributed as $\nu(\cdot/\pi)$. We obtain

$$\mathbf{P}\{R_M \geq r\} = \exp\left(-\frac{\gamma\omega_d}{\alpha}r^\alpha\right)\sum_{n=0}^{\infty}\frac{\left(\frac{\gamma\omega_d}{\alpha}r^\alpha\right)^n}{n!}(1 - P(\nu, n)).$$

This relation is easily generalized if C_0 is conditioned on $\{R_m = r\}$.

We now concentrate on the two-dimensional case. If $d = 2$, the covering probabilities $P(\nu, n)$ can be calculated (Stevens, 1939; Siegel and Holst, 1982), which provides us an expression for $\mathbf{P}\{R_M \geq r\}$. In particular, when $r \to \infty$, we use a basic ordering relation between the covering probabilities (conjectured in (Siegel, 1978) and proved in (Calka, 2002)) in order to "replace" the distribution

ν by its mean $I_\alpha = \frac{1}{\pi}\int_0^{\pi/2} \cos^\alpha(u)\mathrm{d}u$. We obtain that there exist two constants $0 < C_1 < C_2 < \infty$ such that for r large enough,

$$C_1 r^\alpha \exp\left(-\frac{2\pi\gamma I_\alpha}{\alpha} r^\alpha\right) \leq \mathbf{P}\{R_M \geq r\} \leq C_2 r^\alpha \exp\left(-\frac{2\pi\gamma I_\alpha}{\alpha} r^\alpha\right).$$

The calculation of the joint distribution of the couple (R_M, R_m) leads us to obtain an asymptotic estimation of $\mathbf{P}\{R_M \geq r + r^\delta | R_m = r\}$ for $1 - \frac{2}{3}\alpha < \delta < 1$, see section 0.3.2 for a generalization of that result in any dimension. In fact, this probability is proved to decrease exponentially fast to zero, which indicates that the zero-cell converges to a circle when its inradius goes to infinity. It is a particular case of D. G. Kendall's conjecture which will be the center of our next section.

0.3 Asymptotic results

0.3.1 *D. G. Kendall's conjecture*

A very-well known conjecture due to D. G. Kendall states that cells of large area in an isotropic Poisson line tessellation are close to the circular shape, see e.g. the foreword of (Stoyan, Kendall and Mecke, 1987). The conjecture can be rephrased in a modern setting as follows: the conditional distribution of the two-dimensional Crofton cell converges weakly when its area goes to infinity to the degenerate law concentrated at the circular shape.

Works due notably to Miles (1995) or Goldman (1998) were first advancements on the subject (see also (Goldman, 1996; Goldman and Calka, 2003) for an interpretation of the conjecture in terms of the first eigenvalue of the Dirichlet–Laplacian on the cell). Kovalenko (1997, 1998, 1999) proved the conjecture in the case of a two-dimensional isotropic and stationary Poisson line tessellation. Since then, Hug, Schneider and Reitzner (2004*a*, 2004*b*) have obtained far more precise results which generalize D. G. Kendall's conjecture in four different ways: more general Poisson hyperplane tessellations, more general functionals to measure the largeness of a cell, explicit estimates for deviations from asymptotic shapes and identification of the cases where limit shapes do not exist. Their proofs mix precise arguments from geometry of convex bodies (in particular, isoperimetric inequalities and existence of associated extremal bodies) combined with probabilistic estimations which make good use of the Poissonian distribution. This section is devoted to a basic presentation of the main results and the underlying methods contained in papers by Hug, Reitzner and Schneider (2004*a*, 2004*b*) and by Hug and Schneider (2004, 2007*a*).

0.3.1.1 *Context and useful functionals.* We consider the zero-cell C_0 of a hyperplane tessellation in $\mathbb{R}^d$, $d \geq 2$, such that its intensity measure denoted by Θ is defined in spherical coordinates by the equality

$$\mathrm{d}\Theta(t, u) = \gamma t^{\alpha-1}\mathrm{d}t\varphi(\mathrm{d}u),$$

where $\gamma > 0$, $\alpha \geq 1$ and φ is a probability measure on $\mathcal{S}^{d-1}$ such that its support is not contained in a half-sphere. As previously mentioned, this general model interpolates the cases of the Crofton cell of an isotropic Poisson hyperplane tessellation and of the typical Poisson–Voronoi cell.

