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Riemannian geometry of $\mathrm{Diff}(S^1)/S^1$

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Abstract

The main result of the paper is a computation of the Ricci curvature of $\mathrm{Diff}(S^1)/S^1$. Unlike earlier results on the subject, we do not use the Kähler structure symmetries to compute the Ricci curvature, but rather rely on classical finite-dimensional results of Nomizu et al. on Riemannian geometry of homogeneous spaces.

Keywords: Virasoro algebra; Group of diffeomorphisms; Ricci curvature

1. Introduction

Let $\mathrm{Diff}(S^1)$ be the group of orientation-preserving diffeomorphisms of the unit circle. This group is known as the Virasoro group in string theory. Then the quotient space $\mathrm{Diff}(S^1)/S^1$ describes those diffeomorphisms that fix a point on the circle. The geometry of this infinite-dimensional space has been of interest to physicists for a long time in connection with string theory and string field theory (e.g. [7,8,19]). A.A. Kirillov and D.V. Yur'ev in [13], and A.A. Kirillov in [12] showed that the homogeneous space $\mathrm{Diff}(S^1)/S^1$ admits a left-invariant complex structure and can be canonically identified with $\mathcal{M}$, a certain space of univalent functions on the unit disk in $\mathbb{C}$.

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Our motivation comes from stochastic analysis on infinite-dimensional manifolds. In a series of papers written by H. Airault, V. Bogachev, P. Malliavin, A. Thalmaier [2–5], the authors explored several possible approaches to the problem. For example, [5] is a first step in an attempt to construct a Brownian motion on $\mathcal{J}^{\infty}$, the space of smooth Jordan curves of the complex plane which can be described as the double quotient $\mathrm{SU}(1,1)\backslash\mathrm{Diff}(S^1)/\mathrm{SU}(1,1)$. The connection between $\mathcal{J}^{\infty}$ and $\mathrm{Diff}(S^1)$ is given by the conformal welding. A group Brownian motion in $\mathrm{Diff}(S^1)$ has been constructed by P. Malliavin in [14]. S. Fang in [10] described a Brownian motion in $\mathrm{Diff}(S^1)$ corresponding to the $H^{3/2}$-metric on $\mathrm{Diff}(S^1)$, and computed its modulus of continuity. A detailed study of the modulus of continuity was done by H. Airault and J. Ren in [6].

It is well known that the behavior of a Brownian motion on a curved space (finite- or infinite-dimensional) is related to the geometry of this space. In particular, the lower bound of the Ricci curvature controls the growth of the Brownian motion, so it seems that a better understanding of the geometry of $\mathrm{Diff}(S^1)/S^1$ might help in studying a Brownian motion on this homogeneous space.

The approach taken in [7,8,13,19] is to describe the space $\mathrm{Diff}(S^1)/S^1$ as an infinite-dimensional complex manifold with a Kähler metric, find the Riemann tensor corresponding to the Kähler structure, and finally compute the Ricci tensor. These computations use symmetries of the curvature tensor coming from the Kähler structure which are assumed to carry over from finite dimensions to infinite dimensions.

The aim of present article is to compute the Riemannian curvature tensor and the Ricci tensor for this space without appealing directly to the Kähler structure symmetries. Rather we follow the path taken by the first author in [11]. There the Riemannian curvature tensor and the Ricci tensor were computed for a class of infinite-dimensional groups by using finite-dimensional computations of the Riemannian curvature tensor by J. Milnor in [15] as definitions.

We will use the classical finite-dimensional results of K. Nomizu in [16] for homogeneous spaces as our definitions of basic geometric notions in this infinite-dimensional setting. The Virasoro algebra has a natural almost complex structure which has a zero torsion. This allows us to treat this structure as complex. Then using finite-dimensional methods we can find a covariant derivative $\tilde{\nabla}$ compatible with the complex structure. First we compute the original covariant derivative in the natural trigonometric basis of the Virasoro algebra. This is a technical result with respect to the main goal of this paper, and it is easy to check that the Ricci curvature for this covariant derivative is not bounded. Then we compute the Riemannian curvature tensor and the Ricci curvature corresponding to the covariant derivative $\tilde{\nabla}$ compatible with the complex structure. The main result of the paper is Theorem 4.11, which shows that the Ricci curvature for $\tilde{\nabla}$ is finite.

H. Airault in [1] computed the Ricci curvature of $\mathrm{Diff}(S^1)/\mathrm{SU}(1,1)$ using the classical finite-dimensional results of K. Nomizu in [16]. Besides the fact that we study a different homogeneous space, we show that even though the Ricci curvature tensor converges to a finite number, the covariant derivative $\tilde{\nabla}$ is not a Hilbert–Schmidt operator. This fact might pose difficulties if one attempts to define a Brownian motion corresponding to the Riemannian structure of the homogeneous space $\mathrm{Diff}(S^1)/S^1$. For further references to the works exploring the connections between stochastic analysis and Riemannian geometry in infinite dimensions, mostly in loop groups and their extensions such as current groups, path spaces and complex Wiener spaces see [9,11,17,18].

2. Virasoro algebra

In our exposition we follow [3].

Notation 2.1. We denote by $\mathrm{Diff}(S^1)$ the group of orientation preserving C^∞-diffeomorphisms of the unit circle, and by $\mathrm{diff}(S^1)$ its Lie algebra. The elements of $\mathrm{diff}(S^1)$ will be identified with the left-invariant vector fields $f(t)\frac{d}{dt}$, with the Lie bracket given by

$$[f,g]=f\dot{g}-\dot{f}g,\quad f,g\in\mathrm{diff}(S^1).$$

The Lie algebra $\mathrm{diff}(S^1)$ has a natural basis

$$f_k=\cos kt,\quad g_m=\sin mt,\quad k=0,1,2,\dots,\ m=1,2\dots. \tag{2.1}$$

The Lie bracket in this basis satisfies the following identities:

$$\begin{aligned}
[f_m,f_n]&=\frac{1}{2}\left((m-n)g_{m+n}+(m+n)\frac{m-n}{|m-n|}g_{|m-n|}\right),\quad m\neq n,\\
[g_m,g_n]&=\frac{1}{2}\left((n-m)g_{m+n}+(m+n)\frac{m-n}{|m-n|}g_{|m-n|}\right),\quad m\neq n,\\
[f_m,g_n]&=\frac{1}{2}\big((n-m)f_{m+n}+(m+n)f_{|m-n|}\big).
\end{aligned} \tag{2.2}$$

Definition 2.2. Suppose c,h are positive constants. Then the Virasoro algebra $\mathcal{V}_{c,h}$ is the vector space $\mathbb{R}\oplus\mathrm{diff}(S^1)$ with the Lie bracket given by

$$[a\kappa+f,b\kappa+g]_{\mathcal{V}_{c,h}}=\omega_{c,h}(f,g)\kappa+[f,g], \tag{2.3}$$

where k is the central element, and ω is the bilinear symmetric form

$$\omega_{c,h}(f,g)=\int_0^{2\pi}\left(\left(2h-\frac{c}{12}\right)f'(t)-\frac{c}{12}f^{(3)}(t)\right)g(t)\frac{dt}{2\pi}.$$

Remark 2.3. If $h=0$, $c=6$, then $\omega_{c,h}$ is the fundamental cocycle ω (see [3])

$$\omega(f,g)=-\int_0^{2\pi}\big(f'+f^{(3)}\big)g\frac{dt}{4\pi}.$$

Remark 2.4. A simple verification shows that $\omega_{c,h}$ satisfies the Jacobi identity, and therefore $\mathcal{V}_{c,h}$ with this bracket is indeed a Lie algebra.

Notation 2.5. By $\mathrm{diff}_0(S^1)$ we denote the space of functions having mean 0. This can be viewed as $\mathrm{diff}(S^1)/S^1$, where S^1 is identified with constant vector fields corresponding to rotations of S^1.

