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Sard's Theorem for Hyper-Gevrey Functionals on the Wiener Space

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Let (X, H, μ) be the Wiener space; we define "twisted Sobolev spaces" associated to certain compact operators on H. Wiener functionals belonging to one of these spaces are termed "hypersmooth"; we prove a measure-theoretic analog of Sard's theorem for them. Our result applies in particular to solutions of Itô's SDE's with Gevrey class coefficients. © 1995 Academic Press, Inc.

CONTENTS

INTRODUCTION, NOTATIONS AND PREREQUISITES

Let X be the Wiener space, μ the Wiener measure on X, $f: X \to \mathbf{R}$ a smooth map in Malliavin's sense [10] and $\mathscr{C}_f$ the set of critical points of f; a classical result [10, p. 385] allows us to state that the measure $f_*[(\mathbf{1} - \mathbf{1}_{\mathscr{C}_f})\mu]$ is absolutely continuous relatively to the Lebesgue measure m on $\mathbf{R}$. It is now rather natural to study the residual part $\rho_f = f_*(\mathbf{1}_{\mathscr{C}_f}\mu)$ of the image measure $\nu_f = f_*\mu$. Should X be replaced by a finite-dimensional Gaussian space, Sard's well-known theorem [14] would imply that ρ_f would be singular with respect to m; such is not the case here. Y. Katznelson and P. Malliavin [8] have even produced an example of an

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f such that ρ_f has a nonzero $\mathbf{C}^\infty$ density with respect to m. The classical proof of Sard's theorem is related to covering properties. It is well known that for $\mathbf{C}^k$-functions defined on $\mathbf{R}^n$, the theorem fails to be true if n is large enough. On the Wiener space there are essentially two distinct manifestations of the "lack of compactness":

(i) The unit ball of linear forms on $L^2([0, 1])$ is not compact, and the analogous fact is true for all the components of the Wiener chaos (linear forms being canonically identified to the elements of the first-order Wiener chaos).

(ii) Most interesting Wiener functionals are not differentiable in the Banach space sense.

In finite dimension, Ascoli's theorem ensures that the unit ball of the space of Lipschitz functions is compact in the space of continuous functions. No equivalent exists on X according to (i). Recently Da Prato, Malliavin, and Nualart [4] have given a full characterization of the compact subsets of the set of smooth functionals. We shall prove that solutions of certain stochastic differentials equations are "hyper-Gevrey functionals" in a sense to be made precise below. For such functionals, we shall prove Sard's theorem by a covering argument based on an estimation of the oscillation of the function in a fiber containing a critical point. We shall generally follow the terminology and notation of [10]: (X, H, μ) will denote the usual Wiener space, ∇, the gradient in Malliavin's sense, $\mathscr{L}$ the Ornstein–Uhlenbeck operator, $\mathbf{C}_n$ the nth Wiener chaos; it is well known that $L^2(X)$ is the orthogonal direct sum of the $\mathbf{C}_n$. We shall denote by $\mathbf{j}$ the canonical isomorphism

$$\mathbf{j}\colon H \to L^2([0, 1])$$

given by

$$\mathbf{j}(x)(t) = x'(t).$$

Sard's theorem in infinite dimension has recently been investigated in [13]; the setting, however, is rather different there.

A sketch of this paper appeared as part 5 of my Ph.D. thesis (Université Paris VI, 1993) written under the guidance of Professor Paul Malliavin; elements of part 4 of the thesis also appear in Section 3. The bulk of the writing-up was done while I was visiting the Institut für Angewandte Mathematik in Bonn (October 1993—February 1994); I take the opportunity to thank Michaël Röckner and Hans Föllmer for inviting me and for helping in many ways to make my stay in Germany pleasant and fruitful. Sergio Albeverio and Niels Jacob greatly honored me by their invitations to lecture upon this material in Bielefeld and Erlangen, respectively; I then benefitted a lot from comments by members of the audience, notably Z. Brzezniak.

1. Itô–Taylor Formula for Gevrey Functionals

Theorem 1.1. *Let* $g \in \mathscr{W}^{\infty,2,\alpha}(X)$ *be*

$$\forall m \in \mathbf{N},\ \|\nabla^m g\|_{L^2(X;H^{\otimes m})} \leqslant c\gamma^m (m!)^\alpha \qquad (\mathscr{H}')$$

where $\alpha \geqslant 0$ *and* $\gamma \geqslant 1$, *then, denoting by* $\mathscr{C}_g$ *the set of critical points of* g, *i.e.*,

$$\mathscr{C}_g = \{x \in X \mid \nabla g(x) = 0\},$$

one has, setting $\alpha' = \max(\alpha, \frac{1}{2})$:

$$\mathscr{E}(|g(x_\omega(t)) - g(x_\omega(0))|^2\, \mathbf{1}_{\mathscr{C}_g}(x_\omega(0))) \leqslant c_1 \exp(-c_2(\gamma^4 t)^{-1/(4\alpha'-1)})$$

(*where* $x_\omega(t)$ *is the Ornstein–Uhlenbeck process on* X([9])), *with* c_1 *and* c_2 *depending only upon* c *and* α.

A number of preliminary results will be useful.

Lemma 1.2 (Stroock's commutation formula [15]). $\mathscr{L}\nabla = \nabla(\mathscr{L} + 1)$.

Proof. It is obvious *via* the chaos expansion. ∎

Lemma 1.3. *Let* $f \in \mathscr{W}^{\infty,2,\alpha}(X)$; *then*

$$\|\mathscr{L}^n \nabla f\|_{L^2(X;H)} \leqslant \|\mathscr{L}^{n+1} f\|_{L^2(X)}.$$

Proof. Let $f = \sum_{m \geqslant 0} f_m$ the decomposition of f on the Wiener chaos; one can write:

$$\begin{aligned}\mathscr{L}^n \nabla f &= \sum_{m \geqslant 0} \mathscr{L}^n \nabla f_m \\ &= \sum_{m \geqslant 0} (1-m)^n \nabla f_m,\end{aligned}$$

whence

$$\begin{aligned}\|\mathscr{L}^n \nabla f\|^2_{L^2(X;H)} &= \sum_{m \geqslant 0} (m-1)^{2n} \|\nabla f_m\|^2_{L^2(X;H)} \\ &= \sum_{m \geqslant 0} (m-1)^{2n}\, m\, \|f_m\|^2_{L^2(X)} \\ &\leqslant \sum_{m \geqslant 0} m^{2n+2} \|f_m\|^2_{L^2(X)} \\ &= \|\mathscr{L}^{n+1} f\|^2_{L^2(X)}. \quad ∎\end{aligned}$$

By an *elementary operator of type* (r, s) we shall mean the product, in an arbitrary order, of r factors $\mathscr{L}$ and s factors ∇. The next two lemmas allow us to reduce the study of such operators to that of powers of $\mathscr{L}$.

LEMMA 1.4. *Let A be an elementary operator of type (r, s); then*

$$\forall f \in \mathscr{H}^{2,\infty}(X)\ \|Af\|_{L^2(X; H^{\otimes s})} \leqslant \|\mathscr{L}^{r+s}f\|_{L^2(X)}.$$

Proof. Let us proceed by induction over $r+s$, as the result is obvious for $r+s=0$. In the other case, there are two possibilities:

(a) A is of the shape $B\mathscr{L}$, where B is of type $(r-1, s)$. The induction hypothesis applied to B now gives us

$$\begin{aligned}\|Af\|^2_{L^2(X; H^{\otimes s})} &= \|B\mathscr{L}f\|^2_{L^2(X; H^{\otimes s})}\\ &\leqslant \|\mathscr{L}^{r+s-1}(\mathscr{L}f)\|^2_{L^2(X)}\\ &= \|\mathscr{L}^{r+s}f\|^2_{L^2(X)},\end{aligned}$$

whence the desired conclusion.

(b) A is of the shape $B\nabla$, where B is of type $(r, s-1)$. We can then apply the induction hypothesis to B and to each of the components $f_k = (\nabla f \mid e_k)$; we get

$$\|Bf_k\|^2_{L^2(X; H^{\otimes(s-1)})} \leqslant \|\mathscr{L}^{r+s-1}f_k\|^2_{L^2(X)}.$$

But one has

$$\nabla f = \sum_{k=0}^{+\infty} f_k e_k,$$

whence

$$Af = \sum_{k=0}^{+\infty} Bf_k \otimes e_k$$

$$\mathscr{L}^{r+s-1}\nabla f = \sum_{k=0}^{+\infty} (\mathscr{L}^{r+s-1}f_k)\, e_k.$$

Summing over k in the above inequalities we thus get

$$\begin{aligned}\|Af\|^2_{L^2(X; H^{\otimes s})} &\leqslant \|\mathscr{L}^{r+s-1}\nabla f\|^2_{L^2(X; H)}\\ &\leqslant \|\mathscr{L}^{r+s}f\|^2_{L^2(X)},\end{aligned}$$

where we have used Lemma 1.3. The result follows. ∎

LEMMA 1.5. *Let $f \in \mathscr{W}^{2,\alpha}(X)$; then one has*

$$\|\mathscr{L}^n f\|^2_{L^2(X)} \leqslant (3n)^{2n} \|f\|^2_{L^2(X)} + 3^{2n} \|\nabla^{2n} f\|^2_{L^2(X;H^{\otimes(2n)})}.$$

Proof. Let us write the expansion of f using the "generalized Hermite polynomials":

$$f = \sum c_\phi H_\phi.$$

We have

$$\|\mathscr{L}^n f\|^2_{L^2(X;\mathbf{R})} = \sum \|\phi\|^{2n} c_\phi^2$$

and

$$\|\nabla^{2n} f\|^2_{L^2(X;H^{\otimes(2n)})} = \sum \|\phi\| \cdots (\|\phi\| - 2n + 1)\, c_\phi^2.$$

For $\|\phi\| \leqslant 3n$, one majorizes $\|\phi\|^{2n}$ by $(3n)^{2n}$; for $\|\phi\| > 3n$, one can write

$$\prod_{j=1}^{2n} (\|\phi\| - j + 1) \geqslant (\|\phi\| - 2n)^{2n}$$

$$\geqslant \left(\frac{\|\phi\|}{3}\right)^{2n},$$

whence

$$\|\phi\|^{2n} \leqslant 3^{2n} \prod_{j=1}^{2n} (\|\phi\| - j + 1).$$

The result follows. ∎

COROLLARY 1.6. *Let $g \in \mathscr{W}^{2,\alpha}(X)$ satisfying the hypotheses of the theorem; then*

$$\forall n \in \mathbf{N},\ \|\mathscr{L}^n g\|^2_{L^2(X)} \leqslant 2c^2 (9 \cdot \gamma^4 \cdot 2^{4\alpha})^n\, n^{4n\alpha'},$$

where $\alpha' = \max(\alpha, \frac{1}{2})$.