Here we follow the scheme developed in the papers cited above. Three types of functionals are used to study size and shape of the zero-cell C_0. They are defined on the set denoted by $\mathcal{K}_0$ of all convex bodies K such that K contains 0 and is the intersection of its supporting halfspaces which have an outer unit normal vector in the support of φ.

- The *parameter functional* Φ (already used in section 0.2.1) is defined by the equality
$$\Phi(K) = \frac{1}{\alpha} \int h(K,u)^\alpha \varphi(\mathrm{d}u), \quad K \in \mathcal{K}_0.$$
It is a continuous function related to the intensity measure in such a way that the probability $\mathbf{P}\{K \subset C_0\}$ is equal to $\exp(-\gamma\Phi(K))$.
- The *size functional* denoted by Σ is a function aimed at "measuring" the size of C_0. The only properties that Σ has to satisfy are that it has to be continuous, increasing and homogeneous of some degree $k > 0$ (i.e. $\Sigma(rK) = r^k\Sigma(K)$ for every $K \in \mathcal{K}$ and $r > 0$). Volume, surface area or inradius (see section 0.3.2) are basic examples of such a function.
- The *deviation functional* $\mathcal{V}$ is related to the two previous functionals Φ and Σ and will measure the distance between C_0 and the potential limit shape. It is defined in the following way: since Φ is homogeneous of degree α, the two previous functionals Φ and Σ satisfy an isoperimetric inequality of the form
$$\Phi(K) \geq \tau\Sigma(K)^{\alpha/k}, \quad K \in \mathcal{K}_0, \tag{0.9}$$
where τ is some positive constant which can be chosen such that there exist convex bodies $K \in \mathcal{K}_0$, called *extremal bodies*, for which the equality holds. The functional $\mathcal{V}$ is then introduced as a continuous, non-negative and homogeneous of degree zero function such that $\mathcal{V}(K) = 0$ implies that K is extremal. For instance, $\mathcal{V}$ can be defined by the equality
$$\mathcal{V}(K) = \frac{\Phi(K)}{\tau\Sigma(K)^{\alpha/k}} - 1, \quad K \in \mathcal{K}. \tag{0.10}$$

Let us remark that the isoperimetric inequality (0.9) can be strengthened as follows: there exists a non-negative continuous function f on $\mathbb{R}_+$ with a unique zero at zero such that if $\mathcal{V}(K) \geq \varepsilon > 0$ for $K \in \mathcal{K}_0$, K must satisfy the inequality $\Phi(K) \geq \tau(1 + f(\varepsilon))\Sigma(K)^{\alpha/k}$.

0.3.1.2 *Estimates of conditional probabilities.* If $\varepsilon > 0$ and $a > 0$ are fixed, the aim is now to evaluate conditional probabilities
$$\mathbf{P}\{\mathcal{V}(C_0) \geq \varepsilon | \Sigma(C_0) \geq a\} = \frac{\mathbf{P}\{\mathcal{V}(C_0) \geq \varepsilon; \Sigma(C_0) \geq a\}}{\mathbf{P}\{\Sigma(C_0) \geq a\}} \tag{0.11}$$

for a fixed $\varepsilon > 0$ and a sufficiently large.

A first remark is that it is easier to give a lower bound of the denominator in (0.11). Indeed, for an extremal body K, the probability of having $K \subset C_0$ is equal to $\exp(-\tau\gamma\Sigma(K)^{\alpha/k})$. Consequently, it suffices to compare C_0 with an extremal body included in it of size a to obtain that $\mathbf{P}\{\Sigma(C_0) \geq a\} \geq \exp(-\tau\gamma a^{\alpha/k})$ for every $a > 0$.