Then any element of $f \in \mathrm{diff}_0(S^1)$ can be written

$$f(t) = \sum_{k=1}^{\infty} (a_k f_k + b_k g_k).$$

There is a natural endomorphism J of $\mathrm{diff}_0(S^1)$ such that $J^2 = -I$, namely,

$$J(f)(t) = \sum_{k=1}^{\infty} (b_k f_k - a_k g_k). \tag{2.4}$$

Notation 2.6. For any $k \in \mathbb{Z}$

$$\theta_k = 2hk + \frac{c}{12}\big(k^3 - k\big).$$

Remark 2.7. Note that $\theta_{-k} = -\theta_k$, for any $k \in \mathbb{Z}$. Let $b_0 = 0$, then

$$\omega_{c,h}(f, Jf) = \int_0^{2\pi} \left(\left(2h - \frac{c}{12}\right) f'(t) - \frac{c}{12} f^{(3)}(t) \right) (Jf)(t) \frac{dt}{2\pi}$$
$$= \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \theta_k (b_k f_k - a_k g_k) \right) \left(\sum_{m=1}^{\infty} (b_m f_m - a_m g_m) \right) \frac{dt}{2\pi} = \frac{1}{2} \sum_{k=1}^{\infty} \theta_k \big(a_k^2 + b_k^2\big).$$

3. Riemannian geometry of $\mathrm{Diff}(S^1)/S^1$: definitions and preliminaries

We use the finite-dimensional results of [16] as our definitions with the following convention.

Notation 3.1. Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra equipped with an inner product $(\cdot,\cdot)$. We assume that $\mathfrak{g}$ is complete. Suppose that there are two subspaces, $\mathfrak{m}$ and $\mathfrak{h}$, of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ as vector spaces. We assume that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, and that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Note that $\mathfrak{m}$ is not assumed to be a Lie subalgebra of $\mathfrak{g}$.

In our setting $\mathfrak{g} = \mathrm{diff}(S^1)$, $\mathfrak{m} = \mathrm{diff}_0(S^1)$, $\mathfrak{h} = f_0\mathbb{R}$. Note that the assumptions in Notation 3.1 are satisfied since for any $n \in \mathbb{N}$

$$[f_0, f_n] = -n g_n \in \mathfrak{m}, \qquad [g_0, g_n] = n f_n \in \mathfrak{m}.$$

Let $G = \mathrm{Diff}(S^1)$ with the associated Lie algebra $\mathrm{diff}(S^1)$, the subgroup $H = S^1$ with the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, then $\mathfrak{m}$ is a Lie algebra naturally associated with the quotient $\mathrm{Diff}(S^1)/S^1$. For any $g \in \mathfrak{g}$ we denote by $g_{\mathfrak{m}}$ (respectively $g_{\mathfrak{h}}$) its $\mathfrak{m}$- (respectively $\mathfrak{h}$-) component, that is, $g = g_{\mathfrak{m}} + g_{\mathfrak{h}}$, $g_{\mathfrak{m}} \in \mathfrak{m}$, $g_{\mathfrak{h}} \in \mathfrak{h}$. By the assumptions in Notation 3.1 for any $h \in \mathfrak{h}$ the adjoint representation $\mathrm{ad}(h) = [h, \cdot] : \mathfrak{g} \to \mathfrak{g}$ maps $\mathfrak{m}$ into $\mathfrak{m}$. We will abuse notation by using $\mathrm{ad}(h)$ for the corresponding endomorphism of $\mathfrak{m}$. Define

$$B(f, g) = \omega_{c,h}(f, Jg) = \omega_{c,h}(g, Jf).$$

Proposition 3.2. $\langle f, g\rangle = B(f, g)$ *is an inner product on* $\mathrm{diff}_0(S^1)$.

Proof. It follows from properties of $\omega_{c,h}$ as stated in Remark 2.7. In particular, for any $f \in \mathrm{diff}_0(S^1)$

$$B(f, f) = \frac{1}{2}\sum_{k=1}^{\infty}\theta_k\big(a_k(f)^2 + b_k(f)^2\big). \qquad \square$$

Notation 3.3. Let α be an affine connection defined by

$$\alpha(x, y) = \frac{1}{2}[x, y]_{\mathfrak{m}} + U(x, y),$$

where U is defined by

$$B\big(U(x, y), z\big) = \frac{1}{2}\big(B\big([x, z]_{\mathfrak{m}}, y\big) + B\big(x, [y, z]_{\mathfrak{m}}\big)\big)$$

for any $x, y, z \in \mathfrak{m}$. The relation between the covariant derivative $\nabla : \mathfrak{m} \to \mathrm{End}(\mathfrak{m})$ and α is given by

$$\nabla_x y = \alpha(x, y) = \frac{1}{2}[x, y]_{\mathfrak{m}} + U(x, y).$$

The covariant derivative ∇ is not our main interest. In Definition 4.4 we will introduce another covariant derivative, $\tilde{\nabla}$, which corresponds to the Kähler structure on $\mathrm{Diff}(S^1)/S^1$.

Lemma 3.4. *Let* $\lambda_{m,n} = \frac{(2n+m)\theta_m}{2\theta_{m+n}}$ *for any* $n, m \in \mathbb{Z}$. *Then*

$$\lambda_{m,n} = \lambda_{n,m} + \frac{m-n}{2}.$$

Proof.

$$\begin{aligned}\lambda_{m,n} - \lambda_{n,m} &= \frac{(2n+m)\theta_m - (2m+n)\theta_n}{\theta_{m+n}}\\ &= \frac{2hm(2n+m) + \frac{c}{12}(m^3-m)(2n+m) - 2hn(2m+n) + \frac{c}{12}(n^3-n)(2m+n)}{\theta_{m+n}}\\ &= \frac{2h(m-n)(m+n) + \frac{c}{12}(m-n)((m+n)^3-(m+n))}{\theta_{m+n}} = \frac{m-n}{2}. \qquad \square\end{aligned}$$

Proposition 3.5.

$$\begin{aligned}U(f_m, f_n) &= \frac{1}{2}\big[(\lambda_{n,m} + \lambda_{m,n})g_{m+n} + (\lambda_{n,-m} - \lambda_{-m,n})g_{n-m}\big]\\ &= \frac{1}{2}\left[(\lambda_{n,m} + \lambda_{m,n})g_{m+n} + \frac{m+n}{2}g_{n-m}\right], \quad n > m,\end{aligned}$$

$$
\begin{aligned}
U(f_m, f_n) &= \frac{1}{2}\big[(\lambda_{n,m} + \lambda_{m,n})g_{m+n} + (\lambda_{n,-m} - \lambda_{-m,n})g_{m-n}\big] \\
&= \frac{1}{2}\left[(\lambda_{n,m} + \lambda_{m,n})g_{m+n} + \frac{m+n}{2}g_{m-n}\right], \quad m > n, \\
U(f_n, f_n) &= \lambda_{n,n}g_{2n} = -U(g_n, g_n), \\
U(f_m, g_n) &= \frac{1}{2}\big[(\lambda_{-m,n} - \lambda_{n,-m})f_{n-m} - (\lambda_{n,m} + \lambda_{m,n})f_{m+n}\big] \\
&= \frac{1}{2}\left[-\frac{m+n}{2}f_{n-m} - (\lambda_{n,m} + \lambda_{m,n})f_{m+n}\right], \quad n > m, \\
U(f_m, g_n) &= \frac{1}{2}\big[(\lambda_{n,-m} - \lambda_{-m,n})f_{m-n} - (\lambda_{n,m} + \lambda_{m,n})f_{m+n}\big] \\
&= \frac{1}{2}\left[\frac{m+n}{2}f_{m-n} - (\lambda_{n,m} + \lambda_{m,n})f_{m+n}\right], \quad m > n, \\
U(f_n, g_n) &= -\lambda_{n,n}f_{2n}, \\
U(g_m, g_n) &= \frac{1}{2}\big[(\lambda_{n,-m} - \lambda_{-m,n})g_{n-m} - (\lambda_{n,m} + \lambda_{m,n})g_{m+n}\big] \\
&= \frac{1}{2}\left[\frac{m+n}{2}g_{n-m} - (\lambda_{n,m} + \lambda_{m,n})g_{m+n}\right], \quad n > m, \\
U(g_m, g_n) &= \frac{1}{2}\big[(\lambda_{n,-m} - \lambda_{-m,n})g_{m-n} - (\lambda_{n,m} + \lambda_{m,n})g_{m+n}\big] \\
&= \frac{1}{2}\left[\frac{m+n}{2}g_{m-n} - (\lambda_{n,m} + \lambda_{m,n})g_{m+n}\right], \quad m > n.
\end{aligned}
$$

