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Isovectors for the Hamilton-Jacobi-Bellman Equation, Formal Stochastic Differentials and First Integrals in Euclidean Quantum Mechanics

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Abstract. The study of Riemannian stochastic differential geometry is almost as old as the theory of stochastic differential equations themselves. But almost nothing is known about stochastic symplectic geometry. Using only the geometrical content of the heat equation (in a sense inspired by E. Cartan) and its relations with quantum dynamics, we present here the first steps toward such a stochastic version of symplectic geometry.

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1. Introduction

The purpose of this paper is to provide proofs for all the statements made in our *Comptes Rendus* note [7].

The Schrödinger equation for a particle of mass m in a potential V in $\mathbf{R}^n$:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi,$$

can be written, for $n = 1$, $V = 0$ and $m = 1$, as:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial q^2} \equiv H_0\psi.$$

In Euclidean Quantum Mechanics ([3, 10, 11, 12, 13]) this equation splits into:

$$\frac{\partial\psi}{\partial t} = -\frac{\hbar}{2}\frac{\partial^2\psi}{\partial q^2} \qquad (C_1)$$

and

$$\frac{\partial\psi}{\partial t} = \frac{\hbar}{2}\frac{\partial^2\psi}{\partial q^2}, \qquad (\mathcal{C}_2)$$

the probability density being given, not, as usual, by $\psi\bar{\psi}$, but by $\eta\eta_*$, η et η_* denoting respectively an everywhere strictly positive solution of $(\mathcal{C}_1)$ and an everywhere strictly positive solution of $(\mathcal{C}_2)$. A *Bernstein process* z is then naturally associated with the situation; it satisfies the stochastic differential equation:

$$dz(t) = \sqrt{\hbar}dw(t) + \tilde{B}(t, z(t))dt \qquad (\mathcal{B}_1)$$

relatively to the canonical increasing filtration for the Brownian w, and the stochastic differential equation

$$d_*z(t) = \sqrt{\hbar}d_*w_*(t) + \tilde{B}_*(t, z(t))dt \qquad (\mathcal{B}_2)$$

relatively to the canonical decreasing filtration for the Brownian w_*, where

$$\tilde{B} \equiv_{def} \hbar\frac{\frac{\partial\eta}{\partial q}}{\eta}, \qquad \text{and} \qquad \tilde{B}_* \equiv_{def} -\hbar\frac{\frac{\partial\eta_*}{\partial q}}{\eta_*}.$$

Images of Bernstein processes under symmetries of the heat equation can be obtained by a purely algebraic concept of *stochastic quadrature*: if we consider an *isovector* (see §2) N, then $U_\alpha^N = e^{\alpha\hat{N}}$ maps η to a solution $\eta_\alpha = U_\alpha^N(\eta)$ of $(\mathcal{C}_1)$, and z to a process z_α associated with η_α whose law P_α is absolutely continuous with respect to P, with density $\frac{dP^\alpha}{dP} = h_\alpha = \frac{\eta_\alpha}{\eta}$; more precisely:

Theorem 1.1 (Theorem 4 of [7]). *Let $\hat{N} = -V_N$ correspond to an isovector, so that $U_\alpha = e^{\alpha\hat{N}}$ preserve positivity, and η be the particular solution of $(\mathcal{C}_1)$ defining the process $z(t)$, $t \in [t_0, t_1]$, with law P. By definition of the symmetry group of $(\mathcal{C}_1)$, $\eta_\alpha(q,t) = (U_\alpha\eta)(q,t)$ solves the same equation. Then $h_\alpha = \frac{\eta_\alpha}{\eta}$ is a strictly positive martingale of $z(t)$, and, if $z^\alpha(t)$ denotes the process associated with η_α, its law P_α is absolutely continuous with respect to P with $\frac{dP^\alpha}{dP} = h_\alpha$. If*

$$\mathbb{E}(exp(\frac{1}{2}\int_t^T |\tilde{B}^\alpha - \tilde{B}|^2 d\tau)) < +\infty,$$

then $z^\alpha(t)$ is an h-transform of $z(t)$, with drift

$$\tilde{B}^\alpha(q,t) = \tilde{B}(q,t) + \hbar\frac{\partial}{\partial q}\ln h_\alpha(q,t).$$

Proof. The only property to check is that h_α is a (strictly positive) martingale. Now, according to $(\mathcal{B}_1)$ and the definition of the drift B, the generator of the process can be written

$$\tilde{D} = \frac{\partial}{\partial t} + \mathcal{L}, \qquad \text{with} \qquad \mathcal{L} = \tilde{B}\frac{\partial}{\partial q} + \frac{\hbar}{2}\frac{\partial^2}{\partial q^2}.$$

Equivalently, for f of $\mathcal{C}^\infty$-class,

$$\tilde{D}f(q,t) = \frac{1}{\eta}(\frac{\partial}{\partial t} - \frac{1}{\hbar}H_0)(f\eta).$$

In particular

$$\tilde{D}h_\alpha = \frac{1}{\eta}(\frac{\partial}{\partial t} - \frac{1}{\hbar}H_0)(\frac{\eta_\alpha}{\eta}\eta) = 0$$

by construction. □

Applying this result first to N_4, and then to $-\frac{1}{2}N_6$ (with the notations of §2), we recover, for the special case of the Brownian motion, *i.e.* the trivial solution $\eta = 1$ of $(\mathcal{C}_1)$, firstly the scaling invariance of Brownian motion, and secondly the expression of the "Brownian bridge" in terms of Brownian motion; the details are given in §7.

Theorem 1.1 has a counterpart with respect to the decreasing filtration. This is the case of all the ulterior results, formulated only for $(\mathcal{C}_1)$, without lack of generality.