The estimation of the numerator in (0.11) is a much more delicate matter. Following the structure of the proofs contained in (Hug, Reitzner and Schneider, 2004*a*; Hug and Schneider, 2004), we describe below some of the key arguments in order to do it.

- The range of $\Sigma(C_0)$ in the event has first to be limited to an interval $[a, a(1+h)]$ for some h "not too large" (afterwards, the range is extended by a covering argument).
- An additional condition is added in the event in order to guarantee that the diameter of C_0 is bounded and that C_0 is included in a deterministic ball B (afterwards, the sum over all possible intervals for the diameter is taken).
- The cell C_0 is defined as the intersection of all half-spaces coming from the initial hyperplane process. Hyperplanes which have a non-empty intersection with the boundary of C_0 must also intersect the deterministic ball B introduced above. Consequently, the event we are interested in can be rewritten in terms of the set $\mathcal{G}_B$ of all hyperplanes which hit B. Fortunately the distribution of $\mathcal{G}_B$ is known: its cardinality is shown to be Poisson distributed, of mean $\gamma\Phi(B)$ and all hyperplanes in $\mathcal{G}_B$ are i.i.d. and distributed as $(\gamma\Phi(B))^{-1}\mathbf{1}_{\{H_x \cap B \neq \emptyset\}}\mathrm{d}\Theta$.
- By an argument of convex geometry, for a fixed $\alpha > 0$, the polyhedron C_0 can be replaced by the convex hull $\widetilde{C}_0$ of a finite number of its vertices in such a way that the ratio $\Phi(\widetilde{C}_0)/\Phi(C_0)$ is more than $1 - \alpha$.
- The numerator in (0.11) is finally overestimated by the probability that $\widetilde{C}_0$ is not hit by the majority of the hyperplanes involved.

0.3.1.3 *Results and examples.* Various extensions of the estimations presented above imply the following general result: there exists a constant c_0 depending only on the dimension d such that for every $\varepsilon > 0$ and $0 < a < b \leq \infty$ with $a^{\alpha/k}\gamma \geq \sigma_0$, we have

$$\mathbf{P}\{\mathcal{V}(C_0) \geq \varepsilon | \Sigma(C_0) \in [a, b)\} \leq c \exp(-c_0 f(\varepsilon) a^{\alpha/k}\gamma), \tag{0.12}$$

where c is a constant depending only on the measure Θ and the choices of Σ, f, ε and σ_0.

Moreover, the question of the existence of a limit shape has also been investigated: the shape of a convex set K is defined as the equivalence class of K under the action of a subgroup of similarities. The zero-cell C_0 is said to have a limit shape if the distribution of the shape of C_0 conditioned on $\{\Sigma(C_0) \geq a\}$

converges weakly to a Dirac measure when $a \to \infty$. In particular, if there exists a subgroup of similarities which preserves $\mathcal{K}_0$ and the function defined in (0.10) and such that all extremal bodies are in the same equivalence class, then the zero-cell C_0 admits a limit shape. It should be noticed that this limit shape depends not only on the distribution Θ but also on the chosen size functional.

The general result (0.12) can be applied in the particular case where C_0 is the typical Poisson–Voronoi cell ($\gamma = 2^d\omega_d$, $\alpha = d$ and φ is the uniform probability measure on $\mathbb{S}^{d-1}$). As previously seen in section 0.2.1, Φ is the Lebesgue measure of the union of all the balls $B_{\|x\|/2}(x/2)$ where x is any point of C_0. If the size functional is the k-th intrinsic volume, the limit shape is a ball and a convenient choice of deviation functional is $\mathcal{V}(K) = (R_M - R_m)/(R_M + R_m)$ where the radii R_m and R_M are the same as in section 0.2.3. In particular, if $\Sigma(C_0) = \lambda_d(C_0)$, the estimation (0.12) becomes

$$\mathbf{P}\left\{\frac{R_M - R_m}{R_M + R_m} \geq \varepsilon \middle| \lambda_d(C_0) \in [a, b)\right\} \leq c \exp(-c_0 \varepsilon^{(d+1)/2} a\gamma). \tag{0.13}$$