Proof. First,

$$
\begin{aligned}
\omega_{c,h}(f_m, f_n) &= -\int_0^{2\pi} \theta_m g_m f_n \frac{dt}{2\pi} = 0, \\
\omega_{c,h}(f_m, g_n) &= -\int_0^{2\pi} \theta_m g_m g_n \frac{dt}{2\pi} = -\frac{1}{2}\theta_m \delta_{m,n}, \\
\omega_{c,h}(g_m, f_n) &= \int_0^{2\pi} \theta_m f_m f_n \frac{dt}{2\pi} = \frac{1}{2}\theta_m \delta_{m,n}, \\
\omega_{c,h}(g_m, g_n) &= \int_0^{2\pi} \theta_m f_m g_n \frac{dt}{2\pi} = 0,
\end{aligned}
$$

and therefore

$$
B(f_m, f_n) = B(g_m, g_n) = \frac{\theta_m}{2}\delta_{m,n}, \qquad B(f_m, g_n) = B(g_m, f_n) = 0.
$$

By the commutation relations (2.2)

$$\begin{aligned}
&B\big(U(f_m, f_n), f_k\big)\\
&\quad = \frac{1}{2}\big(B\big([f_m, f_k]_{\mathfrak{m}}, f_n\big) + B\big(f_m, [f_n, f_k]_{\mathfrak{m}}\big)\big)\\
&\quad = \frac{1}{4}\left[B\left((m-k)g_{m+k} + \frac{m^2-k^2}{|m-k|}g_{|m-k|}, f_n\right) + B\left(f_m, (n-k)g_{n+k} + \frac{n^2-k^2}{|n-k|}g_{|n-k|}\right)\right]\\
&\quad = 0.
\end{aligned}$$

$$\begin{aligned}
B\big(U(f_m, f_n), g_k\big) &= \frac{1}{2}\big(B\big([f_m, g_k]_{\mathfrak{m}}, f_n\big) + B\big(f_m, [f_n, g_k]_{\mathfrak{m}}\big)\big)\\
&= \frac{1}{4}\big((k-m)B(f_{m+k}, f_n) + (m+k)B(f_{|m-k|}, f_n)\\
&\quad + (k-n)B(f_m, f_{n+k}) + (n+k)B(f_m, f_{|n-k|})\big)\\
&= \frac{1}{8}\big((k-m)\theta_n\delta_{m+k,n} + (m+k)\theta_n\delta_{|m-k|,n}\\
&\quad + (k-n)\theta_m\delta_{m,n+k} + (n+k)\theta_m\delta_{|n-k|,m}\big)\\
&= \frac{1}{8}\big((n-2m)\theta_n\delta_{k,n-m} + (2m+n)\theta_n\delta_{k,m+n} + (2m-n)\theta_n\delta_{k,m-n}\\
&\quad + (m-2n)\theta_m\delta_{k,m-n} + (2n+m)\theta_m\delta_{k,m+n} + (2n-m)\theta_m\delta_{k,n-m}\big)\\
&= \frac{1}{8}\big(\big((n-2m)\theta_n + (2n-m)\theta_m\big)\delta_{k,n-m} + \big((2m+n)\theta_n\\
&\quad + (2n+m)\theta_m\big)\delta_{k,m+n} + \big((2m-n)\theta_n + (m-2n)\theta_m\big)\delta_{k,m-n}\big)
\end{aligned}$$

with the assumption that all the indices are positive. Thus

$$\begin{aligned}
U(f_m, f_n) &= \frac{(2m+n)\theta_n + (2n+m)\theta_m}{4\theta_{m+n}}g_{m+n} + \frac{(n-2m)\theta_n + (2n-m)\theta_m}{4\theta_{n-m}}g_{n-m}, \quad n > m,\\
U(f_m, f_n) &= \frac{(2m+n)\theta_n + (2n+m)\theta_m}{4\theta_{m+n}}g_{m+n} + \frac{(2m-n)\theta_n + (m-2n)\theta_m}{4\theta_{m-n}}g_{m-n}, \quad m > n,\\
U(f_n, f_n) &= \frac{3n\theta_n}{2\theta_{2n}}g_{2n}.
\end{aligned}$$

$$\begin{aligned}
&B\big(U(f_m, g_n), f_k\big)\\
&\quad = \frac{1}{2}\big(B\big([f_m, f_k]_{\mathfrak{m}}, g_n\big) + B\big(f_m, [g_n, f_k]_{\mathfrak{m}}\big)\big)\\
&\quad = \frac{1}{4}\left(B\left((m-k)g_{m+k} + \frac{m^2-k^2}{|m-k|}g_{|m-k|}, g_n\right) - B\big(f_m, (n-k)f_{k+n} + (k+n)f_{|k-n|}\big)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left((m-k)\theta_n\delta_{k,n-m} + \frac{m^2-k^2}{|m-k|}\theta_n\delta_{n,|m-k|} - (n-k)\theta_m\delta_{k,m-n} - (k+n)\theta_m\delta_{m,|k-n|}\right) \\
&= \frac{1}{8}\big(\big((2m-n)\theta_n - (2n-m)\theta_m\big)\delta_{k,n-m} + \big(-(2n-m)\theta_m + (2m-n)\theta_n\big)\delta_{k,m-n} \\
&\quad + \big(-(2m+n)\theta_n - (m+2n)\theta_m\big)\delta_{k,m+n}\big).
\end{aligned}
$$

$$
\begin{aligned}
&B\big(U(f_m,g_n),g_k\big) = \frac{1}{2}\big(B\big([f_m,g_k]_{\mathfrak{m}},g_n\big) + B\big(f_m,[g_n,g_k]_{\mathfrak{m}}\big)\big) \\
&\quad = \frac{1}{4}\left(B\big((k-m)f_{m+k} + (m+k)f_{|m-k|},g_n\big) - B\left(f_m,(n-k)g_{k+n} + \frac{k^2-n^2}{|k-n|}g_{|k-n|}\right)\right) = 0.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&U(f_m,g_n) = \frac{1}{4}\left[\frac{(2m-n)\theta_n - (2n-m)\theta_m}{\theta_{n-m}}f_{n-m} - \frac{(2m+n)\theta_n + (m+2n)\theta_m}{\theta_{m+n}}f_{m+n}\right], \quad n>m, \\
&U(f_m,g_n) = \frac{1}{4}\left[\frac{(2m-n)\theta_n - (2n-m)\theta_m}{\theta_{m-n}}f_{m-n} - \frac{(2m+n)\theta_n + (m+2n)\theta_m}{\theta_{m+n}}f_{m+n}\right], \quad m>n, \\
&U(f_n,g_n) = -\frac{3n\theta_n}{2\theta_{2n}}f_{2n}, \quad m=n.
\end{aligned}
$$

$$
\begin{aligned}
&B\big(U(g_m,g_n),f_k\big) \\
&\quad = \frac{1}{2}\big(B\big([g_m,f_k]_{\mathfrak{m}},g_n\big) + B\big(g_m,[g_n,f_k]_{\mathfrak{m}}\big)\big) \\
&\quad = \frac{1}{4}\big(-B\big((m-k)f_{k+m} + (k+m)f_{|k-m|},g_n\big) - B\big(g_m,(n-k)f_{k+n} + (k+n)f_{|k-n|}\big)\big) \\
&\quad = \frac{1}{4}\big((k-m)B(f_{k+m},g_n) - (k+m)B(f_{|k-m|},g_n) \\
&\qquad - (n-k)B(g_m,f_{k+n}) - (k+n)B(g_m,f_{|k-n|})\big) = 0.
\end{aligned}
$$

$$
\begin{aligned}
&B\big(U(g_m,g_n),g_k\big) \\
&\quad = \frac{1}{2}\big(B\big([g_m,g_k]_{\mathfrak{m}},g_n\big) + B\big(g_m,[g_n,g_k]_{\mathfrak{m}}\big)\big) \\
&\quad = \frac{1}{4}\left(B\left((k-m)g_{m+k} + \frac{m^2-k^2}{|m-k|}g_{|m-k|},g_n\right) + B\left(g_m,(k-n)g_{n+k} + \frac{n^2-k^2}{|n-k|}g_{|n-k|}\right)\right) \\
&\quad = \frac{1}{8}\left((k-m)\theta_n\delta_{k,n-m} + \frac{m^2-k^2}{|m-k|}\theta_n\delta_{n,|m-k|} + (k-n)\theta_m\delta_{k,m-n} + \frac{n^2-k^2}{|n-k|}\theta_m\delta_{m,|n-k|}\right)
\end{aligned}
$$

$$= \frac{1}{8}\Big(\big((n-2m)\theta_n + (2n-m)\theta_m\big)\delta_{k,n-m} + \big((2m-n)\theta_n + (m-2n)\theta_m\big)\delta_{k,m-n} \\ - \big((2m+n)\theta_n + (2m+n)\theta_m\big)\delta_{k,m+n}\Big).$$