Proof. By Lemma 1.5, we have

$$\begin{aligned}
\|\mathscr{L}^n g\|^2_{L^2(X)} &\leqslant (3n)^{2n} \|g\|^2_{L^2(X)} + 3^{2n} \|\nabla^{2n} g\|^2_{L^2(X;H^{\otimes(2n)})} \\
&\leqslant (3n)^{2n} c^2 + 3^{2n} c^2 \gamma^{4n} (2n)!^{2\alpha} \\
&\leqslant 3^{2n} n^{4n\alpha'} c^2 + 3^{2n} c^2 \gamma^{4n} (2n)^{4n\alpha} \\
&\leqslant c^2 3^{2n} \gamma^{4n} 2^{4n\alpha} n^{4n\alpha'} + c^2 3^{2n} \gamma^{4n} 2^{4n\alpha} n^{4n\alpha'} \\
&= 2c^2 (9 \cdot \gamma^4 \cdot 2^{4\alpha})^n\, n^{4n\alpha'}. \quad ∎
\end{aligned}$$

LEMMA 1.7. *Let $f \in \mathscr{W}^{2,\infty}(X)$, A an elementary operator of type (r, s), and n an integer at least equal to $r+s$. Then one has, $\forall t \in [0, 1]$,*

$$\mathscr{E}(\|Af(x_\omega(t)) - Af(x_\omega(0))\|^2_{H^{\otimes s}} \mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))$$
$$\leqslant 4^{n-r-s+1} \frac{t^{n-r-s}}{(n-r-s)!} \|\mathscr{L}^n f\|^2_{L^2(X)}.$$

Proof. n being fixed, we shall use induction on $n-r-s$. For $r+s=n$ we simply write

$$\mathscr{E}(\|Af(x_\omega(t)) - Af(x_\omega(0))\|^2_{H^{\otimes s}} \mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))$$
$$\leqslant \mathscr{E}(\|Af(x_\omega(t)) - Af(x_\omega(0))\|^2_{H^{\otimes s}})$$
$$\leqslant 2(\mathscr{E}(\|Af(x_\omega(t))\|^2_{H^{\otimes s}}) + \mathscr{E}(\|Af(x_\omega(0))\|^2_{H^{\otimes s}}))$$
$$= 4\, \|Af\|^2_{L^2(X; H^{\otimes s})}$$

because the Ornstein–Uhlenbeck process has the property that $(x_s)_* P = \mu$ for each $s \in \mathbf{R}$. It is now enough to apply Lemma 1.4.

Let us now assume that $r+s<n$, and let $(w_j)_{j \in \mathbf{N}}$ be a Hilbertian basis of $H^{\otimes s}$. Let us set

$$Af(x) = \sum_{j=0}^{+\infty} f_j(x)\, w_j.$$

Each f_j belongs to $\mathscr{W}^{2,\infty}(X)$, and thus satisfies Itô's formula [3, p. 207],

$$f_j(x_\omega(t)) - f_j(x_\omega(0)) = \int_0^t \mathscr{L} f_j(x_\omega(u))\, du + M_{f_j}(\omega, t),$$

where

$$M_{f_j}(\omega, t)^2 - \int_0^t \|\nabla f_j(x_\omega(u))\|^2_H\, du$$

is a martingale. It follows that

$$\mathscr{E}((f_j(x_\omega(t)) - f_j(x_\omega(0)))^2\, \mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))$$
$$\leqslant 2\mathscr{E}\left(\left(\int_0^t \mathscr{L} f_j(x_\omega(u))\, du\right)^2 \mathbf{1}_{\mathscr{C}_f}(x_\omega(0)) + M_{f_j}(\omega, t)^2\, \mathbf{1}_{\mathscr{C}_f}(x_\omega(0))\right).$$

But $\mathscr{L}Af(x_\omega(0))=0$ and $\nabla Af(x_\omega(0))=0$ for almost all ω such that $x_\omega(0)\in\mathscr{C}_f$ by [7, Théorème 4]. Schwarz's inequality and the definition of M_f, thus allow us to write

$$\begin{aligned}
&\mathscr{E}((f_j(x_\omega(t))-f_j(x_\omega(0)))^2\,\mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))\\
&\quad\leqslant 2t\int_0^t \mathscr{E}((\mathscr{L}f_j(x_\omega(u))-\mathscr{L}f_j(x_\omega(0)))^2\,\mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))\,du\\
&\qquad+2\int_0^t \mathscr{E}(\|\nabla f_j(x_\omega(u))-\nabla f_j(x_\omega(0))\|_H^2\,\mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))\,du.
\end{aligned}$$

By the dominated convergence theorem one may, while summing on j, exchange integration and summation; moreover, one has

$$\mathscr{L}Af=\sum_{j\in\mathbf{N}}(\mathscr{L}f_j)\,w_j$$

and

$$\nabla Af=\sum_{j\in\mathbf{N}}\nabla f_j\otimes w_j.$$

It appears that

$$\begin{aligned}
&\mathscr{E}(\|Af(x_\omega(t))-Af(x_\omega(0))\|_{H^{\otimes s}}^2\,\mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))\\
&\quad\leqslant 2t\int_0^t \mathscr{E}(\|\mathscr{L}Af(x_\omega(u))-\mathscr{L}Af(x_\omega(0))\|_{H^{\otimes s}}^2\,\mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))\,du\\
&\qquad+2\int_0^t \mathscr{E}(\|\nabla Af(x_\omega(u))-\nabla Af(x_\omega(0))\|_{H^{\otimes(s+1)}}^2\,\mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))\,du.
\end{aligned}$$

The induction hypothesis now gives us

$$\begin{aligned}
&\mathscr{E}(\|Af(x_\omega(t))-Af(x_\omega(0))\|_{H^{\otimes s}}^2\,\mathbf{1}_{\mathscr{C}_f}(x_\omega(0)))\\
&\quad\leqslant 2\,\|\mathscr{L}^n f\|_{L^2(X)}^2\frac{4^{n-r-s}}{(n-r-s-1)!}\left(\int_0^t u^{n-r-s-1}\,du+t\int_0^t u^{n-r-s-1}\,du\right)\\
&\quad=\frac{2\cdot 4^{n-r-s}}{(n-r-s-1)!}\frac{t^{n-r-s}}{n-r-s}(1+t)\,\|\mathscr{L}^n f\|_{L^2(X)}^2\\
&\quad\leqslant\frac{4^{n-r-s+1}}{(n-r-s)!}t^{n-r-s}\,\|\mathscr{L}^n f\|_{L^2(X)}^2. \quad\blacksquare
\end{aligned}$$

Proof of Theorem 1.1. Let us apply Lemma 1.7 to $f = g$ and $A = \mathrm{Id}$; we get

$$\forall t \in [0, 1], \forall n \in \mathbf{N}, \mathscr{E}(|g(x_\omega(t)) - g(x_\omega(0))|^2 \mathbf{1}_{\mathscr{C}_g}(x_\omega(0))) \leqslant 4^{n+1} \frac{t^n}{n!} \|\mathscr{L}^n g\|^2_{L^2(X)}.$$

By Corollary 1.6, this is less than

$$4^{n+1} \frac{t^n}{n!} 2c^2 (9 \cdot \gamma^4 \cdot 2^{4\alpha})^n n^{4n\alpha'}.$$

But $n! \geqslant (n/e)^n$, whence $\forall t \in [0, 1]$, $\forall n \in \mathbf{N}$,

$$\mathscr{E}(|g(x_\omega(t)) - g(x_\omega(0))|^2 \mathbf{1}_{\mathscr{C}_g}(x_\omega(0))) \leqslant 8c^2 (36 \cdot e \cdot \gamma^4 \cdot 2^{4\alpha} \cdot t)^n n^{(4\alpha' - 1)n}.$$

Choosing $n = E((1/e)(36 \cdot e \cdot \gamma^4 \cdot 2^{4\alpha} \cdot t)^{-1/(4\alpha' - 1)})$, we see that the right-hand side is less than

$$\begin{aligned} &8c^2 (36 \cdot e \cdot \gamma^4 \cdot 2^{4\alpha} \cdot t)^n e^{-(4\alpha' - 1)n} (36 \cdot e \cdot \gamma^4 \cdot 2^{4\alpha} \cdot t)^{-n} \\ &\quad = 8c^2 e^{-(4\alpha' - 1)n} \\ &\quad \leqslant 8c^2 e^{4\alpha' - 1} e^{-((4\alpha' - 1)/e)(36 \cdot e \cdot \gamma^4 \cdot 2^{4\alpha} \cdot t)^{-1/(4\alpha' - 1)}}. \end{aligned}$$

Whence the result with $c_1 = 8c^2 e^{4\alpha' - 1}$ and

$$c_2 = \frac{4\alpha' - 1}{e} (36 \cdot e \cdot 16^\alpha)^{-1/(4\alpha' - 1)}. \quad \blacksquare$$

Remark. All infinite dimensional separable Gaussian probability spaces are canonically isomorphic; the results of this section are therefore valid for any of them. We shall use that remark in Section 4.

