Symmetry algebras of certain partial differential equations related to stochastic processes

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The talk is based on a joint work with Hélène Quintard and Jean-Claude Zambrini ([12, 13]) and on L. Valade's ongoing PhD Thesis ([19]).

- The method of isovectors
- General results
- The Hamilton-Jacobi-Bellman equation
- The Black-Scholes equation
- Frey's equation
- The heat equation for the square of the Laplacian
- Bernstein processes
- Parametrization of a one-factor affine model
- Generalized Brownian Bridges
- References

The method of isovectors was introduced in [6] in order to classify up to equivalence (systems of) partial differential equations appearing in mathematical physics.
Given a system of partial differential equations, one expresses it, adding if necessary some of the derivatives of the unknown function as auxiliary unknowns, as the vanishing of a family of first-order differential forms. An isovector is then defined as a vector field in all the variables preserving the differential ideal generated by the forms.
For the one–dimensional heat equation, it was determined (using a different language) by Bluman and Cole ([2]).
Olver's *prolongation method* ([18]) provides a somewhat different approach.

Let us now give some details. We shall consider an equation of the shape

$$\frac{\partial u}{\partial t} = G(t, q, u, \frac{\partial u}{\partial q}, ..., \frac{\partial^{n-1} u}{\partial q^{n-1}}) + \lambda \frac{\partial^n u}{\partial q^n}$$

for $n \geq 2$, $t \in J$ (an interval of $\mathbf{R}$) and $q \in O$ (an open set in $\mathbf{R}^n$).
In order to study the symmetries of the equation, we shall temporarily consider $u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial q}, ..., \frac{\partial^{n-1} u}{\partial q^{n-1}}$ as *independent* variables.
We shall take as state space $M := J \times O \times \mathbf{R}^{n+1}$, the generic point of which will be denoted by $(t, q, u, A, B_1, ..., B_{n-1})$.
All of our computations will take place in the differential algebra $\wedge T^* M$.

Setting

$$\alpha := du - Adt - B_1 dq$$

and, for $1 \leq i \leq n-2$,

$$\beta_i = dB_i dt - B_{i+1} dqdt,$$

we have

$$d\alpha = -dAdt - dB_1 dq$$

and

$$d\beta_i = -dB_{i+1} dqdt.$$

Defining

$$\gamma := dudq - G(t, q, u, B_1, ..., B_{n-1})dtdq - \lambda dtdB_{n-1},$$

we see that $(\mathcal{E})$ is equivalent to the vanishing, on a 2–dimensional submanifold of M, of α, $d\alpha$, the β_i, the $d\beta_i$ and γ.

We define I as the ideal of $\wedge T^*M$ generated by α, $d\alpha$, the β_i, the $d\beta_i$, and γ. It turns out that I is a *differential ideal*, that is $d(I) \subseteq I$; to establish this, it is obviously enough to show that $d\gamma \in I$.
We shall denote by $\mathcal{G}$ the *isovector algebra* of $(\mathcal{E})$; this is the set of vector fields $N \in TM$ such that

$$\mathcal{L}_N(I) \subseteq I.$$

Due to the formal properties of the Lie derivative, these isovectors constitute a Lie algebra for the usual bracket of vector fields.
We shall write each $N \in \mathcal{G}$ as

$$N = N^t \frac{\partial}{\partial t} + N^q \frac{\partial}{\partial q} + N^u \frac{\partial}{\partial u} + N^A \frac{\partial}{\partial A} + \sum_i N^{B_i} \frac{\partial}{\partial B_i}.$$

Theorem

([14]) For each $N \in \mathcal{G}$, N^t depends only on t, N^q on t and q and N^u on (t, q, u). Setting

$$\mathcal{J}_0 := \{N \in \mathcal{G} | N^t = N^q = 0\},$$

it follows that $\mathcal{J}_0$ is an ideal of $\mathcal{G}$.