Thus

$$U(g_m, g_n) = \frac{1}{4}\left[\frac{(n-2m)\theta_n + (2n-m)\theta_m}{\theta_{n-m}} g_{n-m} - \frac{(2m+n)\theta_n + (2n+m)\theta_m}{\theta_{m+n}} g_{m+n}\right], \quad n > m,$$
$$U(g_m, g_n) = \frac{1}{4}\left[\frac{(2m-n)\theta_n + (m-2n)\theta_m}{\theta_{m-n}} g_{m-n} - \frac{(2m+n)\theta_n + (2n+m)\theta_m}{\theta_{m+n}} g_{m+n}\right], \quad m > n,$$
$$U(g_n, g_n) = -\frac{3n\theta_n}{2\theta_{2n}} g_{2n}. \qquad \square$$

Proposition 3.6.

$$\begin{aligned}
\nabla_{f_m} f_n &= \lambda_{m,n} g_{m+n}, \quad n > m,\\
\nabla_{f_m} f_n &= \lambda_{m,n} g_{m+n} + \frac{m+n}{2} g_{m-n}, \quad n < m,\\
\nabla_{f_n} f_n &= \lambda_{n,n} g_{2n} = -\nabla_{g_n} g_n,\\
\nabla_{f_m} g_n &= -\lambda_{m,n} f_{m+n}, \quad n > m,\\
\nabla_{f_m} g_n &= -\lambda_{m,n} f_{m+n} + \frac{m+n}{2} f_{m-n}, \quad n < m,\\
\nabla_{f_n} g_n &= -\lambda_{n,n} f_{2n} = \nabla_{g_n} f_n,\\
\nabla_{g_n} f_m &= -\lambda_{n,m} f_{m+n} - \frac{m+n}{2} f_{n-m}, \quad n > m,\\
\nabla_{g_n} f_m &= -\lambda_{n,m} f_{m+n}, \quad n < m,\\
\nabla_{g_m} g_n &= -\lambda_{m,n} g_{m+n}, \quad n > m,\\
\nabla_{g_m} g_n &= \frac{m+n}{2} g_{m-n} - \lambda_{m,n} g_{m+n}, \quad n < m.
\end{aligned}$$

Proof. The main ingredients of the proof are the commutation relations (2.2), Proposition 3.5 and Lemma 3.4. First, note that

$$\nabla_{f_m} f_n = \frac{1}{2}[f_m, f_n]_{\mathfrak{m}} + U(f_m, f_n),$$

and therefore if $n > m$, then

$$\begin{aligned}
\nabla_{f_m} f_n &= \frac{1}{4}\big((m-n)g_{m+n} - (m+n)g_{n-m}\big) + \frac{1}{2}\left[(\lambda_{n,m} + \lambda_{m,n})g_{m+n} + \frac{m+n}{2} g_{n-m}\right]\\
&= \frac{1}{4}\big[(m-n) + 2(\lambda_{n,m} + \lambda_{m,n})\big]g_{m+n} = \frac{1}{4}\left[(m-n) + 2\left(\frac{n-m}{2} + 2\lambda_{m,n}\right)\right]g_{m+n}\\
&= \lambda_{m,n} g_{m+n};
\end{aligned}$$

if $m > n$, then

$$\begin{aligned}
\nabla_{f_m} f_n &= \frac{1}{4}\big((m-n)g_{m+n} + (m+n)g_{m-n}\big) + \frac{1}{2}\left[(\lambda_{n,m} + \lambda_{m,n})g_{m+n} + \frac{m+n}{2}g_{m-n}\right] \\
&= \frac{m+n}{2}g_{m-n} + \lambda_{m,n}g_{m+n}, \\
\nabla_{f_n} f_n &= U(f_n, f_n) = \lambda_{n,n}g_{2n}.
\end{aligned}$$

Similarly

$$\nabla_{f_m} g_n = \frac{1}{2}[f_m, g_n]_{\mathfrak{m}} + U(f_m, g_n),$$

and so for $n > m$

$$\begin{aligned}
\nabla_{f_m} g_n &= \frac{1}{4}\big((n-m)f_{m+n} + (m+n)f_{n-m}\big) + \frac{1}{2}\left[-\frac{m+n}{2}f_{n-m} - (\lambda_{n,m} + \lambda_{m,n})f_{m+n}\right] \\
&= -\lambda_{m,n}f_{m+n};
\end{aligned}$$

and for $m > n$

$$\begin{aligned}
\nabla_{f_m} g_n &= \frac{1}{4}\big((n-m)f_{m+n} + (m+n)f_{m-n}\big) + \frac{1}{2}\left[\frac{m+n}{2}f_{m-n} - (\lambda_{n,m} + \lambda_{m,n})f_{m+n}\right] \\
&= -\lambda_{m,n}f_{m+n} + \frac{m+n}{2}f_{m-n}, \\
\nabla_{f_n} g_n &= \frac{1}{2}[f_n, g_n]_{\mathfrak{m}} + U(f_n, g_n) = -\lambda_{n,n}f_{2n}.
\end{aligned}$$

Third,

$$\nabla_{g_n} f_m = \frac{1}{2}[g_n, f_m]_{\mathfrak{m}} + U(g_n, f_m),$$

and therefore for $n > m$

$$\begin{aligned}
\nabla_{g_n} f_m &= -\frac{1}{4}\big((n-m)f_{m+n} + (m+n)f_{n-m}\big) - \frac{1}{2}\left[\frac{m+n}{2}f_{n-m} + (\lambda_{n,m} + \lambda_{m,n})f_{m+n}\right] \\
&= -\frac{1}{4}(n-m)f_{m+n} - \frac{m+n}{2}f_{n-m} - \frac{\lambda_{n,m} + \lambda_{m,n}}{2}f_{m+n} \\
&= -\frac{1}{4}(n-m)f_{m+n} - \frac{m+n}{2}f_{n-m} - \frac{\frac{n-m}{2} + 2\lambda_{m,n}}{2}f_{m+n} \\
&= -\frac{m+n}{2}f_{n-m} - \frac{n-m}{2}f_{m+n} - \lambda_{m,n}f_{m+n} = \nabla_{f_m} g_n + [g_n, f_m];
\end{aligned}$$

for $m > n$

$$\begin{aligned}\nabla_{g_n} f_m &= -\frac{1}{4}\big((n-m) f_{m+n} + (m+n) f_{m-n}\big) + \frac{1}{2}\left[\frac{m+n}{2} f_{m-n} - (\lambda_{n,m} + \lambda_{m,n}) f_{m+n}\right] \\ &= -\frac{n-m}{2} f_{m+n} - \lambda_{m,n} f_{m+n} = \nabla_{f_m} g_n + [g_n, f_m], \\ \nabla_{f_n} g_n &= \frac{1}{2}[f_n, g_n]_{\mathfrak{m}} + U(f_n, g_n) = -\lambda_{n,n} f_{2n}.\end{aligned}$$

Finally,

$$\nabla_{g_m} g_n = \frac{1}{2}[g_m, g_n]_{\mathfrak{m}} + U(g_m, g_n),$$

and so if $n > m$, then

$$\begin{aligned}\nabla_{g_m} g_n &= \frac{1}{4}\big((n-m) g_{m+n} - (m+n) g_{n-m}\big) + \frac{1}{2}\left[\frac{m+n}{2} g_{n-m} - (\lambda_{n,m} + \lambda_{m,n}) g_{m+n}\right] \\ &= \frac{n-m}{4} g_{m+n} - \frac{1}{2}(\lambda_{n,m} + \lambda_{m,n}) g_{m+n} = -\lambda_{m,n} g_{m+n},\end{aligned}$$

and if $m > n$, then

$$\begin{aligned}\nabla_{g_m} g_n &= \frac{1}{4}\big((n-m) g_{m+n} + (m+n) g_{m-n}\big) + \frac{1}{2}\left[\frac{m+n}{2} g_{m-n} - (\lambda_{n,m} + \lambda_{m,n}) g_{m+n}\right] \\ &= \frac{n-m}{4} g_{m+n} + \frac{m+n}{2} g_{m-n} - \frac{1}{2}\left(\frac{n-m}{2} + 2\lambda_{m,n}\right) g_{m+n} = \frac{m+n}{2} g_{m-n} - \lambda_{m,n} g_{m+n}, \\ \nabla_{g_n} g_n &= U(g_n, g_n) = -\lambda_{n,n} g_{2n}. \qquad \square\end{aligned}$$