2. Classes of Hyper-Gevrey Functionals

The definitions in this paragraph can be considered as the axiomatic basis for the sequel of the paper. Let (X, H, μ) be an abstract Wiener space; a positive definite self-adjoint compact operator $\mathscr{A}$ on H will be called of order a if

$$\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant \cdots$$

being its eigenvalues in decreasing order, there are $c > 0$ and $c' > 0$ such that:

$$\forall n > 0, \qquad c' n^{-a} \leqslant \lambda_n \leqslant c n^{-a}.$$

Now, such an $\mathscr{A}$ being given, we will define the Sobolev spaces $\mathscr{D}^p_{r,\mathscr{A}}$ by their norm:

$$\|\phi\|_{\mathscr{D}^p_{r,\mathscr{A}}} = \sum_{j=0}^{r} \|(\mathscr{A}^{\otimes j})^{-1} \nabla^j \phi\|_{L^p(X; H^{\otimes j})}.$$

Then we shall the class $\mathscr{G}^\alpha_{p,a}(X)$ of *hypersmooth functionals of order* (p, a, α) by the condition that there exists a compact operator $\mathscr{A}$ of order a and real numbers $c_1(p)$ and $c_2(p)$ such that $\forall \in \mathbf{N}$:

$$\|g\|_{\mathscr{D}^p_{r,\mathscr{A}}} \leqslant c_1(p)\, c_2(p)^r\, (r!)^\alpha.$$

It is a consequence of Appendix A1 that Itô functionals provided by solutions of SDE's with coefficients of Gevrey class α are elements of $\mathscr{G}^\alpha_{p,a}(X)$ for each $p \geqslant 1$ and each $a < \frac{1}{2}$: the operator $\mathscr{A}$ can be taken to be $\mathcal{A}_a$, and $c_1(p)$ and $c_2(p)$ are explicitly computable from p, a, and the derivatives of the σ_i. We define

$$\mathscr{G}_{a,\alpha}(X) = \bigcap_{p \geqslant 1} \mathscr{G}^\alpha_{p,a}(X).$$

3. Heat Propagation for the Ornstein–Uhlenbeck Operator

LEMMA 3.1. *Given $x \in]0, 1[$ and $r \in]0, 1[$ there exists $c_{r,x}$ such that, whenever $A \subset X$ is a Borelian set with $\mu(A) \geqslant x$ then one has*

$$\forall t \geqslant r, \qquad E((P_t \mathbf{1}_A)^{-r}) \leqslant c_{r,x}.$$

Proof. We shall use the symmetrization procedure in the sense of Ehrard [5] and Borrell [1]. Let us denote by A^0 the symmetrization of A in the sense of [1, p. 3] (with $c = g = 0$, $n = 1$, $f = \mathbf{1}_A$). Then for any $p\colon \mathbf{R}^+ \to \mathbf{R}$ convex and increasing, we have

$$E(p(P_t \mathbf{1}_A)) \leqslant E(p(P_t \mathbf{1}_{A^0})).$$

For $\varepsilon > 0$ let us denote

$$p_\varepsilon(\xi) = (\varepsilon + \xi)^{-r} + r\varepsilon^{-r-1}\xi.$$

Then p_ε is convex and increasing; thus

$$E(p_\varepsilon(P_t \mathbf{1}_A)) \leqslant E(p_\varepsilon(P_t \mathbf{1}_{A^0})).$$

Furthermore,

$$E(P_t \mathbf{1}_A) = \mu(A)$$
$$E(P_t \mathbf{1}_{A^\theta}) = \mu(A^\theta) = \mu(A).$$

Therefore (i) implies that

$$E((\varepsilon + P_t \mathbf{1}_A)^{-r}) \leqslant E((\varepsilon + P_t \mathbf{1}_{A^\theta})^{-r}).$$

Letting $\varepsilon \to 0$ we get

$$E((P_t \mathbf{1}_A)^{-r}) \leqslant E((P_t \mathbf{1}_{A^\theta})^{-r})$$

(this is obvious when the right-hand side is infinite; in the other case, Lebesgue's dominated convergence theorem can be applied.)

The computation of the right-hand side for a half-space reduces the problem to a problem *in one dimension*. On $\mathbf{R}$ we shall use the Mehler formula,

$$p_t(\xi_0, \eta) = (2\pi)^{-1/2} \beta_t^{1/2} e^{-1/2(e^{-t}\xi_0 - \eta)^2 \beta_t},$$

where

$$\beta_t = (1 - e^{-2t})^{-1}$$

and p_t is defined by

$$P_t f(\xi_0) = \int_{\mathbf{R}} f(\eta)\, p_t(\xi_0, \eta)\, d\eta.$$

We determine λ_x by the condition

$$\int_{-\infty}^{\lambda_x} d\mu_{\mathbf{R}}(\eta) = x,$$

where

$$d\mu_{\mathbf{R}}(\eta) = (2\pi)^{-1/2} e^{-\eta^2/2}\, d\eta.$$

One has trivially:

$$-\tfrac{1}{2}(e^{-t}\xi_0 - \eta)^2 \geqslant -e^{-2t}\xi_0^2 - \eta^2.$$

Therefore, by an elementary computation, we get the result whenever

$$t > \tfrac{1}{2} \log(1 + 2r).$$

But $\log(1 + 2r) < 2r$. ∎

LEMMA 3.2. *Let G be a Hilbert space; for each $p \geqslant 1$, each $n \in \mathbf{N}$, and each $g \in \mathscr{W}^{\infty, p, \infty}(X; G)$, one has $\forall t \geqslant 0$*

$$E(\|g - P_t g\|_G^p \mathbf{1}_{\mathscr{C}_g}) \leqslant \frac{2^p}{\prod_{j=0}^{n-1}(1+jp)} t^{np} \|\mathscr{L}^n g\|_{L^p(X;G)}^p,$$

where

$$\mathscr{C}_g = \{x \in X \mid \nabla g(x) = 0\}.$$

Proof. We shall proceed by induction over n. For $n = 0$ we have

$$\begin{aligned} E(\|g - P_t g\|_G^p \mathbf{1}_{\mathscr{C}_g}) &\leqslant E(\|g - P_t g\|_G^p) \\ &= \|g - P_t g\|_{L^p(X;G)}^p \\ &\leqslant (\|g\|_{L^p(X;G)} + \|P_t g\|_{L^p(X;G)})^p \\ &\leqslant (2\|g\|_{L^p(X;G)})^p \\ &= 2^p \|g\|_{L^p(X;G)}^p \end{aligned}$$

because P_t is a contraction on $L^p(X; G)$; the result follows.

Let us assume the result to be true at rank n and let $g \in \mathscr{W}^{\infty, p, \infty}(X; G)$; we have

$$g - P_t g = \int_0^t (-P_u \mathscr{L} g)\, du$$

(it is enough to check the equality for g a scalar function, and thus for $g \in \mathbf{C}_m$; but then it reduces to the obvious:

$$1 - e^{-mt} = \int_0^t m e^{-mu}\, du.)$$

But $\mathscr{L} g = 0$ almost surely on $\mathscr{C}_g$ by Théorème 4 of [7, p. 86], whence

$$\mathbf{1}_{\mathscr{C}_g}(g - P_t g) = \int_0^t \mathbf{1}_{\mathscr{C}_g}(\mathscr{L} g - P_u \mathscr{L} g)\, du.$$

One thus has

$$\|(g - P_t g)\mathbf{1}_{\mathscr{C}_g}\|_G \leqslant \int_0^t \|\mathscr{L} g - P_u \mathscr{L} g\|_G \mathbf{1}_{\mathscr{C}_g}\, du.$$

Hölder's inequality now implies that

$$\|(g - P_t g)\,\mathbf{1}_{\mathscr{C}_g}\|_G \leq \left(\int_0^t 1^q\,du\right)^{1/q} \left(\int_0^t \|\mathscr{L}g - P_u \mathscr{L}g\|_G^p\,\mathbf{1}_{\mathscr{C}_g}\,du\right)^{1/p},$$

where q is the conjugate exponent of p. It now follows that

$$\|(g - P_t g)\,\mathbf{1}_{\mathscr{C}_g}\|_G^p \leq t^{p-1} \int_0^t \|\mathscr{L}g - P_u \mathscr{L}g\|_G^p\,\mathbf{1}_{\mathscr{C}_g}\,du.$$