In the same way, if $\Sigma(C_0) = R_m$, we have

$$\mathbf{P}\left\{\frac{R_M - R_m}{R_M + R_m} \geq \varepsilon \middle| R_m \in [a, b)\right\} \leq c \exp(-c_0 \varepsilon^{(d+1)/2} a^d\gamma). \tag{0.14}$$

When C_0 is the Crofton cell ($\gamma = \omega_d$, $\alpha = 1$ and φ is the uniform probability measure on $\mathbb{S}^{d-1}$), D. G. Kendall's conjecture (i.e. convergence to the limit shape of a ball) is solved in any dimension with the particular choice of the d-dimensional Lebesgue measure as the size functional and a deviation function $\mathcal{V}$ defined in the following way: the quantity $\mathcal{V}(K)$ is the infimum of $(s/r - 1)$ over all couples $(s, r) \in (\mathbb{R}_+^*)^2$ such that there exists a translate of K which is between $B_r(0)$ and $B_s(0)$ for the inclusion. Interestingly, the limit shape is not a ball but a segment if the size is measured by the diameter.

These results for the Crofton cell can be extended to the typical cell of a stationary Poisson hyperplane tessellation through the use of a new realization of the typical cell based on the choice of the lowest points of the cells as centroids (Hug and Schneider, 2007*b*), see section 0.1.2.4. Other extensions of this work concern the determination of a logarithmic equivalent for the distribution tail of $\Sigma(C_0)$ (Goldman, 1998; Hug and Schneider, 2007*a*) and large typical cells in Poisson–Delaunay tessellations (Hug and Schneider, 2005).

0.3.2 *Cells with a large inradius*

We go back to the model introduced in section 0.2.1, i.e. we concentrate on the particular case where the hyperplane process is isotropic, of intensity measure $\mathrm{d}\Theta = t^{\alpha-1}\mathrm{d}t\mathrm{d}\sigma_d(u)$ where $1 \leq \alpha \leq d$. We suppose now that the inball centered at the origin is large. The preceding results show that the cell is close to a ball but some specifications can be added. Indeed, the boundary of the cell can be proved to be inside an annulus around the origin with a decreasing thickness when the

inradius goes to infinity. Moreover, limit theorems can be deduced from this fact for both the number N_{d-1} of hyperfaces and the volume V_d between the boundary of the cell and the inball. We give below a description of the methods involved and of the main asymptotic results. The results and methods developed below are almost direct generalizations of a joint work with Schreiber (Calka and Schreiber, 2005; Calka and Schreiber, 2006).

In any dimension, an asymptotic estimation of the probability $\mathbf{P}\{R_M \geq r+s|R_m = r\}$ $(s>0)$ when r goes to infinity, is made possible by a method introduced in (Calka and Schreiber, 2006). This procedure allows us to estimate the quantities N_{d-1} and V_d as well and can be described as follows:

- *Step 1.* After an homothetic transformation on the zero-cell C_0 conditioned on $\{R_m = r\}$, we obtain the zero-cell associated with a deterministic hyperplane at distance one from 0 and a hyperplane process outside $B_1(0)$ of intensity measure $r^\alpha \mathbf{1}_{\mathbb{R}\setminus B_1(0)}\mathrm{d}\Theta$. The number N_{d-1} is preserved whereas V_d is multiplied by r^{-d}.
- *Step 2.* Let us apply the inversion defined by $I(x) = x/\|x\|^2$ for every $x \in \mathbb{R}^d \setminus \{0\}$. It transforms the zero-cell into a germ-grain model inside the unit-ball. More precisely, the image of the hyperplane process outside $B_1(0)$ is a process of spheres centered at $y/2$ and of radius $\|y\|/2$ where y is an element of a Poisson point process Ψ inside $B_1(0)$ of intensity measure $r^\alpha \mathbf{1}_{B_1(0)} t^{-(\alpha+1)}\mathrm{d}t\mathrm{d}\sigma_d(u)$. The number N_{d-1} can be seen as the number of extreme points of the convex hull of Ψ and the volume V_d as $r^{-d}\mu(B_1(0) \setminus \cup_{y\in\Psi\cup\{y_0\}} B_{\|y\|/2}(y/2))$ where y_0 is a deterministic point on $\mathbb{S}^{d-1}$ and $\mu = \mathbf{1}_{B_1(0)} t^{-(d+1)}\mathrm{d}t\mathrm{d}\sigma_d(u)$.
- *Step 3.* We consider a Poisson point process Ψ in $B_1(0)$ of intensity measure λdx $(\lambda > 0)$ or in a more general context $\lambda f(t)dtd\sigma_d(u)$ with $\lim_{t\to 1} f(t) = 1$. Then
  - the convex hull of this process contains the ball $B_{(1-Kt^{-\delta})}(0)$ (for a fixed constant $K>0$ and with $0<\delta<2/(d+1)$) with a probability going to 1 exponentially fast (Calka and Schreiber, 2006).
  - the number of vertices of the convex hull follows a law of large numbers (Rényi and Sulanke, 1963; Reitzner, 2003), as well as a central limit theorem (Reitzner, 2005) and a large deviation-type result (Calka and Schreiber, 2006; Vu, 2005).
  - the volume or the μ-measure of the set between the unit-sphere and the union $\cup_{y\in\Psi} B_{\|y\|/2}(y/2)$ satisfies a law of large numbers, a central limit theorem and a moderate deviation principle (Schreiber, 2002; Schreiber, 2003).

We obtain from the three steps above the following results:

1. There exists a constant $c>0$ such that for every $\frac{d+1-2\alpha}{d+1} < \delta < 1$, we have when r goes to infinity

$$\mathbf{P}\{R_M \geq r + r^\delta | R_m = r\} = \mathcal{O}(\exp(-cr^\beta)) \tag{0.15}$$

where $\beta = \frac{1}{2}\left[(2\alpha - (d+1)) + \delta(d+1)\right]$.
In other words, the boundary of the zero-cell conditioned on $\{R_m = r\}$ is typically included in an annulus of thickness $r^{(d+1-2\alpha)/(d+1)}$.

2. There exists a constant $a > 0$ (known explicitly) depending only on d and α such that

$$N_{d-1}(ar^{\frac{\alpha(d-1)}{d+1}})^{-1} \longrightarrow 1 \quad \text{in } L^1 \text{ and a.s. when } r \to \infty.$$

3. The number N_{d-1} satisfies a central limit theorem when $r \to \infty$ as well as a moderate-deviation result: for every $\varepsilon > 0$,

$$\liminf_{r\to\infty} \frac{1}{\log(r)} \log\left(-\log\left(\mathbf{P}\left\{\left|\frac{N_{d-1}}{\mathbf{E}N_{d-1}} - 1\right| \geq \varepsilon\right\}\right)\right) \geq \frac{\alpha(d-1)}{3d+5}.$$

4. The same type of limit theorems holds for the quantity V_d which grows as $r^{\frac{\alpha(d-1)}{d+1}}$ (up to an explicit multiplicative constant).

It emerges that in the context of a large inradius, supplementary informations on the growth of the number of hyperfaces and of the volume outside the inball are obtained to specify the convergence of the random polyhedron to the ball-shape. Besides, we may notice that the asymptotic result (0.15) could not be deduced from the previous estimation (0.14) since the constant c depends on ε in the latter inequality.

The last part is independent with the rest of the chapter: it concerns different types of tessellation called iterated tessellations, which are natural models in several concrete situations and have been recently investigated for applicational purposes.

0.4 Iterated tessellations

0.4.1 *Tessellations stable with respect to iteration*

Real tessellations may present "hierarchical" structures, which occur in some crack structures, as the "craquelé" on pottery surfaces. In order to provide a good approximation, W. Nagel and V. Weiss have investigated the iteration of tessellations. They aim at determining the existence of tessellations which are stable (in distribution) with respect to iteration (STIT) and at characterizing these tessellations.