4. Diff$(S^1)/S^1$ as a Kähler manifold

The goal of this section is to introduce an almost complex structure on $\mathrm{diff}_0(S^1)$, and then show that it is actually complex for an appropriately chosen connection. Recall that $J : \mathrm{diff}_0(S^1) \to \mathrm{diff}_0(S^1)$ is an endomorphism defined by (2.4), or equivalently, in the basis $\{f_m, g_n\}$, $m, n = 1, \ldots$ by

$$J f_m = -g_m, \qquad J g_n = f_n.$$

The next result is an analogue of the Newlander–Nirenberg theorem in our setting. This statement also appears in [1, p. 255] as was communicated to us by H. Airault after we submitted the present paper.

Proposition 4.1. *The Nijenhuis tensor N (the torsion of the almost complex structure J) defined by*

$$N(X, Y) = 2\big([JX, JY]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - J[X, JY]_{\mathfrak{m}} - J[JX, Y]_{\mathfrak{m}}\big)$$

vanishes on $\mathfrak{m} = \mathrm{diff}_0(S^1)$. Therefore J is a complex structure.

Proof. If $m \neq n$, then by (2.2)

$$\begin{aligned} N(f_m, f_n) &= 2\big([Jf_m, Jf_n]_{\mathfrak{m}} - [f_m, f_n]_{\mathfrak{m}} - J[f_m, Jf_n]_{\mathfrak{m}} - J[Jf_m, f_n]_{\mathfrak{m}}\big) \\ &= 2\big([g_m, g_n]_{\mathfrak{m}} - [f_m, f_n]_{\mathfrak{m}} + J[f_m, g_n]_{\mathfrak{m}} + J[g_m, f_n]_{\mathfrak{m}}\big) \\ &= 2\big((n-m)g_{m+n} + (n-m)Jf_{m+n}\big) = 0. \end{aligned}$$

Then we can use

$$\begin{aligned} N(JX, Y) &= 2\big(-[X, JY]_{\mathfrak{m}} + [JX, J(JY)]_{\mathfrak{m}} - J[JX, JY]_{\mathfrak{m}} - J[X, J(JY)]_{\mathfrak{m}}\big) \\ &= N(X, JY), \\ N(JX, Y) &= 2\big(-[X, JY]_{\mathfrak{m}} + [JX, J(JY)]_{\mathfrak{m}} - J[JX, JY]_{\mathfrak{m}} - J[X, J(JY)]_{\mathfrak{m}}\big) \\ &= -2J\big(-J[X, JY]_{\mathfrak{m}} - J[JX, Y]_{\mathfrak{m}} + [JX, JY]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}}\big) = -JN(X, Y) \end{aligned}$$

to see that

$$N(f_m, f_n) = N(Jg_m, f_n) = N(g_m, Jf_n) = -N(g_m, g_n) = 0,$$

and

$$N(f_m, g_n) = -N(g_n, f_m) = N(Jg_m, g_n) = -JN(g_m, g_n) = 0. \qquad \square$$

Lemma 4.2. *J is a complex structure on $\mathfrak{m} = \mathrm{diff}_0(S^1)$ with the covariant derivative*

$$\begin{aligned} &(\nabla_{f_m} J)(f_n) = (\nabla_{f_m} J)(g_n) = (\nabla_{g_m} J)(f_n) = (\nabla_{g_m} J)(g_n) = 0, \quad n \geqslant m, \\ &(\nabla_{f_m} J)(f_n) = (\nabla_{g_m} J)(g_n) = -(m+n) f_{m-n}, \quad n < m, \\ &(\nabla_{f_m} J)(g_n) = -(\nabla_{g_m} J)(f_n) = (m+n) g_{m-n}, \quad n < m. \end{aligned}$$

Proof. We will use the fact that

$$(\nabla_x J)(y) = \nabla_x(Jy) - J(\nabla_x y).$$

If $n > m$, then

$$\begin{aligned} (\nabla_{f_m} J)(f_n) &= -\nabla_{f_m} g_n - J(\nabla_{f_m} f_n) \\ &= \lambda_{m,n} f_{m+n} - \lambda_{m,n} J(g_{m+n}) = \lambda_{m,n} f_{m+n} - \lambda_{m,n} f_{m+n} = 0. \end{aligned}$$

If $n < m$, then

$$\begin{aligned} (\nabla_{f_m} J)(f_n) &= -\nabla_{f_m} g_n - J(\nabla_{f_m} f_n) \\ &= \lambda_{m,n} f_{m+n} - \frac{m+n}{2} f_{m-n} - \lambda_{m,n} J(g_{m+n}) - \frac{m+n}{2} J(g_{m-n}) \\ &= \lambda_{m,n} f_{m+n} - \frac{m+n}{2} f_{m-n} - \lambda_{m,n} f_{m+n} - \frac{m+n}{2} f_{m-n} = -(m+n) f_{m-n}, \end{aligned}$$

$$(\nabla_{f_n} J)(f_n) = \lambda_{n,n} f_{2n} - \lambda_{n,n} J(g_{2n}) = \lambda_{n,n} f_{2n} - \lambda_{n,n} f_{2n} = 0.$$

If $n > m$, then

$$(\nabla_{f_m} J)(g_n) = \lambda_{m,n} g_{m+n} + \lambda_{m,n} J(f_{m+n}) = \lambda_{m,n} g_{m+n} - \lambda_{m,n} g_{m+n} = 0.$$

If $n < m$, then

$$\begin{aligned}(\nabla_{f_m} J)(g_n) &= \nabla_{f_m} f_n - J(\nabla_{f_m} g_n) \\ &= \lambda_{m,n} g_{m+n} + \frac{m+n}{2} g_{m-n} + \lambda_{m,n} J(f_{m+n}) - \frac{m+n}{2} J(f_{m-n}) \\ &= \lambda_{m,n} g_{m+n} + \frac{m+n}{2} g_{m-n} - \lambda_{m,n} g_{m+n} + \frac{m+n}{2} g_{m-n} = (m+n) g_{m-n}, \\ (\nabla_{f_n} J)(g_n) &= \lambda_{n,n} g_{2n} + \lambda_{n,n} J(f_{2n}) = 0.\end{aligned}$$

If $n > m$, then

$$\begin{aligned}(\nabla_{g_m} J)(f_n) &= -\nabla_{g_m} g_n - J(\nabla_{g_m} f_n) \\ &= \lambda_{m,n} g_{m+n} + \lambda_{m,n} J(f_{m+n}) = \lambda_{m,n} g_{m+n} - \lambda_{m,n} g_{m+n} = 0.\end{aligned}$$

If $n < m$, then

$$\begin{aligned}(\nabla_{g_m} J)(f_n) &= -\nabla_{g_m} g_n - J(\nabla_{g_m} f_n) \\ &= -\frac{m+n}{2} g_{m-n} + \lambda_{m,n} g_{m+n} + \lambda_{m,n} J(f_{n+m}) + \frac{m+n}{2} J(f_{m-n}) \\ &= -\frac{m+n}{2} g_{m-n} - \frac{m+n}{2} g_{m-n} + \lambda_{m,n} g_{m+n} - \lambda_{m,n} g_{m+n} = -(m+n) g_{m-n}, \\ (\nabla_{g_n} J)(f_n) &= \lambda_{n,n} g_{2n} + \lambda_{n,n} J(f_{2n}) = \lambda_{n,n} g_{2n} - \lambda_{n,n} g_{2n} = 0.\end{aligned}$$