But the induction hypothesis *applied to* $\mathscr{L}g$ gives us

$$\forall u \in [0, t], \qquad E(\|\mathscr{L}g - P_u \mathscr{L}g\|_G^p\,\mathbf{1}_{\mathscr{C}_{\mathscr{L}g}}) \leq \frac{2^p}{\prod_{j=0}^{n-1}(1+jp)}\,u^{np}\,\|\mathscr{L}^{n+1}g\|_{L^p(X;G)}^p.$$

But $\mathscr{C}_g \subset \mathscr{C}_{\mathscr{L}g}$ by Théorème 4 of [7], whence

$$\begin{aligned}
&E(\|g - P_t g\|_G^p\,\mathbf{1}_{\mathscr{C}_g})\\
&\quad\leq t^{p-1} \int_0^t E(\|\mathscr{L}g - P_u \mathscr{L}g\|^p\,\mathbf{1}_{\mathscr{C}_g})\,du\\
&\quad\leq t^{p-1} \int_0^t E(\|\mathscr{L}g - P_u \mathscr{L}g\|_G^p\,\mathbf{1}_{\mathscr{C}_{\mathscr{L}g}})\,du\\
&\quad\leq t^{p-1} \int_0^t \frac{2^p}{\prod_{j=0}^{n-1}(1+jp)}\,u^{np}\,\|\mathscr{L}^{n+1}g\|_{L^p(X;G)}^p\,du\\
&\quad= \frac{2^p\,\|\mathscr{L}^{n+1}g\|_{L^p(X;G)}^p}{\prod_{j=0}^{n-1}(1+jp)}\,t^{p-1} \int_0^t u^{np}\,du\\
&\quad= \frac{2^p\,\|\mathscr{L}^{n+1}g\|_{L^p(X;G)}^p}{\prod_{j=0}^{n-1}(1+jp)}\,t^{p-1}\,\frac{t^{np+1}}{np+1}\\
&\quad= \frac{2^p\,\|\mathscr{L}^{n+1}g\|_{L^p(X;G)}^p}{\prod_{j=0}^{n}(1+jp)}\,t^{(n+1)p}.
\end{aligned}$$

The result is now established at rank $n+1$. ∎

4. Majoration of the Oscillation of an Abstract Functional

In the above paragraph an oscillation theorem has been proved for the case of a $\mathscr{W}^{\infty}$-functional. We want to prove an oscillation result for a Gevrey functional having a critical set of large measure in the fiber. For each $\alpha > 0$ and each Gaussian space Y we define $\mathscr{G}_\alpha(Y)$ to be the space of $g: Y \to \mathbf{R}$ satisfying the hypothesis $(\mathscr{H}')$ of Theorem 1.1 for some $\gamma \geq 1$ and some c.

LEMMA 4.1. *Given a Gaussian probability space Y and $g \in \mathscr{G}_\alpha(Y)$, then, for each integer $m \geq 1$, $u_m = \|\nabla g\|^{2m} \in \mathscr{G}_\alpha(Y)$.*

Remark. This would not be true for $u = \|\nabla g\|$, according to the fact that the square root is not a smooth functional.

Proof.

$$\begin{aligned} u_1(y) &= \|\nabla g(y)\|_H^2 \\ &= \int_0^1 \left(\frac{d}{dt}(\nabla g(y))(t)\right)^2 dt \\ &= \int_0^1 D_t g(y)^2 \, dt, \end{aligned}$$

whence

$$D_{t'} u_1(y) = 2 \int_0^1 D_{t',t} g(y) \cdot D_t g(y) \, dt.$$

Thus $u_1 \in \mathscr{G}_\alpha(Y)$, whence the result because $u_m = u_1^m$. ∎

LEMMA 4.2. *For each $p > 1$ there is a constant K_p such that, for each Hilbert space G and each $f \in \mathscr{W}^{p,\infty}(X; G)$, one has, $\forall n \in \mathbf{N}$,*

$$\|\mathscr{L}^n f\|_{L^p(X;G)} \leq K_p^n \, n! \sum_{j=0}^{2n} \|\nabla^j f\|_{L^p(X; H^{\otimes j} \otimes G)}.$$

Proof. By Meyer's inequalities [12] there are constants c_p $(p > 1)$ such that, $\forall f \in \mathscr{W}^{p,2}(X; G)$,

$$\|\mathscr{L} f\|_{L^p(X;G)} \leq c_p(\|f\|_{L^p(X;G)} + \|\nabla^2 f\|_{L^p(X; H^{\otimes 2} \otimes G)}).$$

We set $K_p = 2(1 + c_p)$ and proceed by induction on n. For $n = 0$ it is obvious. Let us assume the hypothesis true at rank $n - 1$ $(n \geq 1)$; then we have

$$\begin{aligned} \|\mathscr{L}^n f\|_{L^p(X;G)} &= \|\mathscr{L}^{n-1}(\mathscr{L} f)\|_{L^p(X;G)} \\ &\leq K_p^{n-1} (n-1)! \sum_{j=0}^{2n-2} \|\nabla^j \mathscr{L} f\|_{L^p(X; H^{\otimes j} \otimes G)}. \end{aligned}$$

But it follows easily from Lemma 1.2 that $\nabla^j \mathscr{L} = (\mathscr{L} - j)\, \nabla^j$, whence

$$
\begin{aligned}
&\|\mathscr{L}^n f\|_{L^p(X;G)} \\
&\leqslant K_p^{n-1}(n-1)! \sum_{j=0}^{2n-2} \left(\|\mathscr{L}\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)} + j\|\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)} \right) \\
&\leqslant K_p^{n-1}(n-1)! \sum_{j=0}^{2n-2} \left(c_p(\|\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)} \right. \\
&\quad \left. + \|\nabla^{j+2} f\|_{L^p(X;H^{\otimes j+2}\otimes G)}) + j\|\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)} \right) \\
&\leqslant K_p^{n-1}(n-1)! \left(c_p \sum_{j=2}^{2n} \|\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)} \right. \\
&\quad \left. + \sum_{j=0}^{2n-2} (c_p + j)\, \|\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)} \right) \\
&\leqslant K_p^{n-1}(n-1)!\, 2(c_p + n) \sum_{j=0}^{2n} \|\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)} \\
&\leqslant K_p^{n-1}(n-1)!\, 2n(1 + c_p) \sum_{j=0}^{2n} \|\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)} \\
&\leqslant K_p^n n! \sum_{j=0}^{2n} \|\nabla^j f\|_{L^p(X;H^{\otimes j}\otimes G)}.
\end{aligned}
$$

The result then follows by induction. ▮

THEOREM 4.3. *Given $p < +\infty$ there exists $c'_p > 0$ and $c''_p > 0$ such that $\forall m \in \mathbf{N}$,*

$$\int_Y \|(A^{\otimes m})^{-1}\, \nabla^m g(y)\|^p_{H^{\otimes m}}\, d\mu(y) \leqslant c'_p e^{-c''_p n^{\alpha/4x}}$$

whenever $\mu(\mathscr{C}_g) \geqslant \frac{1}{2}$ and $g \in \mathscr{G}_{\alpha,\alpha}(Y)$.

Proof. Let $x > p$ be an integer, and let $s = 2x$; let then r be defined by $s = p(1 + 1/r)$. Clearly $r \in\,]0, 1[$. Setting $\beta = r/(r+1)$, $\rho = 1 + 1/r = 1/\beta$ and $\lambda = r + 1$, one has

$$\frac{1}{\lambda} + \frac{1}{\rho} = 1,$$

which will allow us to use Hölder's inequality with the exponents λ and ρ. Setting $q_{t,\mathscr{C}_g} = P_t \mathbf{1}_{\mathscr{C}_g}$, we can write

$$\mathscr{E}(\mathbf{1}_{\mathscr{C}_g}(y_\omega(t)) \, \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g\|^s_{H^{\otimes m}}(y_\omega(0)))$$

$$= \int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g\|^s_{H^{\otimes m}}(y) \, q_{t,\mathscr{C}_g}(y) \, d\mu(y).$$

Now the Schwarz inequality gives us

$$\int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g(y)\|^p_{H^{\otimes m}} \, d\mu(y)$$

$$= \int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g(y)\|^p_{H^{\otimes m}} \, q^{\beta}_{t,\mathscr{C}_g}(y) \, q^{-\beta}_{t,\mathscr{C}_g}(y) \, d\mu(y)$$

$$\leqslant \left(\int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g(y)\|^{p\rho}_{H^{\otimes m}} \, q_{t,\mathscr{C}_g}(y)^{\beta\rho} \, d\mu(y) \right)^{1/\rho} \left(\int_Y \frac{d\mu(y)}{q_{t,\mathscr{C}_g}(y)^{\beta\lambda}} \right)^{1/\lambda}$$

$$= \left(\int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g(y)\|^s_{H^{\otimes m}} \, q_{t,\mathscr{C}_g}(y) \, d\mu(y) \right)^{1/\rho} E(q^{-r}_{t,\mathscr{C}_g})^{1/\lambda}$$

But one has, because of the reversibility of the Ornstein–Uhlenbeck process,

$$\int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g(y)\|^s_{H^{\otimes m}} \, q_{t,\mathscr{C}_g}(y) \, d\mu(y)$$

$$= E(\|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g\|^s_{H^{\otimes m}} \cdot P_t \mathbf{1}_{\mathscr{C}_g})$$

$$= E(P_t(\|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g\|^s_{H^{\otimes m}}) \, \mathbf{1}_{\mathscr{C}_g})$$

$$\leqslant E((P_t u_m - u_m) \, \mathbf{1}_{\mathscr{C}_{u_m}}),$$

where $u_m = \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g\|^s_{H^{\otimes m}}$. Thus one has

$$\int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g\|^p_{H^{\otimes m}} \, d\mu(y) \leqslant E(|u_m - P_t u_m| \, \mathbf{1}_{\mathscr{C}_{u_m}})^{1/\rho} \, E(q^{-r}_{t,\mathscr{C}})^{1/\lambda}$$