Theorem

([14])If, in addition,

$$\frac{\partial^2 G}{\partial B_1 \partial B_{n-1}} = 0,$$

then, for each $N \in \mathcal{G}$, N^u is affine in u.
Setting

$$\mathcal{J} = \mathcal{J}_0 \cap \{N \in \mathcal{G} | \frac{\partial N^u}{\partial u} = 0\},$$

$\mathcal{J}$ is an abelian ideal of $\mathcal{G}$ and the sum

$$\mathcal{H} \oplus \mathcal{J}.$$

is direct.

Theorem

([14])If either $G = 0$ or $n = 2$ and G is of the shape $cB_1 + V(t,q)u$ then $\mathcal{H} \oplus \mathcal{J} = \mathcal{G}$.

For $N \in \mathcal{G}$ and $\alpha \in \mathbf{R}$, $e^{\alpha N}$ maps (t, q, u) to $(t_\alpha, q_\alpha, u_\alpha)$; by definition, u_α considered as a function v_α of (t_α, q_α) is a solution of $(\mathcal{E})$. Setting $e^{\alpha \tilde{N}} u = v_\alpha$

it turns out that $\tilde{N}$ acts by

$$\tilde{N}(u) = -N^t \frac{\partial u}{\partial t} - N^q \frac{\partial u}{\partial q} + N^u$$

(see [11], Lemma 6.1, in the Black–Scholes case)
and that

$$N \mapsto -\tilde{N}$$

is a Lie algebra morphism (see [13],Lemma 2.1, in the HJB case). Therefore

$$\tilde{\mathcal{G}} := \{\tilde{N} | N \in \mathcal{G}\}$$

and

$$\tilde{\mathcal{H}} := \{\tilde{N} | N \in \mathcal{H}\}$$

are Lie algebras.

We shall now demonstrate the working of the method on the example most important to us, following [12, 13, 14, 16].
Let us consider the *backward heat equation with potential V* :

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta}{\partial q^2} + V\eta.$$

Setting $S = -\theta^2 \ln(\eta)$, we obtain for S the Hamilton–Jacobi–Bellman equation $(\mathcal{HJB}^V)$:

$$\frac{\partial S}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 S}{\partial q^2} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 - V.$$

We are in the above case, with $\lambda = -\frac{\theta^2}{2}$ and $G = \frac{B_1^2}{2} - V$.

Setting $E = -\dfrac{\partial S}{\partial t}$ and $B = -\dfrac{\partial S}{\partial q}$ we have:

$$\omega := \alpha = dS + Edt + Bdq \ ,$$

$$\Omega := d\omega = d\alpha = dEdt + dBdq \ ,$$

and

$$\beta := \gamma - \alpha dq = (E + \frac{B^2}{2} - V)dqdt + \frac{\theta^2}{2}dBdt \ .$$

Then

$$\omega_{PC} = Edt + Bdq = \omega - dS$$

is the *Poincaré–Cartan* form, and ω, Ω and β generate I. Due to the linearity of $\mathcal{E}_V$ the ideal $\mathcal{J}$ is infinite–dimensional. It is also abelian.

In the case of the potential

$$V(t, q) = \frac{C}{q^2} + Dq^2 \ ,$$

let $\mathcal{H}_{C,D} := \mathcal{H}_V$. Then

Theorem

For $C \neq 0$, $\mathcal{H}_{C,D} \simeq \mathcal{H}_{1,0}$ *has dimension* 4 *; for* $C = 0$, $\mathcal{H}_{C,D} \simeq \mathcal{H}_{0,0}$ *has dimension* 6. *Furthermore,* $\mathcal{H}_{1,0} \subseteq \mathcal{H}_{0,0}$ *([13]). In addition these Lie algebras possess canonical bases, continuous in D for fixed C, and compatible with the inclusions*

$$\mathcal{H}_{C,D} \subseteq \mathcal{H}_{0,D} .$$

In [19], H. Quintard has determined, the structure of the isovector algebra for the equation

$$\frac{\partial u}{\partial t} = \sigma \Delta u + Vu.$$

for quadratic V.