If $n > m$, then

$$\begin{aligned}(\nabla_{g_m} J)(g_n) &= \nabla_{g_m} f_n - J(\nabla_{g_m} g_n) \\ &= -\lambda_{m,n} f_{m+n} + \lambda_{m,n} J(g_{m+n}) = -\lambda_{m,n} f_{m+n} + \lambda_{m,n} f_{m+n} = 0.\end{aligned}$$

If $n < m$, then

$$\begin{aligned}(\nabla_{g_m} J)(g_n) &= \nabla_{g_m} f_n - J(\nabla_{g_m} g_n) \\ &= -\lambda_{m,n} f_{m+n} - \frac{m+n}{2} f_{m-n} - \frac{m+n}{2} J(g_{m-n}) + \lambda_{m,n} J(g_{m+n}) \\ &= -\lambda_{m,n} f_{m+n} - \frac{m+n}{2} f_{m-n} - \frac{m+n}{2} f_{m-n} + \lambda_{m,n} f_{m+n} = -(m+n) f_{m-n}, \\ (\nabla_{g_n} J)(g_n) &= -\lambda_{n,n} f_{2n} + \lambda_{n,n} J(g_{2n}) = -\lambda_{n,n} f_{2n} + \lambda_{n,n} f_{2n} = 0. \qquad \square\end{aligned}$$

Lemma 4.3. *Let Q be the tensor field of type $(1,2)$ defined by*

$$4Q(x,y) = (\nabla_{Jy}J)x + J\big((\nabla_y J)x\big) + 2J\big((\nabla_x J)y\big).$$

Then

$$\begin{aligned}
&Q(f_m, f_n) = Q(g_m, g_n) = \frac{m+n}{2} g_{|n-m|},\\
&Q(f_m, g_n) = -\frac{m+n}{2} f_{n-m}, \quad n > m,\\
&Q(f_m, g_n) = \frac{m+n}{2} f_{m-n}, \quad n < m,\\
&Q(g_m, f_n) = \frac{m+n}{2} f_{n-m}, \quad n > m,\\
&Q(g_m, f_n) = -\frac{m+n}{2} f_{m-n}, \quad n < m,\\
&Q(f_n, f_n) = Q(f_n, g_n) = Q(g_n, f_n) = Q(g_n, g_n) = 0.
\end{aligned}$$

Proof. First, note that

$$4Q(f_m, f_n) = (\nabla_{Jf_n}J)f_m + J\big((\nabla_{f_n}J)f_m\big) + 2J\big((\nabla_{f_m}J)f_n\big),$$

and therefore

$$\begin{aligned}
&4Q(f_m, f_n) = (m+n)g_{n-m} - (m+n)J(f_{n-m}) = 2(m+n)g_{n-m}, \quad n > m,\\
&4Q(f_m, f_n) = 2(m+n)g_{m-n}, \quad n < m,\\
&4Q(f_n, f_n) = -(\nabla_{g_n}J)f_n + 3J\big((\nabla_{f_n}J)f_n\big) = 0, \quad n = m.
\end{aligned}$$

Second,

$$4Q(f_m, g_n) = (\nabla_{Jg_n}J)f_m + J\big((\nabla_{g_n}J)f_m\big) + 2J\big((\nabla_{f_m}J)g_n\big),$$

and therefore

$$\begin{aligned}
&4Q(f_m, g_n) = -(m+n)f_{n-m} - (m+n)J(g_{n-m}) = -2(m+n)f_{n-m}, \quad n > m,\\
&4Q(f_m, g_n) = 2(m+n)J(g_{m-n}) = 2(m+n)f_{m-n}, \quad n < m,\\
&4Q(f_n, g_n) = (\nabla_{Jg_n}J)f_n + J\big((\nabla_{g_n}J)f_n\big) + 2J\big((\nabla_{f_n}J)g_n\big) = 0.
\end{aligned}$$

Third, note that

$$4Q(g_m, f_n) = (\nabla_{Jf_n}J)g_m + J\big((\nabla_{f_n}J)g_m\big) + 2J\big((\nabla_{g_m}J)f_n\big),$$

and therefore

$$4Q(g_m, f_n) = (m+n) f_{n-m} + (m+n) J(g_{n-m}) = 2(m+n) f_{n-m}, \quad n > m,$$
$$4Q(g_m, f_n) = -(m+n)2J(g_{m-n}) = -(m+n)2 f_{m-n}, \quad n < m,$$
$$4Q(g_n, f_n) = -(\nabla_{g_n} J) g_n + J\big((\nabla_{f_n} J) g_n\big) + 2J\big((\nabla_{g_n} J) f_n\big) = 0.$$

Finally,

$$4Q(g_m, g_n) = (\nabla_{J g_n} J) g_m + J\big((\nabla_{g_n} J) g_m\big) + 2J\big((\nabla_{g_m} J) g_n\big),$$

and so

$$4Q(g_m, g_n) = (m+n) g_{n-m} - (m+n) J(f_{n-m}) = 2(m+n) g_{n-m}, \quad n > m,$$
$$4Q(g_m, g_n) = -2(m+n) J(f_{m-n}) = 2(m+n) g_{m-n}, \quad n < m,$$
$$4Q(g_n, g_n) = (\nabla_{f_n} J) g_n + 3J\big((\nabla_{g_n} J) g_n\big) = 0. \qquad \square$$

Definition 4.4. The new covariant derivative is defined by

$$\tilde{\nabla}_x y = \nabla_x y - Q(x, y).$$

Then combining the results of Proposition 3.6 and Lemma 4.3 we see that

$$\tilde{\nabla}_{f_m} f_n = \nabla_{f_m} f_n - Q(f_m, f_n) = \lambda_{m,n} g_{m+n} - \frac{m+n}{2} g_{n-m}, \quad n > m,$$
$$\tilde{\nabla}_{f_m} f_n = \nabla_{f_m} f_n - Q(f_m, f_n) = \lambda_{m,n} g_{m+n}, \quad n < m,$$
$$\tilde{\nabla}_{f_n} f_n = \nabla_{f_n} f_n - Q(f_n, f_n) = \lambda_{n,n} g_{2n},$$
$$\tilde{\nabla}_{f_m} g_n = \nabla_{f_m} g_n - Q(f_m, g_n) = \frac{m+n}{2} f_{n-m} - \lambda_{m,n} f_{m+n}, \quad n > m,$$
$$\tilde{\nabla}_{f_m} g_n = \nabla_{f_m} g_n - Q(f_m, g_n) = -\lambda_{m,n} f_{m+n}, \quad n < m,$$
$$\tilde{\nabla}_{f_n} g_n = \nabla_{f_n} g_n - Q(f_n, g_n) = -\lambda_{n,n} f_{2n},$$
$$\tilde{\nabla}_{g_n} f_m = \nabla_{g_n} f_m - Q(g_n, f_m) = -\lambda_{n,m} f_{m+n}, \quad n > m,$$
$$\tilde{\nabla}_{g_n} f_m = \nabla_{g_n} f_m - Q(g_n, f_m) = -\lambda_{n,m} f_{m+n} - \frac{m+n}{2} f_{m-n}, \quad n < m,$$
$$\tilde{\nabla}_{g_n} f_n = \nabla_{g_n} f_n - Q(g_n, f_n) = -\lambda_{n,n} f_{2n},$$
$$\tilde{\nabla}_{g_m} g_n = \nabla_{g_m} g_n - Q(g_m, g_n) = -\lambda_{m,n} g_{m+n} - \frac{m+n}{2} g_{n-m}, \quad n > m,$$
$$\tilde{\nabla}_{g_m} g_n = \nabla_{g_m} g_n - Q(g_m, g_n) = -\lambda_{m,n} g_{m+n}, \quad n < m,$$
$$\tilde{\nabla}_{g_n} g_n = \nabla_{g_n} g_n - Q(g_n, g_n) = -\lambda_{n,n} g_{2n}.$$

Theorem 4.5. *The covariant derivative* $\tilde{\nabla}$ *has the following properties*:

(1) $\tilde{\nabla}$ *is the Levi-Civita covariant derivative, that is, it is metric compatible and torsion free*;

(2) $\tilde{\nabla}$ *is not a Hilbert–Schmidt operator.*

Remark 4.6. The original covariant derivative ∇ is also torsion free, which can be checked by a direct computation:

$$\begin{aligned} T_{\nabla}(X,Y) &= \nabla_X(Y) - \nabla_Y(X) - [X,Y]_{\mathfrak{m}} \\ &= \left(\frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y)\right) - \left(\frac{1}{2}[Y,X]_{\mathfrak{m}} + U(Y,X)\right) - [X,Y]_{\mathfrak{m}} \\ &= U(X,Y) - U(Y,X). \end{aligned}$$

Note that $U(X,Y)$ is symmetric in (X,Y) due to the symmetry of B as can be seen from Notation 3.3, and therefore $T_{\nabla} = 0$. Similarly to the finite-dimensional case the new covariant derivative $\tilde{\nabla}$ is torsion free if the almost complex structure J has no torsion. This is indeed the case by Proposition 4.1.