We then apply Lemma 3.2 (with $p = 1$) to u_m. We have

$$\int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g(y)\|^p_{H^{\otimes m}} \, d\mu(y) \leqslant \left(\frac{2t^n}{n!} \, \|\mathscr{L}^n u_m\|_{L^1(X)} \right)^{1/\rho} E(q^{-r}_{t,\mathscr{C}_g})^{1/\lambda}.$$

Taking $t \geq r$ we get, by Lemma 4.2, that

$$\int_Y \|(\mathscr{A}^{\otimes m})^{-1} \nabla^m g(y)\|^p_{H^{\otimes m}} \, d\mu(y)$$
$$\leq \left[2K_1'' t^n \left(\sum_{j=0}^{2n} \|\nabla^j u_m\|_{L^\lambda(X; H^{\otimes j})} \right) \right]^{1/p} E(q_{t,\mathscr{C}_g}^{-r})^{1/\lambda}$$

and we get the result thanks to Lemmas 3.1 and 4.1. ∎

COROLLARY 4.4. *The oscillation of g on each fiber satisfies*

$$\forall p \geq 1, \qquad \int_Y \left| g(y) - \int_Y g(\tilde{y}) \, d\mu(\tilde{y}) \right|^p d\mu(y) \leq c_p''' e^{-c_p'' n^{a/4\alpha}}.$$

Proof. We use Ocone's formula, which is possible because Y is isomorphic to the classical Wiener space,

$$g(y) = \int_Y g(\tilde{y}) \, d\mu(\tilde{y}) + \int_0^1 E^{\mathscr{F}_t}(D_t g)(y) \, dy(t),$$

where $D_t g(y) = j(\nabla g(y))(t)$. The Burkholder–Gundy inequalities [2, Theorem 5.1] tell us that, for each $p > 1$, there is a constant a_p such that, for each Hilbert-space valued adapted martingale $A(y, t)$, one has

$$E\left(\left\| \int_0^1 A(y, t) \, dy(t) \right\|^p \right) \leq a_p \int_0^1 E(\|A(y, t)\|^p) \, dt.$$

Applying this result to $A(y, t) = E^{\mathscr{F}_t}(\nabla g)(y)$, one finds

$$\begin{aligned} \int_Y \left| g(y) - \int_Y g(\tilde{y}) \, d\mu(\tilde{y}) \right|^p d\mu(y) &= E\left(\left| \int_0^1 E^{\mathscr{F}_t}(D_t g)(y) \, dy(t) \right|^p \right) \\ &\leq a_p \int_0^1 E(|E^{\mathscr{F}_t}(D_t g)(y)|^p) \, dt \\ &\leq a_p \int_0^1 E(E^{\mathscr{F}_t}(|D_t g(y)|^p_H)) \, dt \\ &\leq a_p \int_0^1 E(|D_t g(y)|^p) \, dt \\ &= a_p \, E\left(\int_0^1 |D_t g(y)|^p \, dt \right) \end{aligned}$$

and one can then apply Theorem 4.3. ∎

5. A COVERING LEMMA

LEMMA 5.1. *For $a < \frac{1}{2}$ let $\mathscr{A}_a$ denote a compact operator on H having $\{m^{-a}\}_{m=1}^{+\infty}$ for eigenvalues. Denote by $\mathscr{A}_{a,n}$ the restriction of $\mathscr{A}_a$ to the space*

E_n generated by its first n eigenfunctions. We take on E_n the euclidean metric. Denote

$$\mathscr{C}_n = \{\xi \in E_n \mid \|\mathscr{A}_{a,n}^{-1}(\xi)\| \leqslant n^a\}.$$

Then for all $\varepsilon > 0$ we have

$$\operatorname{vol}(\mathscr{C}_n) \leqslant \frac{2\pi^{n/2}}{n\Gamma(n/2)} c_\varepsilon^{an}(n^a)^{n\varepsilon},$$

where c_ε is independent of n and α.

Proof. We may assume that $\varepsilon < 1$; let us then set $\delta = 1 - \varepsilon$ and note that

$$\forall x \in]0, \delta], \qquad \frac{-\log(1-x)}{x} \leqslant \frac{-\log(1-\delta)}{\delta}.$$

Denote

$$\mathscr{B}_{a,n} = \frac{1}{n^a} \mathscr{A}_{a,n}^{-1}.$$

Then

$$\{\xi \mid \|\mathscr{B}_{a,n}(\xi)\| \leqslant 1\} = \mathscr{B}_{a,n}^{-1}(\{\eta \mid \|\eta\| \leqslant 1\}) \quad .$$

and

$$\begin{aligned} \operatorname{vol}(\mathscr{C}_n) &= \operatorname{vol}(\{\xi \mid \|\mathscr{B}_{a,n}(\xi)\| \leqslant 1\}) \\ &= \det(\mathscr{B}_{a,n}^{-1}) \operatorname{vol}(\mathbf{B}_{\mathbf{R}^n}(0, 1)). \end{aligned}$$

But the volume of the unit ball in $\mathbf{R}^n$ is

$$\frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

$\mathscr{B}_{a,n}^{-1}$ has for its eigenvalues the n^a/m^a, for $1 \leqslant m \leqslant n$; thus

$$\begin{aligned} \log \det(\mathscr{B}_{a,n}^{-1}) &= a \sum_{m=1}^{n} \log \frac{n}{m} \\ &= a \sum_{s=1}^{E(n\delta)} \left(-\log\left(1 - \frac{s}{n}\right)\right) + a \sum_{s=1+E(n\delta)}^{n-1} \log\left(\frac{n}{n-s}\right) \\ &\leqslant -a\delta \log(1-\delta)n + an\varepsilon \log n, \end{aligned}$$

whence the result with

$$c_\varepsilon = e^{-\delta \log(1-\delta)} = \varepsilon^{-\delta} = \varepsilon^{\varepsilon - 1}. \quad \blacksquare$$

THEOREM 5.2. *For every $\varepsilon > 0$ there exists a constant $c_{\varepsilon,a}$ independent of n such that it is possible to cover $\mathscr{C}_n$ by $[n^{an\varepsilon} c_{\varepsilon,a} r^{-n}]$ balls of radius r ($r < 1$).*

Proof. For each $\psi: \{1, ..., n\} \to \mathbf{Z}$ we consider the hypercube C_ψ consisting of the points $(\xi_1, ..., \xi_n)$ of $\mathbf{R}^n$ such that

$$\forall j \in \{1, ..., n\}, \xi_j \in \left]\frac{2r\psi(j)}{\sqrt{n}}, \frac{2r(\psi(j)+1)}{\sqrt{n}}\right].$$

For each ψ such that $C_\psi \cap \mathscr{C}_n \neq \varnothing$ we consider a ball of radius r containing C_ψ; such a ball is contained in $(2+8r^2)^{1/2}\,\mathscr{C}_n$, whence their number is at most

$$N(r) = (2+8r^2)^{n/2}\,\mathrm{vol}(\mathscr{C}_n)(2r/\sqrt{n})^{-n}.$$

As

$$\mathrm{vol}(\mathscr{C}_n) \leqslant \frac{2\pi^{n/2}}{n\Gamma(n/2)}\, c_{\varepsilon/2}^{an} n^{an\varepsilon/2}$$

we get

$$N(r) \leqslant (2+8r^2)^{n/2} \frac{2\pi^{n/2}}{n\Gamma(n/2)}\, c_{\varepsilon/2}^{an} n^{an\varepsilon/2} \left(\frac{\sqrt{n}}{2r}\right)^n.$$

By Stirling's formula, one has

$$\frac{n}{2}\Gamma\left(\frac{n}{2}\right) \sim \sqrt{\pi n}\left(\frac{n}{2}\right)^{n/2} e^{-n/2},$$

whence we have a constant $c > 0$ such that

$$\Gamma\left(\frac{n}{2}\right) \geqslant \frac{c}{\sqrt{n}}\, n^{n/2}(2e)^{-n/2},$$

from which follows:

$$\begin{aligned} N(r) &\leqslant 10^{n/2} \frac{2\pi^{n/2}}{n} \frac{\sqrt{n}}{c}\, n^{-n/2}(2e)^{n/2}\, c_{\varepsilon/2}^{an} n^{an\varepsilon/2} n^{n/2} r^{-n} 2^{-n} \\ &= (10\pi)^{n/2}\left(\frac{e}{2}\right)^{n/2} \frac{2}{c} \frac{1}{\sqrt{n}}\, c_{\varepsilon/2}^{an} n^{an\varepsilon/2} r^{-n} \\ &\leqslant c_{\varepsilon,a} n^{an\varepsilon} r^{-n}. \quad \blacksquare \end{aligned}$$

6. SARD THEOREM

THEOREM 6.1. *Let* $a>0$ *and* $\alpha\in\,]0, a/4[$ *be given, and let* $g\in\mathscr{G}_{a,\alpha}(X)$; *define* $\rho_g=g_*(\mathbf{1}_{\mathscr{C}_g}\mu)$. *Then* ρ_g *is carried by a set of zero Hausdorff dimension.*