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2. The complete Lie algebra of isovectors for the Hamilton-Jacobi-Bellman equation

We shall make explicit, in a slightly more general context, the result of calculations analogous (and in fact formally equivalent) to those contained in [6], pp. 657–659.

Let us consider the heat equation $(\mathcal{C}_1)$ (for $\hbar \neq 0$):

$$\frac{\partial\psi}{\partial t} = -\frac{\hbar}{2}\frac{\partial^2\psi}{\partial q^2}. \qquad (1)$$

Setting $S = -\hbar\ln(\psi)$, one obtains the Hamilton-Jacobi-Bellman equation:

$$\frac{\partial S}{\partial t} = -\frac{\hbar}{2}\frac{\partial^2 S}{\partial q^2} + \frac{1}{2}(\frac{\partial S}{\partial q})^2. \qquad (2)$$

In order to deal with differential forms, we have to reduce that equation to a system of first order equations. For that purpose, let $E = -\frac{\partial S}{\partial t}$ (*formal energy*) and $B = -\frac{\partial S}{\partial q}$ (*formal momentum*: cf. the above definition of $\tilde{B}$) ; then any solution of (2) will annul the following differential forms on $\mathbf{M} = \mathbf{R}^5$:

$$\omega = dS + Edt + Bdq,$$

(by definition of B and E) ;

$$d\omega = dEdt + dBdq,$$

(by Schwarz' Theorem) and

$$\beta = (E + \frac{1}{2}B^2)dqdt + \frac{\hbar}{2}dBdt$$

(because of (2)).

From now on we shall always, except when otherwise stated, consider t, q, S, E and B as *independent* variables, that is as coordinates in the state space $\mathbf{M} = \mathbf{R}^5$.

As

$$d\beta = dEdqdt + BdBdqdt = -dqdEdt + BdtdBdq = (-dq + Bdt)d\omega,$$

$d\beta$ belongs to the ideal $\mathcal{I}$ of $\mathcal{A} = \bigwedge T^*(\mathbf{M})$ generated by ω, $d\omega$ and β; therefore $\mathcal{I}$ is the smallest differential ideal of $\mathcal{A}$ containing both ω and β. As usual, we shall term *isovector* for $\mathcal{I}$ a vector field N on $\mathbf{M}$ such that:

$$\mathcal{L}_N(\mathcal{I}) \subset \mathcal{I},$$

where $\mathcal{L}_N$ denotes the Lie derivative with respect to N. By Cartan's results ([1, 2]), these N's constitute the Lie algebra $\mathcal{G}$ of the symmetry group G of the equation.

It is well known that the generic form of an $N \in \mathcal{H}$ is given by :

$$N = N^t \frac{\nabla}{\nabla t} + N^q \frac{\nabla}{\nabla q} + N^S \frac{\nabla}{\nabla S} + N^E \frac{\nabla}{\nabla E} + N^B \frac{\nabla}{\nabla B},$$

where, by definition

$$N^t = 2k_6 t^2 + 2k_4 t + k_1 \tag{3}$$

$$N^q = q(2k_6 t + k_4) - 2k_5 t + k_2 \tag{4}$$

$$N^S = k_6(\hbar t - q^2) + 2k_5 q - \hbar k_3 - \hbar e^{\frac{S}{\hbar}} g$$

$$N^E = -k_6(2qB + 4tE + \hbar) + 2k_5 B - 2k_4 E + e^{\frac{S}{\hbar}}(\hbar g_t - Eg) \tag{5}$$

$$N^B = -k_4 B - 2k_5 + 2k_6(q - tB) + e^{\frac{S}{\hbar}}(\hbar g_q - Bg), \tag{6}$$

$k_1, \ldots, k_6$ denoting arbitrary real constants and g an arbitrary solution of (1). These formulas coïncide, modulo the appropriate identifications, with formula (2.57) in [9], p. 122.

The algebra $\mathcal{G}$ therefore possesses a canonical generating system given by $N_1, \ldots, N_6$ and the N_g(g solution of (1)), each N_i being given by $k_i = 1$, $k_j = 0$ for $j \neq i$ and $g = 0$, and N_g by all $k_i = 0$, *i.e*,

$$N_1 = \frac{\partial}{\partial t}, \; N_2 = \frac{\partial}{\partial q}, \; N_3 = -\hbar \frac{\partial}{\partial S}$$

$$N_4 = 2t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q} - 2E\frac{\partial}{\partial E} - B\frac{\partial}{\partial B} \tag{7}$$

$$N_5 = -2t\frac{\partial}{\partial q} + 2q\frac{\partial}{\partial S} + 2B\frac{\partial}{\partial E} - 2\frac{\partial}{\partial B},$$

$$N_6 = 2t^2\frac{\partial}{\partial t} + 2qt\frac{\partial}{\partial q} + (\hbar t - q^2)\frac{\partial}{\partial S} - (2qB + 4tE + \hbar)\frac{\partial}{\partial E} + 2(q - tB)\frac{\partial}{\partial B},$$

and

$$N_g = -\hbar e^{\frac{S}{\hbar}} g \frac{\partial}{\partial S} + e^{\frac{S}{\hbar}}(\hbar g_t - Eg)\frac{\partial}{\partial E} + e^{\frac{S}{\hbar}}(\hbar g_q - Bg)\frac{\partial}{\partial B}.$$