This is the most famous equation in Mathematical Finance :

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

Here it turns out ([11]) that $\mathcal{H}$ has dimension 6, and is isomorphic to the algebra $\mathcal{H}_{0,0}$ determined in §2. This is not surprising inasmuch that both the Black–Scholes equation and the HJB equation with $V = 0$ can be reduced to the heat equation. Nevertheless our computation does not depend upon that reduction, and would actually suggest it ; notably, the quantities $r + \dfrac{\sigma^2}{2}$ and $r - \dfrac{\sigma^2}{2}$ appear in a natural way.

$$\frac{\partial u}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\frac{\partial^2 u}{\partial S^2}}{(1 - \rho S \lambda(S) \frac{\partial^2 u}{\partial S^2})^2} = 0$$

Here ρ is a real parameter and λ a given function.
It is a nonlinear version of Black–Scholes equation, first considered by Frey ([4]).
That equation doesn't fit directly into the above framework.
Bobrov ([3]), in an unpublished paper, determined the isovectors ; they were computed again in a different way by Valade ([20]).

Defining

$$\tilde{V}_1 = \frac{\partial}{\partial t},$$

$$\tilde{V}_2 = S\frac{\partial}{\partial u},$$

and

$$\tilde{V}_3 = u\frac{\partial}{\partial u},$$

then, when λ is not of the form ωS^k, it appears that $\tilde{\mathcal{G}}$ is generated by $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$; in particular, it is abelian of dimension 3.
When $\lambda(S) :\equiv \omega S^k$, $\tilde{\mathcal{G}}$ is generated by $\tilde{V}_1$, $\tilde{V}_2$, $\tilde{V}_3$ and

$$\tilde{V}_4 := -S\frac{\partial}{\partial S} + (1-k)u\frac{\partial}{\partial u}$$

and it has a far more interesting structure ([3, 20]). The computations can be carried out in a way that preserves all along the connection to the Black–Scholes equation ; in fact, $\tilde{\mathcal{G}}$ is a subalgebra of the symmetry algebra for the Black–Scholes equation.

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial q^4}.$$

We have here $n = 4$, $\lambda = -1$ and

$$\gamma = dudq - dB_3dt \ .$$

The algebra of isovectors has been determined by Vigot ([21]) and Valade ([20]). $\tilde{\mathcal{H}}$ has a basis $(\tilde{X}_i)(1 \leq i \leq 4)$ with

$$\tilde{X}_1 = \frac{\partial}{\partial x}$$

$$\tilde{X}_2 = \frac{\partial}{\partial t}$$

$$\tilde{X}_3 = x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t}$$

$$\tilde{X}_4 = u\frac{\partial}{\partial u}$$

Therefore $\tilde{\mathcal{H}} = \mathbf{R} \oplus \mathcal{L}$, with $\mathcal{L}$ a three–dimensional solvable Lie algebra with 2–dimensional derived algebra.
Once exponentiated, here is how the basis elements act on a solution u of the equation :

$$e^{\alpha \tilde{X}_1} u(t,x) = u(t+\alpha, x)$$

$$e^{\alpha \tilde{X}_2} u(t,x) = u(t, x+\alpha)$$

$$e^{\alpha \tilde{X}_3} u(t,x) = u(e^{4\alpha} t, e^{\alpha} x)$$

$$e^{\alpha \tilde{X}_4} u(t,x) = e^{\alpha} u(t,x)$$

The equation is deeply related to Hochberg's pseudo–process ([8]).

For simplicity, we shall only consider processes with values in **R**.
We fix a real number θ (in the quantum mechanical context, $\theta = \sqrt{\hbar} = \sqrt{\frac{h}{2\pi}}$) and a potential $V \equiv V(t, q)$ subject to some mild conditions.