Proof. In case $\mu(\mathscr{C}_g)=0$, we have $\rho_g=0$ and the statement is obvious; we shall henceforth assume that $\mu(\mathscr{C}_g)>0$. Let $\mathscr{A}$ be the operator that appears in the definition of "hypersmoothness" and let e_i $(i\in\mathbf{N}^*)$ be an orthonormal basis of H in which $\mathscr{A}$ is diagonalised: $\mathscr{A}e_i=\lambda_i e_i$, with the sequence λ_i $(i\in\mathbf{N}^*)$ decreasing. We denote by V_n the subspace generated by the e_i $(1\leqslant i\leqslant n)$. We denote by π_n the canonical orthogonal projection onto V_n:

$$\pi_n(x)=\sum_{k=1}^{n}\langle e_k, x\rangle e_k.$$

It is clear that $\mu_{V_n}=(\pi_n)_*\mu$ is the canonical Gaussian measure on V_n. Let

$$Y_n=\ker\pi_n=\operatorname{Im}(I-\pi_n)$$

and

$$\mu_{Y_n}=(I-\pi_n)_*\mu.$$

Then we have a pseudo-direct sum decomposition $X=V_n\oplus Y_n$, $\mu=\mu_{V_n}\otimes\mu_{Y_n}$ and $(Y_n, V_n^{\perp}, \mu_{Y_n})$ is an abstract Wiener space. For $\xi\in V_n$ we define a measure on X by

$$\nu^{n,\xi}(A)=\mu_{Y_n}((A-\xi)\cap Y_n).$$

It is clear that we have here a disintegration of μ along $(\pi_n)_*\mu=\mu_{V_n}$, i.e.,

$$\forall f\in L^1(X)\qquad \int_X f(x)\,\mu(dx)=\int_{V_n}\left(\int_X f(x)\,\nu^{n,\xi}(dx)\right)\mu_{V_n}(d\xi).$$

Let $\mathscr{C}_n=\pi_n(\mathscr{C}_g)$. Then

$$\mu_{V_n}(\mathscr{C}_n)=[(\pi_n)_*\mu](\mathscr{C}_n)=\mu(\pi_n^{-1}(\pi_n(\mathscr{C}_g)))$$

decreases to $\mu(\mathscr{C}_g)$ when $n\to+\infty$. We can therefore find n_0 such that:

$$\forall n\geqslant n_0,\qquad \mu_{V_n}(\mathscr{C}_n)\leqslant\frac{3}{2}\mu(\mathscr{C}_g).$$

For each $n \geqslant n_0$ there is $\mathscr{C}'_n \subset \mathscr{C}_n$ such that

$$\mu_{V_n}(\mathscr{C}'_n) > \frac{3}{4}\mu_{V_n}(\mathscr{C}_n)$$

and such that

$$E^{V_n}(\|\mathscr{A}^{-1}\nabla g\|)^2(x) \leqslant c'_p e^{-c''_p n^{a 4\alpha}} \qquad (*)$$

for each $x \in \pi_n^{-1}(\mathscr{C}'_n)$. Furthermore, we know, by Section 5, that $\|\nabla_X g\|$ along each fiber is small. Let us denote $g_n = E^{V_n}(g)$; then one can write

$$\nabla^m g_n = E^{V_n}(P_{V_n}^{\otimes m}(\nabla^m g)),$$

whence

$$E(\|\nabla^m g_n\|_{H^{\otimes m}}) \leqslant E(\|\nabla^m g\|_{H^{\otimes m}}).$$

Therefore g_n has its mth derivative on $\mathscr{C}'_n$ bounded by $c'_1 e^{-c''_1 m^{a 4\alpha}}$; by the Sobolev embedding theorem on *the finite-dimensional space* V_n, $g_n \in \mathscr{C}^r(V_n)$. Let us define

$$\mathscr{D}_n = \{\xi \in V_n \mid (\nabla g)^*|_{\pi^{-1}(\xi)} \text{ vanishes on a set of measure} > \tfrac{1}{2}\}.$$

LEMMA 6.2. *$\mathscr{D}_n$ is a closed set.*

Proof. Quasi-sure analysis [11].

LEMMA 6.3. *For each $n \geqslant n_0$, $\mu_{V_n}(\mathscr{D}_n) \geqslant \frac{1}{2}\mu(\mathscr{C}_g)$.*

Proof. We have

$$\begin{aligned}
\mu(\mathscr{C}_g) &= \int_{\mathscr{C}_n} \nu^{n,\xi}(\mathscr{C}_g \cap \pi_n^{-1}(\xi))\, \mu_{V_n}(d\xi) \\
&\leqslant \tfrac{1}{2}\mu_{V_n}(\mathscr{C}_n - \mathscr{D}_n) + \mu_{V_n}(\mathscr{D}_n) \\
&\leqslant \tfrac{3}{4}\mu(\mathscr{C}_g) + \tfrac{1}{2}\mu_{V_n}(\mathscr{D}_n). \quad ∎
\end{aligned}$$

Let $\beta > 0$ and $\varepsilon > 0$ be so chosen that $\beta/(\varepsilon + \beta) > 4\alpha/a$; this is possible because $\alpha < a/4$. Let $r_n = n^{-\beta}$; by Theorem 5.2, it is possible to cover

$$\mathscr{D}_{n,M} = \mathscr{D}_n \cap \mathscr{A}(\mathscr{B}_{V_n}(0, M)),$$

using

$$N_n \leqslant n^{an\varepsilon} c_{\varepsilon,a} \left(\frac{n^a c r_n}{M}\right)^{-n} \qquad (**)$$

balls of radius r_n. According to Corollary 4.4, for each $\xi \in \mathscr{G}_{n,M}$, the oscillation of g on Y_ξ is less than $c_p''' \exp(-c_p'' n^{a/4\alpha})$. Let us apply the finite-dimensional Taylor's formula to g_n on V_n, in a neighborhood of $\xi \in \mathscr{G}_n$; we get

$$g_n(x) - g_n(\xi) = \sum_{j=0}^{n} \frac{1}{j!} \nabla^j g_n(x-\xi, ..., x-\xi) + R_{n,\xi}(x).$$

But, by (*), the oscillation of g_n on each of these balls is at most

$$c_p' e^{-c_p'' n^{a/4\alpha}} r_n.$$

Furthermore, the oscillation of g on each fiber introduces an increase of the length of those intervals which is $\exp(-n^{a/4\alpha})$. Therefore,

$$g(\pi_n^{-1}(\mathscr{G}_{n,M})) \subset \bigcup_{j=1}^{N_n} I_j,$$

where

$$l(I_j) \leqslant \exp(-n^{a/4\alpha}) + c_p' \exp(-c_p'' n^{a/4\alpha}) r_n.$$

We want that

$$\log N_n = o(n^{a/4\alpha}).$$

But this is the case because of (**) and

$$\frac{\beta}{\varepsilon+\beta} > \frac{4\alpha}{a}. \quad \blacksquare$$

APPENDIX. AN EXAMPLE OF HYPER-GEVREY FUNCTIONALS ASSOCIATED TO AN ITÔ SDE

A1. *A Classical Majoration for a Multiplicative Stochastic Integral*

For $A \in \mathbf{M}_n(\mathbf{R})$, we shall denote by $\|A\|_2$ its Hilbert–Schmidt norm,

$$\|A\|_2 = (\operatorname{tr}(A'A))^{1/2},$$

and by $\|A\|_\infty$ its uniform norm,

$$\|A\|_\infty = \sup_{x \in \mathbf{R}^n \setminus \{0\}} \frac{\|Ax\|}{\|x\|}.$$

It is well known that

$$\begin{aligned} &\forall (A, B) \in \mathbf{M}_n(\mathbf{R})^2 && \|AB\|_\infty \leqslant \|A\|_\infty \|B\|_\infty \\ &\forall A \in \mathbf{M}_n(\mathbf{R}), && \|{}^tA\|_\infty = \|A\|_\infty \\ &\forall A \in \mathbf{M}_n(\mathbf{R}), && |\mathrm{tr}(A)| \leqslant n \|A\|_\infty . \end{aligned}$$

The following result is close to Théorème 3 from Ibero [6].