Proof. (1) The torsion can be found by the following formula:

$$\widetilde{T}(x,y) = T_{\tilde{\nabla}}(x,y) = \tilde{\nabla}_x y - \tilde{\nabla}_y x - [x,y]_{\mathfrak{m}}.$$

Let $m \neq n$, then

$$\begin{aligned} \widetilde{T}(f_m,f_n) &= \tilde{\nabla}_{f_m} f_n - \tilde{\nabla}_{f_n} f_m - \frac{m-n}{2} g_{m+n} - \frac{m^2-n^2}{2|m-n|} g_{|m-n|} \\ &= \frac{m-n}{2} g_{m+n} + \frac{m^2-n^2}{2|m-n|} g_{|m-n|} - \frac{m-n}{2} g_{m+n} - \frac{m^2-n^2}{2|m-n|} g_{|m-n|} = 0, \\ \widetilde{T}(f_m,g_n) &= \tilde{\nabla}_{f_m} g_n - \tilde{\nabla}_{g_n} f_m - \frac{n-m}{2} f_{m+n} - \frac{m+n}{2} f_{|m-n|} \\ &= \frac{m+n}{2} f_{|m-n|} + (\lambda_{n,m} - \lambda_{m,n}) f_{m+n} - \frac{n-m}{2} f_{m+n} - \frac{m+n}{2} f_{|m-n|} = 0, \\ \widetilde{T}(g_m,f_n) &= \tilde{\nabla}_{g_m} f_n - \tilde{\nabla}_{f_n} g_m + \frac{m-n}{2} f_{m+n} + \frac{m+n}{2} f_{|m-n|} \\ &= -\frac{m+n}{2} f_{|m-n|} + (\lambda_{n,m} - \lambda_{m,n}) f_{m+n} + \frac{m-n}{2} f_{m+n} + \frac{m+n}{2} f_{|m-n|} = 0, \\ \widetilde{T}(g_m,g_n) &= \tilde{\nabla}_{g_m} g_n - \tilde{\nabla}_{g_n} g_m - \frac{n-m}{2} g_{m+n} - \frac{m^2-n^2}{2|m-n|} g_{|m-n|} \\ &= (\lambda_{n,m} - \lambda_{m,n}) g_{m+n} + \frac{m^2-n^2}{2|m-n|} g_{|m-n|} - \frac{n-m}{2} g_{m+n} - \frac{m^2-n^2}{2|m-n|} g_{|m-n|} = 0. \end{aligned}$$

(2) We have

$$\sum_{m=1}^{\infty}\big(\langle\tilde\nabla_{f_m} f_n, \tilde\nabla_{f_m} f_n\rangle + \langle\tilde\nabla_{g_m} f_n, \tilde\nabla_{g_m} f_n\rangle\big)$$
$$= \sum_{m=1}^{n-1}\left(\frac{\lambda_{m,n}^2\theta_{m+n}}{\theta_n\theta_m} + \frac{(m+n)^2\theta_{n-m}}{4\theta_n\theta_m}\right) + \frac{\lambda_{n,n}^2\theta_{2n}}{\theta_n^2} + \sum_{m=n+1}^{\infty}\frac{\lambda_{m,n}^2\theta_{m+n}}{\theta_n\theta_m} = +\infty. \qquad \square$$

Notation 4.7. Let $n \in \mathbb{N}$, then define

$$L_m = f_m + i g_m, \; L_{-m} = f_m - i g_m, \quad \text{where } i^2 = -1.$$

Lemma 4.8.

$$\begin{aligned}
[L_m, L_n] &= i(n-m)L_{m+n},\\
[L_{-m}, L_n] &= i(m+n)L_{n-m},\\
[L_m, L_{-n}] &= -i(m+n)L_{m-n},\\
[L_{-m}, L_{-n}] &= i(m-n)L_{-m-n}.
\end{aligned}$$

Proof.

$$\begin{aligned}
[L_m, L_n] &= [f_m, f_n] - [g_m, g_n] + i\big([f_m, g_n] + [g_m, f_n]\big)\\
&= (m-n)g_{m+n} + i(n-m)f_{m+n} = i(n-m)L_{m+n},\\
[L_{-m}, L_n] &= [f_m, f_n] + [g_m, g_n] + i\big([f_m, g_n] - [g_m, f_n]\big)\\
&= (m+n)\frac{m-n}{|m-n|}g_{|m-n|} + i(m+n)f_{|m-n|} = i(m+n)L_{n-m},\\
[L_m, L_{-n}] &= [f_m, f_n] + [g_m, g_n] + i\big(-[f_m, g_n] + [g_m, f_n]\big)\\
&= (m+n)\frac{m-n}{|m-n|}g_{|m-n|} - i(m+n)f_{|m-n|} = -i(m+n)L_{m-n},\\
[L_{-m}, L_{-n}] &= [f_m, f_n] - [g_m, g_n] - i\big([f_m, g_n] + [g_m, f_n]\big)\\
&= (m-n)g_{m+n} - i(n-m)f_{m+n} = i(m-n)L_{-m-n}. \qquad \square
\end{aligned}$$

Lemma 4.9.

$$\begin{aligned}
\tilde\nabla_{L_m} L_n &= -\tilde\nabla_{L_{-m}} L_{-n} = -2i\lambda_{m,n}L_{m+n},\\
\tilde\nabla_{L_{-m}} L_n &= -\tilde\nabla_{L_m} L_{-n} = i(m+n)L_{n-m}, \quad n > m,\\
\tilde\nabla_{L_{-m}} L_n &= \tilde\nabla_{L_m} L_{-n} = 0, \quad m > n,\\
\tilde\nabla_{L_n} L_n &= -\tilde\nabla_{L_{-n}} L_{-n} = -2i\lambda_{n,n}L_{2n},\\
\tilde\nabla_{L_{-n}} L_n &= \tilde\nabla_{L_n} L_{-n} = 0.
\end{aligned}$$

Proof. First,

$$\begin{aligned}\tilde{\nabla}_{L_m} L_n &= \tilde{\nabla}_{f_m} f_n - \tilde{\nabla}_{g_m} g_n + i(\tilde{\nabla}_{f_m} g_n + \tilde{\nabla}_{g_m} f_n)\\ &= 2\lambda_{m,n} g_{m+n} - 2i\lambda_{m,n} f_{m+n} = -2i\lambda_{m,n} L_{m+n}.\end{aligned}$$

Second,

$$\begin{aligned}\tilde{\nabla}_{L_{-m}} L_n &= \tilde{\nabla}_{f_m} f_n + \tilde{\nabla}_{g_m} g_n + i(\tilde{\nabla}_{f_m} g_n - \tilde{\nabla}_{g_m} f_n)\\ &= -(m+n) g_{n-m} + i(m+n) f_{n-m} = i(m+n) L_{n-m}, \quad n > m,\\ \tilde{\nabla}_{L_{-m}} L_n &= \tilde{\nabla}_{f_m} f_n + \tilde{\nabla}_{g_m} g_n + i(\tilde{\nabla}_{f_m} g_n - \tilde{\nabla}_{g_m} f_n) = 0, \quad n < m.\end{aligned}$$

Third,

$$\begin{aligned}\tilde{\nabla}_{L_m} L_{-n} &= \tilde{\nabla}_{f_m} f_n + \tilde{\nabla}_{g_m} g_n - i(\tilde{\nabla}_{f_m} g_n - \tilde{\nabla}_{g_m} f_n)\\ &= -(m+n) g_{n-m} - i(m+n) f_{n-m} = -i(m+n) L_{m-n}, \quad n > m,\\ \tilde{\nabla}_{L_m} L_{-n} &= \tilde{\nabla}_{f_m} f_n + \tilde{\nabla}_{g_m} g_n - i(\tilde{\nabla}_{f_m} g_n - \tilde{\nabla}_{g_m} f_n) = 0, \quad n < m.\end{aligned}$$