3. The structure of the Lie algebra $\mathcal{G}$

With the notations of §2, an easy computation shows that:

$$[N_1, N_2] = 0, \; [N_1, N_3] = 0, \; [N_1, N_4] = 2N_1, \; [N_1, N_5] = -2N_2,$$

$$[N_1, N_6] = 2N_4 - N_3, \; [N_1, N_g] = N_{g_t} \text{ (as } g \text{ is a solution of (1), so is } g_t),$$

$$[N_2, N_3] = 0, \; [N_2, N_4] = N_2, \; [N_2, N_5] = -\frac{2}{\hbar}N_3,$$

$$[N_2, N_6] = -N_5, \; [N_2, N_g] = N_{g_q}, \text{ (as } g \text{ satisfies (1), so does } g_q),$$

$$[N_3, N_4] = 0, \; [N_3, N_5] = 0, \; [N_3, N_6] = 0, \; [N_3, N_g] = -N_g = N_{-g},$$

$$[N_4, N_5] = N_5, \; [N_4, N_6] = 2N_6, \; [N_4, N_g] = N_{2tg_t + qg_q}, \; (2tg_t + qg_q \text{ satisfies (1)}),$$

$$[N_5, N_6] = 0, \; [N_5, N_g] = N_{\frac{2}{\hbar}qg - 2tg_q},$$

$$[N_6, N_g] = N_{2t^2 g_t + 2tqg_q + gt - \frac{gq^2}{\hbar}}, \; [N_g, N_{g'}] = 0.$$

Let $\mathcal{H}$ denote the subspace of $\mathcal{G}$ generated by $N_1, \ldots, N_6$, $\mathcal{J}$ the subspace of the N_g (g solution of (1)), $\mathcal{H}_1$ the subspace generated by N_3, and $\mathcal{H}_0$ the subspace generated by N_2, N_3 and N_5. Then it follows from the commutation relations that:

$\mathcal{H}$ is a subalgebra of $\mathcal{G}$,

$\mathcal{J}$ is an abelian subalgebra of $\mathcal{G}$ and $[\mathcal{H}, \mathcal{J}] \subseteq \mathcal{J}$.

As, obviously, $\mathcal{G} = \mathcal{H} \oplus \mathcal{J}$, we have: $[\mathcal{G}, \mathcal{J}] \subseteq \mathcal{J}$.

$\mathcal{G}$ therefore appears as the semi-direct product of a 6-dimensional subalgebra $\mathcal{H}$ and an infinite-dimensional abelian ideal $\mathcal{J}$.

It is easy to analyze the structure of $\mathcal{H}$: the commutation relations yield that:

$$\mathcal{H}_1 = \mathbf{R}N_3 \subset Z(\mathcal{H}),$$

and that

$$\mathcal{H}_0 \text{ is an ideal of } \mathcal{H}.$$

Let $\bar{h}$ denote the class modulo $\mathcal{H}_1$ of $h \in \mathcal{H}$; then:

$$[\bar{N}_1, \bar{N}_2] = 0, \; [\bar{N}_1, \bar{N}_4] = 2\bar{N}_1, \; [\bar{N}_1, \bar{N}_5] = -2\bar{N}_2, \; [\bar{N}_1, \bar{N}_6] = 2\bar{N}_4,$$

$$[\bar{N}_2, \bar{N}_4] = \bar{N}_2, \; [\bar{N}_2, \bar{N}_5] = 0, \; [\bar{N}_2, \bar{N}_6] = -\bar{N}_5,$$

$$[\bar{N}_4, \bar{N}_5] = \bar{N}_5, \; [\bar{N}_4, \bar{N}_6] = 2\bar{N}_6, \; [\bar{N}_5, \bar{N}_6] = 0.$$

In the natural semi-direct product $sl_2(\mathbf{R}) \ltimes \mathbf{R}^2$, let:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x = (1, 0), \; y = (0, 1).$$

Then, as is well known:

$$[h, e] = 2e, \; [h, f] = -2f, \; [e, f] = h, \; [h, x] = h(x) = x,$$

$$[h, y] = h(y) = -y, \; [e, x] = e(x) = 0, \; [e, y] = e(y) = x,$$

$$[f, x] = f(x) = y, \; [f, y] = f(y) = 0, \; [x, y] = 0.$$

Therefore the application:

$$\bar{N_1} \mapsto -e,\ \bar{N_2} \mapsto x,\ \bar{N_4} \mapsto -h,\ \bar{N_5} \mapsto 2y,\ \bar{N_6} \mapsto 2f$$

induces an isomorphism of Lie algebras:

$$\frac{\mathcal{H}}{\mathcal{H}_1} \simeq sl_2(\mathbf{R}) \ltimes \mathbf{R}^2.$$

In this isomorphism, $\frac{\mathcal{H}_0}{\mathcal{H}_1}$ corresponds to $< x, 2y > = \mathbf{R}^2$, whence:

$$\frac{\frac{\mathcal{H}}{\mathcal{H}_1}}{\frac{\mathcal{H}_0}{\mathcal{H}_1}} \simeq sl_2(\mathbf{R}),\ i.e.\ \frac{\mathcal{H}}{\mathcal{H}_0} \simeq sl_2(\mathbf{R}).$$

As

$$[N_2, N_3] = 0,\ [N_2, N_5] = -\frac{2}{\hbar} N_3,\ [N_3, N_5] = 0,$$

one sees that $\mathcal{H}_0$ is a three-dimensional Heisenberg algebra having for its center the subspace $H_1 = \mathbf{R}N_3$ generated by N_3. But

$$\frac{\mathcal{H}}{\mathcal{H}_0} \simeq sl_2(\mathbf{R})$$

has trivial center, hence $Z(\mathcal{H}) \subset \mathcal{H}_0$, whence $Z(\mathcal{H}) \subset Z(\mathcal{H}_0) = \mathcal{H}_1 \subset Z(\mathcal{H})$, and :

$$Z(\mathcal{H}) = Z(\mathcal{H}_0) = \mathcal{H}_1.$$

We set:

$$\mathcal{H}_2 = Vect(N_1, N_2, N_3, N_4);$$

this is a *subalgebra* of $\mathcal{H}$, hence of $\mathcal{G}$.