LEMMA A1.1. *Given a stochastic differential equation defined on* $\mathbf{GL}_m(\mathbf{R})$ $(m \geqslant 1)$,

$$\begin{aligned} dM &= M\left(\sum_{i=0}^{d} A_i\, dx^i\right) \qquad (d \geqslant 1) \\ M(0) &= I \end{aligned}$$

(*we have set* $dx^0(\tau) = d\tau$), *where the* A_i *are adapted matrix-valued functionals with* $\|A_i\|_\infty \leqslant c_0$, *then there exists constants* $c_1 > 0$ *and* $c_2 > 0$ *depending only on* d, m *and* c_0 *such that*:

$$\forall R \geqslant 1, \qquad P\Big(\sup_{\tau \in [0,1]} \|M(\tau)\|_\infty \geqslant R\Big) \leqslant c_1 \exp(-c_2 (\log R)^2).$$

Proof. Let us denote by $\lambda_1, ..., \lambda_m$ the eigenvalues of $M{}^tM$. And let

$$\forall r \in \mathbf{N}, \qquad Q_r(x, \tau) = (M(\tau){}^t M(\tau))^r.$$

Then let us compute the drift term with the help of Itô calculus,

$$\begin{aligned} dQ_r = \Bigg(\sum_{i=1}^{d} \Bigg(&\sum M(A_i \cdots {}^tA_i){}^tM \cdots + \sum \cdots MA_i{}^tM \cdots MA_i{}^tM \cdots \\ &+ \sum \cdots {}^tA_i{}^tM \cdots {}^tA_i{}^tM \cdots \Bigg) + \sum \cdots MA_0^t M \cdots \\ &+ \sum \cdots M{}^tA_0^t M \cdots \Bigg) d\tau, \end{aligned}$$

where the first inner sum has r^2 terms, each of the next two $r(r-1)/2$ and each of the outer sums r. Defining $q_r = \mathrm{tr}(Q_r)$ then one has

$$dq_r \leqslant \|M\|_\infty^{2r} \left(dmc_0^2 r^2 + 2dmc_0^2 \frac{r(r-1)}{2} + 2rc_0 m \right) d\tau.$$

Therefore,

$$dq_r \leqslant \|M\|_\infty^{2r}\, 2mc_0(1 + dc_0)\, r^2\, d\tau.$$

But

$$\|M\|_{\infty}^{2r} = \sup_i \lambda_i^r \leqslant q_r.$$

Therefore $q_r e^{-c_r \tau}$ is a submartingale, where

$$c_r = 2mc_0(1 + dc_0)\, r^2$$

Let

$$T_{r,R} = \inf\{\tau \geqslant 0 \mid q_r(\tau) \geqslant R^{2r}\};$$

then

$$E(\exp(-c_r T_{r,R})\, R^{2r}) \leqslant q_r(0) = m,$$

from which we get

$$P(T_{r,R} \leqslant 1) \exp(-c_r)\, R^{2r} \leqslant m;$$

that is,

$$P(T_{r,R} \leqslant 1) \leqslant m \exp(c_r - 2r \log R).$$

Let $\lambda = 2mc_0(1 + dc_0)$ and let $r_0 = E(\frac{\log R}{\lambda})$; then $c_{r_0} = \lambda r_0^2 \leqslant r_0 \log R$, whence

$$\begin{aligned}
c_{r_0} - 2r_0 \log R &\leqslant -r_0 \log R \\
&\leqslant -\frac{(\log R)^2}{\lambda} + \log R \\
&\leqslant -\frac{(\log R)^2}{\lambda} + \frac{1}{2\lambda}[(\log R)^2 + \lambda^2] \\
&= -\frac{(\log R)^2}{2\lambda} + \frac{\lambda}{2}
\end{aligned}$$

and

$$\begin{aligned}
P(T_{r_0,R} \leqslant 1) &\leqslant m \exp(c_{r_0} - 2r_0 \log R) \\
&\leqslant m \exp\left(\frac{\lambda}{2}\right) \exp\left(-\frac{(\log R)^2}{2\lambda}\right).
\end{aligned}$$

But from $q_r \geqslant \|M\|_{\infty}^{2r}$ it follows that

$$\begin{aligned}
\{\sup_{\tau \in [0,1]} \|M(\tau)\|_{\infty} \geqslant R\} &\subset \{\sup_{\tau \in [0,1]} q_{r_0}(\tau) \geqslant R^{2r_0}\} \\
&\subset \{T_{r_0,R} \leqslant 1\},
\end{aligned}$$

whence we get the result with

$$c_1 = m \exp\left(\frac{\lambda}{2}\right) = m \exp(mc_0(1 + dc_0))$$

and

$$c_2 = \frac{1}{2\lambda} = \frac{1}{4mc_0(1 + dc_0)}. \quad \blacksquare$$

A2. *Best Approximation of an Itô Functional by Functions of a Finite Number of Variables*

Given an increasing sequence V_n ($n \in \mathbf{N}$) of finite-dimensional subspaces of H, then for every $f \in \mathscr{W}^{2,2}(X)$ the martingale $f_n = E^{V_n}(f)$ converges towards f in $\mathscr{W}^{2,2}(X)$. The *reduction* of f consists in choosing a good sequence V_n ($n \in \mathbf{N}$) such that

(1) f_n "approximates well" f

(2) $\dim(V_n)$ has a controlled growth.

For a general smooth functional f, it is not possible to ensure *a priori* the control stated in (2). But in the case of Itô functionals this control is possible. We shall obtain it through the machinery introduced to prove the compactness of certain subsets of $L^2(X)$ in [4]. We shall denote by e_s ($s \in \mathbf{N}$) the Haar basis of $L^2([0, 1])$ defined by

$$\forall k \in \mathbf{N},\ \forall j \in [0, 2^k - 1],\ \forall r \in [0, 1]$$

$$e_{2^k+j}(r) = 2^{k/2}(\mathbf{1}_{[j \cdot 2^{-k}, (2j+1)2^{-k-1}]}(r) - \mathbf{1}_{[(2j+1)2^{-k-1}, (j+1)2^{-k}]}(r))$$

and

$$e_0 = 1.$$

Let

$$A_a\colon L^2([0, 1]) \to L^2([0, 1])$$

be given by $A_a e_s = 2^{ka} e_s$ whenever $s = 2^k + j$ with $j \in [0, 2^k - 1]$ and $A_a e_0 = e_0$. Let us define $\mathcal{A}_a\colon H \to H$ by

$$\mathcal{A}_a = \mathbf{j}^{-1} \circ A_a \circ \mathbf{j}.$$

We shall work under the following hypothesis:

($\mathscr{H}$) *Let us consider on* $\mathbf{R}^n$ *the following Stratonovitch stochastic differential equation*:

$$dv_x(\tau) = \sum_{i=0}^{d} \sigma_i(v_x(\tau))\, dx^i(\tau), \qquad v_x(0) = 0 \tag{i}$$

(*we have again set* $dx^0(\tau) = d\tau$), *and let* $g(x) = v_x(1)$. *We assume that the* σ_i^α ($1 \leqslant \alpha \leqslant n$) *and all their partial derivatives of order at most three are bounded in absolute value by a constant* M *and we denote* $g(x) = v_x(1)$.

A2.1. *Computation of the First Derivative*

The first derivative is given (see [11]) by

$$D_\tau g = J_{1 \leftarrow \tau} = J_{1 \leftarrow 0}(J_{\tau \leftarrow 0})^{-1}$$

where $J_{\tau \leftarrow 0}$ is obtained by solving the *linear* stochastic differential equation,

$$d_\tau J_{\tau \leftarrow 0} = \left(\sum_{i=0}^{d} A_i(v_x(\tau))\, dx^i(\tau) \right) J_{\tau \leftarrow 0}, \qquad J_{0 \leftarrow 0} = I, \tag{ii}$$

where A is defined by

$$A_i^{\alpha,\beta}(\xi) = \frac{\partial \sigma_i^\alpha}{\partial v^\beta}(\xi) \qquad (\alpha, \beta \in \{1, ..., n\}).$$

Then $M(\tau) = (J_{\tau \leftarrow 0})^{-1}$ is given by the following stochastic differential equation:

$$dM(\tau) = -M(\tau) \left(\sum_{i=1}^{d} A_i(v_x(\tau))\, dx^i(\tau) + A_0(v_x(\tau))\, d\tau \right), \qquad M(0) = I.$$

In coordinates (ii) takes the form:

$$d_\tau J^{\alpha,\beta}_{\tau \leftarrow 0} = \sum_{i=0}^{d} \sum_{\gamma=1}^{n} A_i^{\alpha,\gamma}(v_x(\tau))\, J^{\gamma,\beta}_{\tau \leftarrow 0} dx^i(\tau), \qquad J^{\alpha,\beta}_{0 \leftarrow 0} = \delta_{\alpha,\beta}.$$

A2.2. *Computation of the Second Derivative*

We shall use the mechanism of prolongation introduced by Malliavin [9, p. 228]. We have

$$D_{\tau'} D_\tau g = [(D_{\tau'} J_{1 \leftarrow 0}) - J_{1 \leftarrow 0} J^{-1}_{\tau \leftarrow 0}(D_{\tau'} J_{\tau \leftarrow 0})]\, J^{-1}_{\tau \leftarrow 0}.$$

We therefore have to compute, τ being fixed, $D_{\tau'} J_{\tau \leftarrow 0}$. Then we have

$$D_{\tau'} J_{\tau \leftarrow 0} = 0 \qquad \text{if} \quad \tau' > \tau.$$

Let us write $V^0 = \mathbf{R}^n$ and let us denote by V^1 the principal bundle $V^0 \times \mathbf{GL}_n(\mathbf{R})$. Then let us define vector fields σ_i^1 on V^1 by

$$\sigma_i^1(v_0, \gamma) = (\sigma_i(v_0), A_i(v_0)\gamma).$$

This makes sense because $Z_\gamma = A_i(v_0)\gamma$ defines a vector field on $\mathbf{GL}_n(\mathbf{R})$ (the tangent space $\mathbf{gl}_n(\mathbf{R})$ to $\mathbf{GL}_n(\mathbf{R})$ being naturally identified to $\mathbf{M}_n(\mathbf{R})$). Then, setting $v_x^1(\tau) = (v_x(\tau), J_{\tau \leftarrow 0})$, the conjunction of Eqs. (i) and (ii) can be written as

$$dv_x^1(\tau) = \sum_{i=0}^{d} \sigma_i^1(v_x^1(\tau))\, dx^i(\tau), \qquad v_x^1(0) = (0, I). \tag{iii}$$