An explicit model called the crack STIT tessellation is given via an algorithmic construction and it is proved that such a model is indeed STIT and conversely, that any STIT tessellation is a crack STIT tessellation. This section is devoted to a formal description of the work due to W. Nagel and V. Weiss (Nagel and Weiss, 2003; Nagel and Weiss, 2004; Nagel and Weiss, 2005).

0.4.1.1 *Construction of a crack STIT tessellation in a window.* Let φ be a probability measure on $\mathbb{S}^{d-1}$ such that its support contains a basis of $\mathbb{R}^d$. As we previously did, we consider the associated measure $\mathrm{d}\Theta = \mathrm{d}t\phi(\mathrm{d}u)$ on the

set of hyperplanes of $\mathbb{R}^d$ which is supposed to be invariant with respect to the translations of $\mathbb{R}^d$. For a bounded Borel set $C \subset \mathbb{R}^d$, we denote by $[C]$ the set of all hyperplanes that hit C and by Θ_C the probability measure on $[C]$ defined by the relation $\Theta_C = \frac{1}{\Theta([C])}\Theta(\cdot \cap [C])$ (if $0 < \Theta([C]) < \infty$).

Let $W \subset \mathbb{R}^d$ be a d-dimensional compact and convex domain such that $0 < \Theta([W]) < \infty$. The *crack STIT tessellation* $\mathcal{T}(a, W)$ is constructed in W and on a time interval $[0, a]$, $a > 0$, as follows: an i.i.d sequence (τ_i, γ_i), $i \geq 1$, is given where τ_i is a random time which is exponentially distributed with parameter $\Theta([W])$ and γ_i is a random hyperplane with distribution Θ_W.

- If $\tau_1 > a$, the algorithm does not begin and the tessellation is W itself.
- If $\tau_1 \leq a$, the algorithm starts with a first cutting of W at time τ_1 into two parts W_+ and W_-. W_+ and W_- are then treated in the same way, separately and independently. Let us describe the evolution of W_+:
  - If $\tau_1 + \tau_2 > a$, W_+ is conserved as it is and will be a part of the final tessellation.
  - If $\tau_1 + \tau_2 \leq a$, W_+ is divided at time $\tau_1 + \tau_2$ by γ_2 if γ_2 intersects W_+. If W_+ has not been divided by γ_2, the next potential division of W_+ occurs at time $\tau_1 + \tau_2 + \tau_3$ (if that time is less than a). If W_+ has been divided by γ_2, the algorithm goes on with the two subsections $W_{+,+}$ and $W_{+,-}$.

This construction can be fulfilled equivalently through a formal description based on random binary trees.

0.4.1.2 *Extension to a crack STIT tessellation of $\mathbb{R}^d$ and stability with respect to iteration.* The capacity functional of the tessellation $\mathcal{T}(a, W)$,

$$T_{\mathcal{T}(a,W)} : C \longmapsto \mathbf{P}\{\mathcal{T}(a, W) \cap C = \emptyset\}, \qquad C \in \mathcal{K} \cap \mathscr{C},$$

can be calculated if C is connected and recursively if C has a finite number of connected components. Moreover, this computation does not depend on the window W which contains C and is invariant with respect to translations of $\mathbb{R}^d$. Consequently, there exists a random stationary tessellation $\mathcal{T}(a)$ of the whole space $\mathbb{R}^d$ such that its intersection with a compact convex window W is equal in distribution to $\mathcal{T}(a, W)$ (see Satz of 2.3.1 in (Schneider and Weil, 2000)). In particular, $\mathcal{T}(a)$ satisfies the scaling property, i.e. for $a > 0$, $\mathcal{T}(a)$ coincides in distribution with $\frac{1}{a}\mathcal{T}(1)$.

A fundamental property of the tessellation $\mathcal{T}(a)$ is that it is stable with respect to iteration. More precisely, let us define the operation of iteration: an initial stationary tessellation $\mathcal{T}^{(0)}$ of the whole space $\mathbb{R}^d$ and a sequence of i.i.d. tessellations $\mathcal{Y} = \{\mathcal{T}^{(i)} : i \geq 1\}$ are given. We denote by $C_1^{(0)}, C_2^{(0)}, \ldots$ the cells of $\mathcal{T}^{(0)}$. The iterated tessellation $I(\mathcal{T}^{(0)}, \mathcal{Y})$ is then obtained by replacing in $\mathcal{T}^{(0)}$ the interior of each cell $C_i^{(0)}$ by $C_i^{(0)} \cap \mathcal{T}^{(i)}$, $i \geq 1$.