4. The bilinear form

For $N \in \mathcal{H}$, we shall set:

$$\Phi_N = -N^S = k_6(q^2 - \hbar t) - 2k_5 q + k_3 \hbar. \tag{8}$$

This Φ_N coïncides with the *phase* defined in [10], pp. 317–318. It is visible that Φ_N depends only upon q and t. We also set:

$$n_N = N^t E + N^q B - \Phi_N. \tag{9}$$

Definition 4.1. For any $(\delta, \delta') \in (T^1(\mathbf{M})^*)^2$(the space of pairs of C^∞-vector fields on $\mathbf{M} = \mathbf{R}^5$), let:

$$\Omega(\delta, \delta') = (\delta(B)\delta'(q) - \delta(q)\delta'(B)) + (\delta(E)\delta'(t) - \delta(t)\delta'(E)).$$

It is clear that Ω is a bilinear and antisymmetric mapping from $(T^1(\mathbf{M})^*)^2$ to $C^\infty(\mathbf{M})$.

The key to our subsequent computations is:

Proposition 4.2. *For arbitrary $N \in \mathcal{H}$ and δ in $T^1(\mathbf{M})^*$, one has:*

$$\Omega(N, \delta) = -\delta(n_N).$$

Proof. By Definition 4.1 and the definition (9) of n_N, one has, with the notations of §3:

$$\begin{aligned}
\Omega(N,\delta) + \delta(n_N) &= N(B)\delta(q) - N(q)\delta(B) + N(E)\delta(t) - N(t)\delta(E) \\
&\quad + \delta(EN^t + BN^q - \Phi_N) \\
&= N(B)\delta(q) - N(q)\delta(B) + N(E)\delta(t) - N(t)\delta(E) \\
&\quad + \delta(E)N^t + E\delta(N^t) + \delta(B)N^q + B\delta(N^q) - \delta(\Phi_N) \\
&= N^B\delta(q) + N^E\delta(t) + E\delta(N^t) + B\delta(N^q) - \delta(\Phi_N) \\
&= -k_4 B\delta(q) - 2k_5\delta(q) + 2k_6 q\delta(q) - 2k_6 tB\delta(q) - 2k_6 qB\delta(t) \\
&\quad - 4k_6 tE\delta(t) - \hbar k_6\delta(t) + 2k_5 B\delta(t) - 2k_4 E\delta(t) + E\delta(N^t) \\
&\quad + B\delta(N^q) - \delta(\Phi_N) \\
&= E\delta(N^t - 2k_6 t^2 - 2k_4 t) + B\delta(N^q - k_4 q - 2k_6 tq + 2k_5 t) \\
&\quad + \delta(-2k_5 q + k_6 q^2 - \hbar k_6 t - \Phi_N) \\
&= E\delta(k_1) + B\delta(k_2) + \delta(-\hbar k_3) \\
&= 0.
\end{aligned}$$

□

Lemma 4.3. *For each $N \in \mathcal{H}$, one has:*

(i) $N^E = -B\dfrac{\partial N^q}{\partial t} - E\dfrac{dN^t}{dt} - \dfrac{\hbar}{4}\dfrac{d^2N^t}{dt^2}$ *and* $N^B = -\dfrac{B}{2}\dfrac{dN^t}{dt} + \dfrac{\partial N^q}{\partial t}.$

(ii) $\dfrac{dN^t}{dt} = 2\dfrac{\partial N^q}{\partial q}.$

Proof. This is apparent from formulas (3), (4), (5) and (6). □

5. Sectioning

Definition 5.1. We shall call

$$\omega_{PC} = Edt + Bdq = \omega - dS$$

the Poincaré-Cartan form, and

$$L = \frac{B^2}{2}$$

the formal free Lagrangian.

Proposition 5.2. *Let η be a given, everywhere > 0, solution of (1). Then there exists a unique morphism of graded differential algebras $\theta_\eta : \mathcal{A} \to \mathcal{B} = \bigwedge T^*(\mathbf{R}^2)$ such that, denoting by (t,q) the generic point of $\mathbf{R}^2$, one has:*

$$\theta_\eta(t) = t,\ \theta_\eta(q) = q,\ \theta_\eta(S) = -\hbar\ln(\eta),$$

$$\theta_\eta(E) = \frac{\hbar}{\eta}\frac{\partial\eta}{\partial t},\ \theta_\eta(B) = \frac{\hbar}{\eta}\frac{\partial\eta}{\partial q}.$$

In addition $\mathcal{I} \subseteq \ker(\theta_\eta)$, whence θ_η induces a mapping $\bar{\theta}_\eta : \frac{\mathcal{A}}{\mathcal{I}} \to \mathcal{B}$.

Proof. The existence and uniqueness of θ_η follows from the usual universal properties of the exterior algebra. In order to show that $\mathcal{I} \subseteq \ker(\theta_\eta)$, it is enough to see that ω and β belong to $\ker(\theta_\eta)$. For ω it is obvious, and for β it follows from the equivalence of (1) and (2), and the definitions of $\tilde{E}$ and $\tilde{B}$. □

We shall call θ_η the *section map* relative to η.

From now on, we shall fix a solution $\eta > 0$ of (1), and denote $\tilde{S} = \theta_\eta(S)$, $\tilde{E} = \theta_\eta(E)$ and $\tilde{B} = \theta_\eta(B)$.