Finally, we have

$$\begin{aligned}\tilde{\nabla}_{L_{-m}} L_{-n} &= \tilde{\nabla}_{f_m} f_n - \tilde{\nabla}_{g_m} g_n - i(\tilde{\nabla}_{f_m} g_n + \tilde{\nabla}_{g_m} f_n)\\ &= 2\lambda_{m,n} g_{m+n} + 2i\lambda_{m,n} f_{m+n} = 2i\lambda_{m,n} L_{-m-n},\\ \tilde{\nabla}_{L_n} L_n &= \tilde{\nabla}_{f_n} f_n - \tilde{\nabla}_{g_n} g_n + i(\tilde{\nabla}_{f_n} g_n + \tilde{\nabla}_{g_n} f_n)\\ &= 2\lambda_{n,n} g_{2n} - 2i\lambda_{n,n} f_{2n} = -2i\lambda_{n,n} L_{2n},\\ \tilde{\nabla}_{L_{-n}} L_n &= \tilde{\nabla}_{f_n} f_n + \tilde{\nabla}_{g_n} g_n + i(\tilde{\nabla}_{f_n} g_n - \tilde{\nabla}_{g_n} f_n) = 0,\\ \tilde{\nabla}_{L_n} L_{-n} &= \tilde{\nabla}_{f_n} f_n + \tilde{\nabla}_{g_n} g_n - i(\tilde{\nabla}_{f_n} g_n - \tilde{\nabla}_{g_n} f_n) = 0,\\ \tilde{\nabla}_{L_{-n}} L_{-n} &= \tilde{\nabla}_{f_n} f_n - \tilde{\nabla}_{g_n} g_n - i(\tilde{\nabla}_{f_n} g_n + \tilde{\nabla}_{g_n} f_n)\\ &= 2\lambda_{n,n} g_{2n} + 2i\lambda_{n,n} f_{2n} = 2i\lambda_{n,n} L_{-2n}. \qquad \square\end{aligned}$$

Definition 4.10. The curvature tensor $\widetilde{R}_{xy} : \mathfrak{m}_{\mathbb{C}} \to \mathfrak{m}_{\mathbb{C}}$ is defined by

$$\widetilde{R}_{xy} = \tilde{\nabla}_x \tilde{\nabla}_y - \tilde{\nabla}_y \tilde{\nabla}_x - \tilde{\nabla}_{[x,y]_{\mathfrak{m}_{\mathbb{C}}}} - \operatorname{ad}([x,y]_{\mathfrak{h}_{\mathbb{C}}}), \quad x, y \in \mathfrak{g},$$

then the Ricci tensor $\operatorname{Ric}(x, y) : \mathfrak{m}_{\mathbb{C}} \to \mathfrak{m}_{\mathbb{C}}$ is the trace of the map $z \mapsto \widetilde{R}_{zx} y$.

Theorem 4.11. *The only non-zero components of the Ricci tensor are*

$$\operatorname{Ric}\left(\frac{L_n}{\sqrt{|\theta_n|}}, \frac{L_{-n}}{\sqrt{|\theta_n|}}\right) = -\frac{13n^3 - n}{6\theta_n}, \quad n \in \mathbb{Z}.$$

Proof. Note that for any $\alpha, \beta, \gamma \in \mathbb{Z}$ we have $\widetilde{R}_{L_\gamma L_\alpha} L_\beta = C_{\alpha\beta\gamma} L_{\alpha+\beta+\gamma}$ for some $C_{\alpha\beta\gamma} \in \mathbb{C}$. Therefore the only non-zero components of $\mathrm{Ric}(L_\alpha, L_\beta)$ are when $\alpha + \beta = 0$. We will deduce the formula for $\mathrm{Ric}(L_n/\sqrt{|\theta_n|}, L_{-n}/\sqrt{|\theta_n|})$ in the case $n \in \mathbb{N}$, the case $n < 0$ follows from this.

Suppose $m \neq n$, then

$$\begin{aligned}\widetilde{R}_{L_m,L_n} L_{-n} &= \tilde{\nabla}_{L_m} \tilde{\nabla}_{L_n} L_{-n} - \tilde{\nabla}_{L_n} \tilde{\nabla}_{L_m} L_{-n} - \tilde{\nabla}_{[L_m,L_n]_\mathfrak{m}} L_{-n} - \mathrm{ad}\big([L_m, L_n]_\mathfrak{h}\big) L_{-n} \\ &= -\tilde{\nabla}_{L_n} \tilde{\nabla}_{L_m} L_{-n} - i(n-m)\tilde{\nabla}_{L_{m+n}} L_{-n} = -\tilde{\nabla}_{L_n} \tilde{\nabla}_{L_m} L_{-n},\end{aligned}$$

therefore

$$\begin{aligned}&\widetilde{R}_{L_m,L_n} L_{-n} = 0, \quad m > n, \\ &\widetilde{R}_{L_m,L_n} L_{-n} = i(m+n)\tilde{\nabla}_{L_n} L_{m-n} = 0, \quad m < n.\end{aligned}$$

If $m \neq n$, then

$$\begin{aligned}\widetilde{R}_{L_{-m},L_n} L_{-n} &= \tilde{\nabla}_{L_{-m}} \tilde{\nabla}_{L_n} L_{-n} - \tilde{\nabla}_{L_n} \tilde{\nabla}_{L_{-m}} L_{-n} \\ &\quad - \tilde{\nabla}_{[L_{-m},L_n]_\mathfrak{m}} L_{-n} - \mathrm{ad}\big([L_{-m}, L_n]_\mathfrak{h}\big) L_{-n} \\ &= -\tilde{\nabla}_{L_n} \tilde{\nabla}_{L_{-m}} L_{-n} - i(m+n)\tilde{\nabla}_{L_{n-m}} L_{-n} \\ &= -2i\lambda_{m,n} \tilde{\nabla}_{L_n} L_{-m-n} - i(m+n)\tilde{\nabla}_{L_{n-m}} L_{-n} \\ &= -2(m+2n)\lambda_{m,n} L_{-m} - i(m+n)\tilde{\nabla}_{L_{n-m}} L_{-n},\end{aligned}$$

thus

$$\begin{aligned}&\widetilde{R}_{L_{-m},L_n} L_{-n} = -2(m+2n)\lambda_{m,n} L_{-m} + 2(m+n)\lambda_{m-n,n} L_{-m}, \quad m > n, \\ &\widetilde{R}_{L_{-m},L_n} L_{-n} = -2(m+2n)\lambda_{m,n} L_{-m} - (2n-m)(m+n) L_{-m}, \quad m < n.\end{aligned}$$

Finally,

$$\begin{aligned}\widetilde{R}_{L_{-n},L_n} L_{-n} &= \tilde{\nabla}_{L_{-n}} \tilde{\nabla}_{L_n} L_{-n} - \tilde{\nabla}_{L_n} \tilde{\nabla}_{L_{-n}} L_{-n} - \tilde{\nabla}_{[L_{-n},L_n]_\mathfrak{m}} L_{-n} - \mathrm{ad}\big([L_{-n}, L_n]_\mathfrak{h}\big) L_{-n} \\ &= -6n\lambda_{n,n} L_{-n} - 2n^2 L_{-n}.\end{aligned}$$

Thus

$$\begin{aligned}&\mathrm{Ric}\left(\frac{L_n}{\sqrt{\theta_n}}, \frac{L_{-n}}{\sqrt{\theta_n}}\right) \\ &\quad = \sum_{m=n+1}^{\infty} \frac{2(m+n)\lambda_{m-n,n} - 2(m+2n)\lambda_{m,n}}{\theta_n} - \sum_{m=1}^{n} \frac{(m+n)(2n-m) + 2(m+2n)\lambda_{m,n}}{\theta_n}\end{aligned}$$

$$= \sum_{m=1}^{n} \frac{2(m+2n)\lambda_{m,n}}{\theta_n} - \sum_{m=1}^{n} \frac{(m+n)(2n-m) + 2(m+2n)\lambda_{m,n}}{\theta_n}$$

$$= -\sum_{m=1}^{n} \frac{(m+n)(2n-m)}{\theta_n} = -\frac{13n^3 - n}{6\theta_n}. \qquad \square$$

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