In order to preserve the same surface intensity of the tessellation, a rescaling is needed. Consequently, if $\{\mathcal{Y}_m : m \geq 1\}$ is a sequence of i.i.d. sequences

of tessellations (distributed as $\mathcal{T}^{(0)}$), we define $I_2(\mathcal{T}^{(0)}) = I(2\mathcal{T}^{(0)}, 2\mathcal{Y}_1)$ and recursively, for every $m \geq 3$,

$$I_m(\mathcal{T}^{(0)}) = I\left(\frac{m}{m-1} I_{m-1}(\mathcal{T}^{(0)}), m\mathcal{Y}_{m-1}\right).$$

In other words, at step m, the tessellation $\frac{m}{m-1} I_{m-1}(\mathcal{T}^{(0)})$ is "iterated" with the sequence of tessellations $\mathcal{Y} = m\mathcal{Y}_{m-1}$.

A tessellation is said to be *stable with respect to iteration* (STIT) if $I_m(\mathcal{T}^{(0)})$ and $\mathcal{T}^{(0)}$ are equal in law for every $m \geq 2$.

Going back to the crack STIT tessellation $\mathcal{T}(a)$, we can observe that for any $a, b > 0$, the process of iterating $\mathcal{T}(a)$ with an independent sequence $\mathcal{T}(b)$ of i.i.d. tessellations distributed as $\mathcal{T}(b)$ is equivalent to constructing the crack STIT tessellation over the time interval $[0, a+b]$, i.e. $I(\mathcal{T}(a), \mathcal{Y}(b))$ coincides in distribution with $\mathcal{T}(a+b)$. This property comes from the Markov property of $(\mathcal{T}(t, W))_{t>0}$ for a fixed window W and combined with the scaling property of $\mathcal{T}(a)$, it implies that $\mathcal{T}(a)$, $a > 0$, is stable with respect to iteration.

It can be proved conversely that any tessellation which has the STIT property is a crack STIT tessellation. Indeed, a modified version of Korolyuk's theorem on processes of facets (see chapter 3 of (Daley and Vere-Jones, 1988)) can be used to show that for any stationary tessellation $\mathcal{T}$, the sequence $I_m(\mathcal{T})$, $m \geq 1$, converges weakly to a crack STIT tessellation with the same surface intensity and directional distribution as $\mathcal{T}$.

Numerous properties of STIT tessellations have been derived in (Nagel and Weiss, 2004). Let us cite in particular the preservation of the STIT property for every section of a STIT tessellation, as well as the equality in distribution of the interior of the typical cell with the typical cell of a homogeneous Poisson hyperplane tessellation with the same surface intensity and directional distribution. Mean values in dimension two and three have also been calculated.

0.4.2 *Iterated tessellations in telecommunications*

We end this last section with a small introduction on the use of iterated tessellations in telecommunications. Models and results cited below are due notably to Maier, Schmidt and Mayer (2004, 2003) and Heinrich, Schmidt and Schmidt (2006).

Classical tessellations have been of great use in that specific domain of application for several years, see e.g.(Baccelli, Gloaguen and Zuyev, 2000*a*; Baccelli, Tchoumatchenko and Zuyev, 2000*b*; Baccelli and Błaszczyszyn, 2001; Błaszczyszyn and Schott, 2003). In order to make the models more realistic and take into account the fact that a network may contain two levels of roads, R. Maier and V. Schmidt have introduced a stationary iterated tessellation in the following way (Maier and Schmidt, 2003).

- Let $\mathcal{T}^{(0)} = \{C_i^{(0)}\}_{i \geq 1}$ be a random stationary tessellation called the *initial tessellation*.