Corollary 5.3.

$$\theta_\eta(\omega_{PC}) = \tilde{B}dq + \tilde{E}dt.$$

Proof. One has, by the definitions of ω_{PC} and θ_η:

$$\theta_\eta(\omega_{PC}) = \theta_\eta(\omega - dS) = \theta_\eta(\omega) - \theta_\eta(dS) = \theta_\eta(\omega) - d(\theta_\eta(S)).$$

But $\theta_\eta(\omega) = 0$ by the proof of Proposition 5.2, whence:

$$\theta_\eta(\omega_{PC}) = -d(\theta_\eta(S)) = -d(-\hbar\ln(\eta)) = \frac{\hbar}{\eta}d\eta = \tilde{B}dq + \tilde{E}dt.$$ □

Lemma 5.4.

(i) *For each $N \in \mathcal{H}$, $\theta_\eta(n_N) = \hbar\frac{\mathcal{V}_{\mathbf{N}}\eta}{\eta}$, where*

$$\mathcal{V}_{\mathbf{N}} \equiv_{def} N^t\frac{\partial}{\partial t} + N^q\frac{\partial}{\partial q} - \frac{1}{\hbar}\Phi_N. \in T(\mathbf{R}^2)$$

acts on functions of (q,t).

(ii) *For each $N \in \mathcal{H}_2$, $\theta_\eta(N(B)) = \delta_N(\tilde{B})$ and $\theta_\eta(N(E)) = \delta_N(\tilde{E})$, where δ_N denotes the* variation *associated with N in the sense of* [13], *§6, formula* (60).

Proof. (i) One has:

$$\theta_\eta(n_N) = \theta_\eta(N^tE + N^qB - \Phi_N) = \theta_\eta(N^t)\theta_\eta(E) + \theta_\eta(N^q)\theta_\eta(B) - \theta_\eta(\Phi_N)$$
$$= N^t\tilde{E} + N^q\tilde{B} - \Phi_N$$

(as N^t, N^q and Φ_N depend only upon q and t, by formulas (3), (4) and (8)), whence:

$$\theta_\eta(n_N) = N^t\hbar\frac{1}{\eta}\frac{\partial\eta}{\partial t} + N^q\hbar\frac{1}{\eta}\frac{\partial\eta}{\partial q} - \Phi_N = \frac{\hbar}{\eta}(N^t\frac{\partial\eta}{\partial t} + N^q\frac{\partial\eta}{\partial q} - \frac{\Phi_N}{\hbar}\eta)$$
$$= \hbar\frac{\mathcal{V}_{\mathbf{N}}\eta}{\eta},$$

by definition of $\mathcal{V}_{\mathbf{N}}$.

(ii) As $N \in \mathcal{H}_2$, $k_5 = k_6 = 0$, whence one has:

$$\theta_\eta(N(B)) = \theta_\eta(N^B) = \theta_\eta(-k_4B - 2k_5 + 2k_6(q - tB)) = \theta_\eta(-k_4B)$$
$$= -k_4\theta_\eta(B) = -k_4\tilde{B} = -\tilde{B}(2k_6t + k_4) = -\tilde{B}\frac{\partial N^q}{\partial q}$$

(using (4) and the fact that $k_6 = 0$). But this equals $\delta_N(\tilde{B})$ by Lemma 1, p. 401, in [13] (modulo a misprint in the sign). Similarly, one sees that:

$$\theta_\eta(N(E)) = \theta_\eta(N^E) = \theta_\eta(-2k_4E) = -2k_4\theta_\eta(E)$$
$$= -2k_4\tilde{E} = -\tilde{E}\frac{dN^t}{dt} = \delta_N(\tilde{E}).$$ □

Let $\Omega_\eta = \theta_\eta \circ \Omega : \mathcal{H}^2 \to \mathcal{B}$.

Lemma 5.5. *For $N \in \mathcal{H}_2$ and $N' \in \mathcal{H}_2$, $\Omega_\eta(N,N')$ coincides with the bilinear form $\Omega_\eta(\delta_N,\delta_{N'})$ as defined in* [13], *Definition 3, p. 402.*

Proof. One has:

$$\Omega_\eta(N,N') = \theta_\eta(\Omega(N,N'))$$
$$= \theta_\eta((N(B)N'(q) - N(q)N'(B)) + (N(E)N'(t) - N(t)N'(E)))$$

(by Definition 4.1)

$$= (\theta_\eta(N^B)\theta_\eta(N'^q) - \theta_\eta(N^q)\theta_\eta(N'^B))$$
$$+ (\theta_\eta(N^E)\theta_\eta(N'^t) - \theta_\eta(N^t)\theta_\eta(N'^E)).$$

But N^t depends only upon q and t, therefore $\theta_\eta(N^t) = N^t = \delta_N(t)$, and similarly $\theta_\eta(N^q) = N^q = \delta_N(q)$. By Lemma 5.4.(ii) we have $\theta_\eta(N^B) = \delta_N(\tilde{B})$ and $\theta_\eta(N^E) = \delta_N(\tilde{E})$, and similarly for N', whence:
$\Omega_\eta(N,N') = (\delta_N(\tilde{B})\delta_{N'}(q) - \delta_N(q)\delta_{N'}(\tilde{B})) + (\delta_N(\tilde{E})\delta_{N'}(t) - \delta_N(t)\delta_{N'}(\tilde{E})) = \Omega_\eta(\delta_N,\delta_{N'})$. □

Our main result connects the canonical stochastic differential 2-form and the Lie algebra structure of $\mathcal{G}$:

Theorem 5.6. *For $(N,N') \in \mathcal{H}^2$, one has:*

$$\Omega_\eta(N,N') = \hbar\frac{\mathcal{V}_{[\mathbf{N},\mathbf{N}']}\eta}{\eta}.$$