Two partial differential equations, dual to one another, will play an essential role in all that follows : the above–mentioned

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2}\frac{\partial^2 \eta}{\partial q^2} + V\eta \ \ (\mathcal{E}_V)$$

and

$$-\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2}\frac{\partial^2 \eta}{\partial q^2} + V\eta \ \ (\mathcal{E}_V^{'}).$$

Theorem

Let us be given two real numbers $t_0 < t_1$ and two Borel probability measures μ_0 and μ_1 on $\mathbf{R}$. Then there exists an essentially unique triple (η, η_*, z) such that

1. $\eta > 0$ *and* $\eta_* > 0$ *are defined on* $]t_0, t_1[$.
2. η *is a solution of* $\mathcal{E}_V$ *and* η_* *is a solution of* $\mathcal{E}'_V$.
3. $z(t)(t \in [t_0, t_1])$ *is a brownian diffusion satisfying the stochastic differential equation*

$$dz(t) = \theta dw(t) + \tilde{B}(t, z(t))dt$$

   where

$$\tilde{B} := \frac{\theta^2}{\eta}\frac{\partial \eta}{\partial q}.$$

4. *For each* $t \in]t_0, t_1[$, *the law of* $z(t)$ *is* $\eta(t, q)\eta_*(t, q)dq$.
5. *The law of* $z(t_0)$ *is* μ_0, *and the law of* $z(t_1)$ *is* μ_1.

These processes are termed *Bernstein* processes ; in case μ_0 and μ_1 are Dirac measures, they are sometimes referred to as *Schrödinger bridges*.
Following an idea of Schrödinger (1932), this Theorem was established by Zambrini (1986). An important preliminary result was proved by Beurling ([1]) : given μ_0 and μ_1, there is, up to constant factors, a unique couple (η, η_*) such that 1), 2) hold and 4) and 5) *may* hold.
Many familiar stochastic processes can be viewed as Bernstein processes, for instance

1. **The Brownian Motion** (see e.g. [15], p. 200)
   For $t_0 = 0$, $t_1 = T > 0$, $\mu_0 = \delta_0$ and $\mu_1 = \mathcal{N}(0, \theta^2 T)$, $z(t) = \theta w(t)$ is a Bernstein process for $V = 0$, $\eta(t,q) = 1$ and $\eta_*(t,q) = \frac{1}{\theta\sqrt{2\pi t}} e^{-\frac{q^2}{2\theta^2 t}}$.
   Here
   $$dz(t) = \theta dw(t).$$
2. **The Brownian Bridge** (see [15], p. 201).
   Here $t_0 = 0$, $t_1 = 1$, $V = 0$, $\mu_0 = \mu_1 = \delta_0$ and
   $$\eta(t,q) = \frac{1}{\sqrt{1-t}} e^{-\frac{q^2}{2\theta^2(1-t)}}.$$
   Then
   $$dz(t) = \theta dw(t) - \frac{z(t)}{1-t} dt.$$

3. **The Ornstein–Uhlenbeck process** starting from 0
Here $t_0 = 0$, $t_1 = T > 0$, $\mu_0 = \delta_0$,
$\mu_1 = \mathbf{N}(0, \frac{\theta^2}{2\omega}(1 - e^{-2\omega T}))$, $V = \frac{\omega^2 q^2}{2}$ and

$$\eta(t, q) = e^{\frac{\omega}{2\theta^2}(\theta^2 t - q^2)}.$$

We have

$$dz(t) = \theta dw(t) - \omega z(t) dt.$$

A *one–factor affine interest rate model* is characterized by an instantaneous rate $r(t)$, satisfying a stochastic differential equation of the following type :

$$dr(t) = \sqrt{\alpha r(t) + \beta}\, dw(t) + (\phi - \lambda r(t))\, dt\ ,$$

under the risk–neutral probability Q ($\alpha = 0$ corresponds to the Vasicek model, and $\beta = 0$ corresponds to the Cox–Ingersoll–Ross model ; see [5], [7]).
Assuming $\alpha \neq 0$, let us set