We shall treat the computation of the second derivative by the procedure we have already used for (i); we have to differentiate along the vertical component $\mathbf{GL}_n(\mathbf{R})$ of the fibre bundle. We choose a basis $(e_{\alpha,\beta})_{1 \leqslant \alpha \leqslant n, 1 \leqslant \beta \leqslant n}$ of the Lie algebra $\mathbf{gl}_n(\mathbf{R})$ and we differentiate on the right:

$$\partial_e u(v, \gamma) = \frac{d}{d\varepsilon} u(v, \gamma \exp(\varepsilon e)).$$

Then the vector fields $(A_i(v)\gamma)_{1 \leqslant i \leqslant n}$ are invariant under this differentiation *because it is performed on the right*; therefore, defining $A^1 = \partial\sigma^1/\partial v_\alpha$ we have

$$A_i^1 \in \mathrm{End}(\mathbf{R}^n \times \mathbf{gl}_n(\mathbf{R})), \qquad A_i^1 = \begin{pmatrix} A_i^0 & Q_i \\ 0 & I_{gl_n(\mathbf{R})} \end{pmatrix}, \tag{iv}$$

where $Q_i \in \mathrm{End}(\mathbf{R}^n \times \mathbf{gl}_n(\mathbf{R}))$ is defined by

$$Q_i^{\alpha,\beta,\gamma} = \frac{\partial^2 \sigma_i^\alpha}{\partial v^\beta\, \partial v^\gamma}.$$

Then we introduce the jacobian matrix

$$J_{\tau' \leftarrow 0}^1 \in \mathrm{End}(\mathbf{R}^n \times \mathbf{gl}_n(\mathbf{R}))$$

defined by

$$d_{\tau'} J_{\tau' \leftarrow 0}^1 = \left(\sum_{i=0}^{d} A_i^1(v_x(\tau'))\, dx^i(\tau') \right) J_{\tau' \leftarrow 0}^1, \qquad J_{0 \leftarrow 0}^1 = I. \tag{v}$$

Taking (iv) into account we have

$$J^1_{\tau' \leftarrow 0} = \begin{pmatrix} J_{\tau' \leftarrow 0} & Q^{\alpha,\beta,\gamma}_{\tau'} \\ 0 & I_{gl_n(\mathbf{R})} \end{pmatrix}.$$

Then

$$J^1_{\tau \leftarrow 0}(J^1_{\tau' \leftarrow 0})^{-1} = \begin{pmatrix} * & D_\tau D_{\tau'} g \\ * & * \end{pmatrix}.$$

Denoting $M^1_{\tau'} = (J^1_{\tau' \leftarrow 0})^{-1}$ we have again that $M^1_{\tau'}$ is given by

$$d_{\tau'} M^1_{\tau'} = -M^1_{\tau'} \left(\sum_{i=0}^{d} A^1_i(v_x(\tau'))\, dx^i(\tau') \right), \qquad M^1_0 = I. \qquad \text{(vi)}$$

A2.3. *A Tensor Product Norm*

Let G be a Hilbert space; for each $a \in [0, \frac{1}{2}]$ we shall define

$$\mathscr{H}_a(G) = \{ p: [0, 1] \to G \mid p(0) = 0 \text{ and } \|p\|_{\mathscr{H}_a(G)} < +\infty \}$$

with

$$\|p\|^2_{\mathscr{H}_a(G)} = \iint_{[0,1]^2} \frac{\|p(t) - p(t')\|^2_G}{|t - t'|^{1+2a}}\, dt\, dt'$$

with the natural scalar product

$$(p \mid q)_{\mathscr{H}_a(G)} = \iint_{[0,1]^2} \frac{(p(t) - p(t') \mid q(t) - q(t'))_G}{|t - t'|^{1+2a}}\, dt\, dt'.$$

We shall abbreviate $\mathscr{H}_a(\mathbf{R})$ in $\mathscr{H}_a$. For $h: [0, 1]^2 \to \mathbf{R}$, we shall denote, for each $x \in [0, 1]$, by h_x the partial function,

$$h_x: [0, 1] \to \mathbf{R}$$
$$y \to h(x, y).$$

We consider the space $\mathscr{H}_a([0, 1]; \mathscr{H}_a([0,1]))$. Its norm is defined by

$$\int_{[0,1]^2} \|h_x - h_{x'}\|^2_{\mathscr{H}_a} \frac{dx\, dx'}{|x - x'|^{1+2a}} = \|h\|^2_{\mathscr{H}_a \otimes \mathscr{H}_a}.$$

Thus it is easily seen that

$$\|h\|^2_{\mathscr{H}_a([0,1];\, \mathscr{H}_a([0,1]))} = \int_{[0,1]} A_h(s, s', t, t') \frac{ds\, ds'\, dt\, dt'}{|s - s'|^{1+2a}\, |t - t'|^{1+2a}},$$

where

$$A_h(s, s', t, t') = (h(s, t) - h(s, t') - h(s', t) + h(s', t'))^2.$$

THEOREM A2.3.1. *For any $a < \frac{1}{2}$ there exists a constant c_a depending only on n, d, and the uniform norm of the first three derivatives of the σ_i such that:*

$$E(\|D^{(2)}_{\cdot,\cdot} g\|^2_{\mathscr{H}_a([0,1];\, \mathscr{H}_a([0,1]))\otimes \mathbf{R}^n}) \leqslant c_a. \tag{vii}$$

Remark. $D^{(2)}_{\tau,\tau'} g$ is $\mathbf{R}^n$-valued, which explains the presence of $\mathbf{R}^n$ in the tensor product.

Proof. By the above, we have

$$D_\tau g = J_{1\leftarrow 0}(J_{\tau\leftarrow 0})^{-1},$$

whence

$$D_{\tau'} D_\tau g = (D_{\tau'} J_{1\leftarrow 0})\, J^{-1}_{\tau\leftarrow 0} - J_{1\leftarrow 0} J^{-1}_{\tau\leftarrow 0}(D_{\tau'} J_{\tau\leftarrow 0})\, J^{-1}_{\tau\leftarrow 0}.$$

The finiteness of the norm in (vii) results by a direct computation from 2.1, the Burkholder–Gundy inequalities [2, Theorem 5.1] and Lemma A.1.1. ▮

A2.4. *The Reduction Theorem*

We denote by V_n the subspace of $L^2([0, 1])$ generated by the e_k for $0 \leqslant k \leqslant n$. The Hilbert space splitting,

$$H = V_n \oplus V_n^\perp,$$

induces the following decomposition of X:

$$X = \mathrm{Seg}(V_n) \otimes \mathrm{Seg}(V_n^\perp).$$

Let u be a G-valued function on X, G being an Hilbert space, and let x correspond to (v, y) as above; then we shall set

$$u(v, y) = u_v(y).$$

LEMMA A2.4.1. *Let G be an Hilbert space, let u be a G-valued function on X; then one has*

$$\int_{V_n} \|\nabla_Y u_v\|^p_{L^p(Y;\, H\otimes G)}\, d\mu_{V_n}(v) = \int_X \left(\sum_{k=n+1}^{+\infty} \|D_{e_k} u(x)\|^2_G \right)^{p/2} d\mu(x).$$

Proof. That is a result of Cruzeiro [3, pp. 210–211].

COROLLARY A2.4.2. *Let $u = \nabla g$; then*

$$E(\|u_v\|^2_{V_n \otimes H}) \leqslant c_a n^{-2a},$$

where c_a is the constant depending only on a and M appearing in the hypothesis $(\mathscr{H})$.

Proof. From Lemma A2.4.1 we get

$$E\left(\sum_{k=n}^{+\infty} (D_{e_k} D_{e_s} g)^2\right) \leqslant n^{-2a} \|D_{\cdot,\cdot} g\|^2_{\mathscr{W}_a \times \mathscr{W}_a}. \quad \blacksquare$$

LEMMA A2.4.3. *Let Y be a Gaussian space, and let $\phi \in \mathscr{W}^{2,1}(Y)$; then we have*

$$\|\phi - E(\phi)\|_{L^2(Y)} \leqslant \|\nabla\phi\|_{L^2(Y;H)}.$$

Proof. Let $\mathbf{C}_n$ denote the nth Wiener chaos on Y (i.e., the closed subspace of $L^2(Y)$ spanned by the Hermite polynomials of degree n in the elements of an orthonormal basis of H contained in Y'). It is well known that $L^2(Y)$ is the orthogonal direct sum of $\mathbf{C}_n$; let $\phi = \sum_{n=0}^{+\infty} \phi_n$ correspond to this decomposition. We can write

$$\nabla\phi = \sum_{n=1}^{+\infty} \nabla\phi_n$$

with $\nabla\phi_n \in \mathbf{C}_{n-1} \otimes H$. By well-known facts concerning the Hermite polynomials, we have

$$\forall n \in \mathbf{N}, \qquad \|\nabla\phi_n\|^2_{L^2(Y,H)} = n\, \|\phi_n\|^2_{L^2(Y)}$$

which implies the theorem. $\blacksquare$

Remark. The same inequality holds in each L^p $(p > 1)$ up to a multiplicative constant that depends on p by using the Clark–Ocone representation formula (see the proof of Corollary 4.4 above).

THEOREM A2.4.4. *Let g be as in Theorem 1.1, and let $u = \nabla g$; then*

$$E(\|E^{V_n} g - g\|^2_H) \leqslant c_a n^{-2a}.$$

Proof. Applying Corollary A2.4.2 and Lemma A2.4.3, we get

$$E(\|E^{V_n} g - g\|^2_H) \leqslant E(\|u_v\|^2_{V_n \otimes H})$$
$$\leqslant c_a n^{-2a}. \quad \blacksquare$$

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