Proof. One has:

$$[N,N']^t=[N,N'](t)=(NN'-N'N)(t)$$
$$=N(N'(t))-N'(N(t))=N(N'^t)-N'(N^t)$$

and similarly

$$[N,N']^q=N(N'^q)-N'(N^q)\quad\text{and}\quad [N,N']^S=N(N'^S)-N'(N^S),$$

that is:

$$\Phi_{[N,N']}=N(\Phi_{N'})-N'(\Phi_N).$$

But, by Proposition 4.2, one may write:

$$\Omega(N,N')=-\hbar N'(n_N),\quad\text{where}\quad n_N=N^tE+N^qB-\Phi_N.$$

Therefore:

$$\Omega(N,N')=N'(\Phi_N-N^tE-N^qB)$$
$$=N'(\Phi_N)-N'(N^t)E-N^tN'(E)-N'(N^q)B-N^qN'(B)$$
$$=N'(\Phi_N)-N'(N^t)E-N^t(-E\frac{dN'^t}{dt}-B\frac{\partial N'^q}{\partial t}-\frac{\hbar}{4}\frac{d^2N'^t}{dt^2})$$
$$-N'(N^q)B-N^q(-\frac{B}{2}\frac{dN'^t}{dt}+\frac{\partial N'^q}{\partial t}),$$

according to Lemma 4.3.(i) applied to N'.
Whence:

$$2\Omega(N,N')=\Omega(N,N')-\Omega(N',N)$$
$$=N'(\Phi_N)-N(\Phi_{N'})-EN'(N^t)+EN(N'^t)$$
$$+E(N^t\frac{dN'^t}{dt}-N'^t\frac{dN^t}{dt})$$
$$+B(N^t\frac{\partial N'^q}{\partial t}-N'^t\frac{\partial N^q}{\partial t})+\frac{\hbar}{4}(N^t\frac{d^2N'^t}{dt^2}-N'^t\frac{d^2N^t}{dt^2})$$
$$-B(N'^t\frac{\partial N^q}{\partial t}+N'^q\frac{\partial N^q}{\partial q})+B(N^t\frac{\partial N'^q}{\partial t}+N^q\frac{\partial N'^q}{\partial q})$$
$$+\frac{B}{2}(N^q\frac{dN'^t}{dt}-N'^q\frac{dN^t}{dt})-N^q\frac{\partial N'^q}{\partial t}+N'^q\frac{\partial N^q}{\partial t}$$
$$=N'(\Phi_N)-N(\Phi_{N'})+2E[N,N']^t+\frac{\hbar}{4}\frac{d[N,N']^t}{dt}$$
$$+2BN(N'^q)-2BN'(N^q)+(N'^q\frac{\partial N^q}{\partial t}-N^q\frac{\partial N'^q}{\partial t})$$
$$+\frac{B}{2}(N^q\frac{dN'^t}{dt}-N'^q\frac{dN^t}{dt})+B(N'^q\frac{\partial N^q}{\partial q}-N^q\frac{\partial N'^q}{\partial q})$$
$$=N'(\Phi_N)-N(\Phi_{N'})+2E[N,N']^t+2B[N,N']^q+\frac{\hbar}{4}\frac{d[N,N']^t}{dt}$$
$$+(N'^q\frac{\partial N^q}{\partial t}-N^q\frac{\partial N'^q}{\partial t}),$$

where we have used, once more, that N^t and N'^t depend only upon t, and N^q and N'^q only upon q and t, and also Lemma 4.3.(ii), applied to N and N'. But:

$$\frac{\partial N^q}{\partial t}=\frac{\partial \Phi_N}{\partial q},\quad\text{and}\quad \frac{d^2N^t}{dt^2}=-\frac{4}{\hbar}\frac{\partial\Phi_N}{\partial t}$$

(this follows from (3),(4) and (8), and also from [10], Lemma 3.5, p. 318), whence:

$$\frac{d[N,N']^t}{dt}=N^t\frac{d^2N'^t}{dt^2}-N'^t\frac{d^2N^t}{dt^2}$$
$$=N^t(-\frac{4}{\hbar}\frac{\partial\Phi_{N'}}{\partial t})-N'^t(-\frac{4}{\hbar}\frac{\partial\Phi_N}{\partial t})$$

and

$$N'^q\frac{\partial N^q}{\partial t}-N^q\frac{\partial N'^q}{\partial t}=N'^q\frac{\partial\Phi_N}{\partial q}-N^q\frac{\partial\Phi_{N'}}{\partial q},$$

from which follows

$$\frac{\hbar}{4}\frac{d[N,N']^t}{dt}+(N'^q\frac{\partial N^q}{\partial t}-N^q\frac{\partial N'^q}{\partial t})$$
$$=-N^t\frac{\partial\Phi_{N'}}{\partial t}+N'^t\frac{\partial\Phi_N}{\partial t}+N'^q\frac{\partial\Phi_N}{\partial q}-N^q\frac{\partial\Phi_{N'}}{\partial q}$$
$$=N'(\Phi_N)-N(\Phi_{N'}),$$

as Φ_N and $\Phi_{N'}$ depend only upon q and t.