$$\theta = \frac{\alpha}{2}\ ,$$

$$\tilde{\phi} := \phi + \frac{\lambda\beta}{\alpha}\ ,$$

$$\delta := \frac{4\tilde{\phi}}{\alpha} \ ,$$

$$\begin{aligned} C &:= \frac{\alpha^2}{8}(\tilde{\phi} - \frac{\alpha}{4})(\tilde{\phi} - \frac{3\alpha}{4}) \\ &= \frac{\alpha^4}{128}(\delta - 1)(\delta - 3) \ , \end{aligned}$$

$$D := \frac{\lambda^2}{8} \ ,$$

and define the potential

$$V(t, q) = \frac{C}{q^2} + Dq^2 \ .$$

Theorem (see [13], Theorem 5.4) Consider the process

$$z(t) = \sqrt{\alpha r(t) + \beta}$$

and the stopping time

$$\tilde{T} = \inf\{t > 0 | \alpha r(t) + \beta = 0\}.$$

1) One has $\tilde{T} = +\infty$ *a.s.* for $\delta \geq 2$, and $\tilde{T} < +\infty$ *a.s.* for $\delta < 2$.
2) There exists a process $y(t)$ satisfying the stochastic differential equation

$$\forall t > 0$$

$$dy(t) = \theta dw(t) + \tilde{B}(t, y(t))dt$$

relatively to the canonical increasing filtration of the Brownian w, where

$$\tilde{B} \equiv_{def} \theta^2 \frac{\frac{\partial \eta}{\partial q}}{\eta} \ .$$

for a certain solution η of $(\mathcal{E}_V)$
For each given $t > 0$, the law of $y(t)$ is $\eta(t,q)\eta_*(t,q)dq$, where η_* satisfies the dual equation $(\mathcal{E}_V^{'})$. One has

$$\forall t \in [0, \tilde{T}[\ z(t) = y(t) \ .$$

In particular, in case $\delta \geq 2$, z itself is a Bernstein process on any interval $[0, T_0]$ ($T_0 > 0$).

Proposition

The isovector algebra associated with the affine model has dimension 6 *if and only if* $\delta \in \{1, 3\}$ *; in the opposite case, it has dimension* 4*.*

Let us analyze more closely the first case ; the general case is considered in [13].

1) $\tilde{\phi} = \frac{\alpha}{4}$, *i.e.* $\delta = 1$. Then $y(t)$ is a solution of

$$dy(t) = \frac{\alpha}{2} dw(t) - \frac{\lambda}{2} y(t) dt \ ,$$

i.e. $y(t)$ is an Ornstein–Uhlenbeck process, and the potential V is quadratic, in agreement with the result of §3.

Here

$$\eta(t,q) = e^{\frac{\lambda t}{4} - \frac{\lambda q^2}{\alpha^2}} \ .$$

From

$$\begin{aligned} y(t) &= e^{-\frac{\lambda t}{2}}(z_0 + \frac{\alpha}{2}\int_0^t e^{\frac{\lambda s}{2}} dw(s)) \\ &= e^{-\frac{\lambda t}{2}}(z_0 + \tilde{w}(\frac{\alpha^2(e^{\lambda t}-1)}{4\lambda})) \ , \end{aligned}$$

($\tilde{w}$ denoting another Brownian motion),

it appears that $y(t)$ follows a normal law ν_t with mean $e^{-\frac{\lambda t}{2}} z_0$ and variance $\frac{\alpha^2(1-e^{-\lambda t})}{4\lambda}$.
For each $T > 0$, $(y(t))_{0 \leq t \leq T}$ is a Bernstein process with $\mu_0 = \delta_{z_0}$ and $\mu_1 = \nu_T$.
The law of $y(t)$ therefore has density

$$\rho_t(q) = \frac{2\sqrt{\lambda}}{\alpha\sqrt{2\pi(1-e^{-\lambda t})}} e^{\left(-\frac{2\lambda(q-e^{-\frac{\lambda t}{2}} z_0)^2}{\alpha^2(1-e^{-\lambda t})}\right)} .$$