- Let $\{\mathcal{T}^{(n)}\}_{n\geq 1}$ be a sequence of random stationary tessellations which is independent with $\mathcal{T}^{(0)}$ and such that the $\mathcal{T}^{(n)}$, $n \geq 1$, are i.i.d. or at least exchangeable. The $\mathcal{T}^{(n)} = \{C_i^{(n)}\}_{i\geq 1}$ are called the *component tessellations*.

Then the tessellation $\mathcal{T}$ constituted with all the intersections $C_i^{(n)} \cap C_n^{(0)}$, $n \geq 1$, $i \geq 1$, with a non-empty interior, is the associated *stationary iterated tessellation*. Basic examples which are concretely used are obtained when the initial and component tessellations are distributed as Poisson–Voronoi tessellations or stationary Poisson hyperplane tessellations (Gloaguen, Fleischer, Schmidt and Schmidt, 2006).

Let us denote by $\mathscr{C}^{(0)}$ and $\lambda^{(0)}$ (resp. $\mathscr{C}$ and λ) the typical cell and the intensity of $\mathcal{T}^{(0)}$ (resp. of $\mathcal{T}$). The use of Neveu's exchange formula (Neveu, 1977) provides a precise link between $\mathscr{C}^{(0)}$ and $\mathscr{C}$, i.e. for every bounded and measurable function $f : \mathcal{K} \to \mathbb{R}$,

$$\mathbf{E}(f(\mathscr{C})) = \frac{\lambda^{(0)}}{\lambda}\mathbf{E}\sum_{i\geq 1} f(C_i^{(1)} \cap \mathscr{C}^{(0)})\mathbf{1}_{\{\mathrm{Int}(C_i^{(1)})\cap \mathrm{Int}(\mathscr{C}^{(0)})\neq\emptyset\}}.$$

Quantities of interest for such an iterated tessellation are measurements of inner structure of the initial cells, such as the number or the k-dimensional Hausdorff measure of the k-faces of the component tessellation inside an initial cell. In (Heinrich, Schmidt and Schmidt, 2006), L. Heinrich, H. Schmidt and V. Schmidt have obtained a more general law of large numbers and a multivariate central limit theorem which can be applied to the quantities above. Indeed, for a fixed $m \geq 1$, they consider a sequence of i.i.d. vectors $J_i = (J_{i,1}, \ldots, J_{i,m})$, $i \geq 1$, whose coordinates are stationary random measures. In particular, J_i is the "descriptor" of the inner structure of the i-th cell of the initial tessellation and it is supposed to have a finite intensity vector $(\lambda_1, \ldots, \lambda_m)$. For a fixed window W which is a convex set of $\mathbb{R}^d$ with a non-empty interior, we denote by $Z_{k,\rho}$ the quantity $\sum_{i\geq 0} J_{i,k}(C_i^{(0)} \cap \rho W)$ where $1 \leq k \leq m$ and $\rho > 0$. If the initial tessellation is ergodic, under some integrability conditions upon the typical cell of $\mathcal{T}^{(0)}$ and J_i, we have for every $1 \leq k \leq m$

$$\frac{1}{\lambda_d(\rho W)} Z_{k,\rho} \longrightarrow \lambda_k, \quad \text{as } \rho \to \infty.$$

The proof of this result is based on classical methods related to Wiener's ergodic theorem associated with a precise treatment of the contribution of the cells hitting the boundary of the window.

Moreover, the authors use a refinement of the Berry–Esseen inequality to prove under certain conditions of integrability related to J_i and the typical cell $\mathscr{C}^{(0)}$ that the vector

$$\frac{1}{\sqrt{\lambda_d(\rho W)}}(Z_{1,\rho} - \lambda_1\lambda_d(\rho W), \ldots, Z_{m,\rho} - \lambda_m\lambda_d(\rho W))$$

converges to a mean-zero normal distribution with an explicit covariance matrix.

These convergence results are used to estimate the quantities λ_k and decide which model of iterated tessellation fits the best in concrete situations, see (Gloaguen, Fleischer, Schmidt and Schmidt, 2006).

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