Therefore:

$$2\Omega(N,N')=2(N'(\Phi_N)-N(\Phi_{N'}))+2E[N,N']^t+2B[N,N']^q$$
$$=-2\Phi_{[N,N']}+2E[N,N']^t+2B[N,N']^q,$$

that is:

$$\Omega(N,N')=[N,N']^tE+[N,N']^qB-\Phi_{[N,N']}$$

and:

$$\Omega_\eta(N,N')=\theta_\eta(\Omega(N,N'))$$
$$=\theta_\eta([N,N']^tE+[N,N']^qB-\Phi_{[N,N']})$$
$$=[N,N']^t\tilde{E}+[N,N']^q\tilde{B}-\Phi_{[N,N']}$$
$$=\frac{\hbar}{\eta}([N,N']^t\frac{\partial\eta}{\partial t}+[N,N']^q\frac{\partial\eta}{\partial q}-\frac{\Phi_{[N,N']}}{\hbar}\eta)$$
$$=\hbar\frac{\mathcal{V}_{[\mathbf{N},\mathbf{N}']}\eta}{\eta}.$$

□

Corollary 5.7. *For $(N,N') \in \mathcal{H}_2 \times \mathcal{H}_2$, one has:*

$$\Omega_\eta(\delta_N, \delta_{N'}) = \hbar \frac{[N,N']\eta}{\eta}.$$

Proof. By the proof of Theorem 5.6, one has:

$$\Phi_{[N,N']} = N(\Phi_{N'}) - N'(\Phi_N) = 0,$$

as Φ_N and $\Phi_{N'}$ are constant (this can also be deduced from the commutation relations in §3, which imply that $[N,N'] \in Vect(N_1, N_2, N_4)$). Therefore:

$$\begin{aligned}\mathcal{V}_{[\mathbf{N},\mathbf{N}']}\eta &= [N,N']^t \frac{\partial \eta}{\partial t} + [N,N']^q \frac{\partial \eta}{\partial q} - \Phi_{[N,N']}\\ &= [N,N']^t \frac{\partial \eta}{\partial t} + [N,N']^q \frac{\partial \eta}{\partial q}\\ &= [N,N']\eta,\end{aligned}$$

as η depends only upon q and t. The result now follows from Theorem 5.6. □

Remark 5.8. The above Corollary is essentially equivalent to the formula used in the proof of Theorem 1, p. 402 in [13].

Theorem 5.9. *The differential* 2*-form* Ω_η *is closed,* i.e. $d\Omega_\eta = 0$.

Proof. As is well known ([4]), the 3-form $d\Omega_\eta$ is defined by the following formula, for all $(N,N',N'') \in \mathcal{H}^3$:

$$\begin{aligned}d\Omega_\eta(N,N',N'') &= N(\Omega_\eta(N',N'')) - N'(\Omega_\eta(N,N'')) + N''(\Omega_\eta(N,N'))\\ &\quad - \Omega_\eta([N,N'],N'') + \Omega_\eta([N,N''],N') - \Omega_\eta([N',N''],N).\end{aligned}$$

According to Theorem 5.6, the last three terms add up to

$$\begin{aligned}&\frac{\hbar}{\eta}[-\mathcal{V}_{[[N,N'],N'']}\eta + \mathcal{V}_{[[N,N''],N']}\eta - \mathcal{V}_{[[N',N''],N]}\eta]\\ &= \frac{\hbar}{\eta}\mathcal{V}_{[N,[N',N'']]+[N',[N'',N]]+[N'',[N,N']]}\eta\\ &= \frac{\hbar}{\eta}\mathcal{V}_0\eta \text{ (by the Jacobi identity)}\\ &= 0.\end{aligned}$$

Therefore, by the antisymmetry of Ω_η, one finds:

$$d\Omega_\eta(N,N',N'') = N(\Omega_\eta(N',N'')) + N'(\Omega_\eta(N'',N)) + N''(\Omega_\eta(N,N')).$$

But this is easily computed to be zero. □

Corollaries 5.3 and 5.7 and Theorem 5.9 together yield Theorem 1 of [7], *modulo* the correction of a misprint there: in the first line of "Théorème 1", one has to read "$\theta_\eta(\omega_{PC})$"instead of "$\theta_\eta(\omega)$".

6. An algebraic version of the Itô differential

The following two results provide, in a purely algebraic way, stochastic analogue of some classical formulas of Analytical Mechanics. Let us denote by D the *form* *total derivative along the Bernstein process associated with* η (cf. the introductio and [10, 11]):

$$D = \frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + \frac{\hbar}{2}\frac{\partial^2}{\partial q^2}.$$

Theorem 6.1. *For each* $N \in \mathcal{H}$, $\mathcal{L}_N(\omega_{PC}) = d\Phi_N$.

Proof. By the well-known formal properties of the Lie derivative (see for in stance [6],p. 654), one has:

$$\begin{aligned}\mathcal{L}_N(\omega_{PC}) &= \mathcal{L}_N(Edt + Bdq)\\ &= \mathcal{L}_N(E)dt + E\mathcal{L}_N(dt) + \mathcal{L}_N(B)dq + B\mathcal{L}_N(dq)\\ &= N^E dt + Ed(\mathcal{L}_N(t)) + N^B dq + Bd(\mathcal{L}_N(q))\\ &= N^E dt + EdN^t + N^B dq + BdN^q.\end{aligned}$$

But, using formulas (3), (4), (5) and (6), this is easily seen to equal:

$$-\hbar k_6 dt - 2k_5 dq + 2k_6 q dq = d(k_6 q^2 - 2k_5 q - k_6 \hbar t) = d\Phi_N.$$ □

We have established yet another coïncidence between our results and com putations using Itô's formula in [10].