Whence

$$\eta_*(t,q) = \frac{\rho_t(q)}{\eta(t,q)} = \frac{1}{\alpha}\sqrt{\frac{\lambda}{\pi \sinh\left(\frac{\lambda t}{2}\right)}}\; e^{\left(\frac{-\lambda q^2 - \lambda q^2 e^{-\lambda t} + 4\lambda q z_0 e^{-\frac{\lambda t}{2}} - 2\lambda z_0^2 e^{-\lambda t}}{\alpha^2(1-e^{-\lambda t})}\right)}$$

and one may check that, as was to be expected, η_* satisfies $(\mathcal{E}'_V)$.

2) $\tilde{\phi} = \dfrac{3\alpha}{4}$, *i.e.* $\delta = 3$.
Then

$$\eta(t,q) = q\ e^{\dfrac{\lambda}{\alpha^2}\left(\dfrac{3\alpha^2 t}{4} - q^2\right)}\ .$$

Let us define

$$s(t) = e^{\frac{\lambda t}{2}} \frac{1}{y(t)} \ ;$$

then an easy computation using Itô's formula gives

$$ds(t) = -\frac{\alpha}{2} e^{\frac{\lambda t}{2}} s(t)^2 dw(t) \ ;$$

in particular, $s(t)$ is a martingale.

It may be seen that, in case $X_0 = 0$,

$$X_t = e^{-\lambda t} Y(\frac{\alpha^2(e^{\lambda t} - 1)}{4\lambda})$$

where Y is a $BESQ^3$(squared Bessel process with parameter 3) with $Y(0) = 0$. But, for fixed $t > 0$, Y_t has the same law as tY_1, and $Y_1 = ||B_1||^2$ is the square of the norm of a 3–dimensional Brownian motion (see [5]) ; the law of Y_1 is therefore

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} \sqrt{u} \mathbf{1}_{u \geq 0} du \ .$$

Therefore the density $\rho_t(q)$ of the law of $y(t)$ is given by :

$$\rho_t(q) = \frac{1}{\sqrt{2\pi}} \frac{16\lambda^{\frac{3}{2}}}{\alpha^3(1 - e^{-\lambda t})^{\frac{3}{2}}} q^2 e^{-\dfrac{2\lambda q^2}{\alpha^2(1 - e^{-\lambda t})}}$$

and

$$\eta_*(t,q) = \frac{\rho_t(q)}{\eta(t,q)} = \frac{16\lambda^{\frac{3}{2}}}{\alpha^3\sqrt{2\pi}}(1-e^{-\lambda t})^{-\frac{3}{2}} q e^{\frac{3\lambda t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})}} .$$

For $X_0 \neq 0$, a Bessel function appears in the expression for η_*.
M. Houda ([9]) has extended these computations.

Let us fix $T > 0$ and $\beta \in \mathbf{R}$. Mansuy's generalized Brownian Bridge ([17]) $X^{(\beta,\gamma,T,x)}$ as the solution $X(t)$ of the stochastic differential equation

$$dX(t) = \theta dw(t) - \beta \frac{X(t)}{T-t} dt$$

with initial value $X(0) = x$.
It is a Bernstein process for

$$\eta(t,q) := (T-t)^{-\frac{\beta}{2}} e^{-\frac{\beta q^2}{2\theta^2 (T-t)}}$$

and

$$V(t,q) := \frac{\beta(\beta-1)q^2}{2(T-t)^2} \ ,$$

an example of *semi–classical potential.*

Clearly, for $T = \beta = 1$, one recovers the Brownian Bridge mentioned in §3.
More generally, using the notations of [15], 7.2, one has

$$\forall \alpha > 0 \; X^{(1,\sqrt{\hbar},\frac{1}{\alpha},0)} = \sqrt{\hbar} w_{0,0}^{0,\alpha^{-1}}(t)$$

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