Theorem 6.2. *For each* $N \in \mathcal{H}$,

$$\mathcal{L}_N(L) + L\frac{dN^t}{dt} = D\Phi_N.$$

Proof. One has:

$$\begin{aligned}\mathcal{L}_N(L) &= N(L) = N(\frac{1}{2}B^2) = BN(B) = BN^B = B(-k_4 B - 2k_5 + 2k_6 q - 2k_6 tB\\ &= -k_4 B^2 - 2k_5 B + 2k_6 qB - 2k_6 tB^2 = -\frac{B^2}{2}(2k_4 + 4k_6 t) + 2k_6 qB - 2k_5\\ &= -\frac{B^2}{2}\frac{d}{dt}(2k_6 t^2 + 2k_4 t + k_1) - k_6\hbar + B(2k_6 q - 2k_5) + \frac{\hbar}{2}(2k_6)\\ &= -L\frac{dN^t}{dt} + \frac{\partial \Phi_N}{\partial t} + B\frac{\partial \Phi_N}{\partial q} + \frac{\hbar}{2}\frac{\partial^2 \Phi_N}{\partial q^2} = -L\frac{dN^t}{dt} + D\Phi_N.\end{aligned}$$ □

By Theorems 6.1 and 6.2 are justified all the assertions made in [7], Theorem 2 and 3, *modulo* a misprint in "Théorème 3": instead of "$\tilde{B}$", read "B".

The probabilistic interpretation of all our results is founded on the observa tion that

$$\tilde{S}(q,t) = E_{q,t}(\int_t^T \omega_{PC})$$

where $E_{q,t}$ denotes the conditional expectation given that $z(t) = q$, and z is the Bernstein process of the Introduction. Using Definition 5.1 we have, more explicitly, under integrability condition

$$\tilde{S}(q,t) = E_{q,t}\left(\int_t^T L(Dz(\tau))d\tau\right) = E_{q,t}\left(\int_t^T \frac{1}{2}\tilde{B}^2(z(\tau),\tau)d\tau\right).$$

Of course, ω_{PC} when evaluated along the paths of $z(\tau)$, has to be interpreted in the Stratonovich sense. Theorem 6.2, for instance, is a stochastic condition of invariance of the Lagrangian L under the symmetry group built from the one of $(\mathcal{C}_1)$. It was originally found ([10, 11]) at the expense of long sessions of Itô calculus, both in the flat and in the Riemannian case ([12]).

In conclusion, let us observe that Euclidean Quantum Mechanics is in no way restricted to diffusion processes ([14]). So the geometrical methods introduced here should hold more generally.

7. Two interesting particular cases

7.1. Case 1: Scaled Brownian motion

Let us consider the generator N_4 (cf. (7)), let $\alpha \in \mathbf{R}$, let $U_\alpha = e^{\alpha\hat{N}_4}$ and let η be a solution of (1). Then, as

$$\hat{N}_4 = -\mathcal{V}_{N_4} = -2t\frac{\partial}{\partial t} - q\frac{\partial}{\partial q},$$

it is readily seen that $U_\alpha^{N_4} = e^{\alpha\hat{N}_4}$ maps $\eta(q,t)$ to $\eta(e^{-\alpha}q, e^{-2\alpha}t)$.

Moreover, $e^{\alpha N_4}$ maps (T,Q) to $(e^{2\alpha}T, e^{\alpha}Q)$, whence

$$z^\alpha(t) = e^\alpha z(e^{-2\alpha}t).$$

In the case $\eta = 1$, we find $\eta_\alpha = 1$. Applying $(\mathcal{B}_1)$ to η and w,we find that

$$z(t) = \sqrt{\hbar}w(t);$$

setting now $e^{-2\alpha} = \epsilon$, it appears that, whenever w is a Brownian motion,

$$w^\epsilon(t) \equiv_{def} \epsilon^{-\frac{1}{2}}w(\epsilon t)$$

is also one, for any $\epsilon > 0$. Thus have we (re)established the scaling invariance of the Brownian motion, as claimed in [7], p. 266, and in §2 above.

As a matter of fact, N_4 provides the complete collection of Bernstein processes satisfying such a property. Of course for $\eta \neq 1$, we obtain a transformation of the starting diffusion and not an invariance.

7.2. Case 2: Brownian bridge

Let us here set $\eta = 1$ and

$$N = -\frac{1}{2}N_6$$

whence

$$\hat{N} = -\frac{1}{2}\hat{N}_6 = \frac{1}{2}\mathcal{V}_{N_6} = t^2\frac{\partial}{\partial t} + qt\frac{\partial}{\partial q} - \frac{1}{2\hbar}(q^2 - \hbar t).$$

Let $\alpha \in \mathbf{R}$; then U_α^N maps $\eta(q,t)$ to

$$\frac{1}{\sqrt{1-\alpha t}}e^{-\frac{\alpha q^2}{2\hbar(1-\alpha t)}}\eta\left(\frac{q}{1-\alpha t}, \frac{t}{1-\alpha t}\right).$$

Also, $e^{\alpha N}$ maps (T,Q) to $(\frac{T}{1+\alpha T}, \frac{Q}{1+\alpha T})$ whence , in particular:

$$z^\alpha(t) = (1-\alpha t)z\left(\frac{t}{1-\alpha t}\right).$$

In the case $\eta = 1$, one has:

$$h_\alpha = \frac{\eta_\alpha}{\eta} = \eta_\alpha = U_\alpha^N(1) = \frac{1}{\sqrt{1-\alpha t}}e^{-\frac{\alpha q^2}{2\hbar(1-\alpha t)}},$$

and the drift associated with z^α is:

$$\tilde{B}^\alpha = \hbar\frac{\partial}{\partial q}(\ln\eta_\alpha) = \hbar\frac{\partial}{\partial q}\left(-\frac{1}{2}\ln(1-\alpha t) - \frac{\alpha q^2}{2\hbar(1-\alpha t)}\right) = -\frac{\alpha q}{1-\alpha t},$$

whence z^α satisfies the stochastic differential equation:

$$dz^\alpha(t) = \sqrt{\hbar}dw(t) - \frac{\alpha z^\alpha(t)}{1-\alpha t}dt,$$

and (for $\alpha > 0$), this defines the usual Brownian bridge, *i.e.* the standard Brownian motion $w(t)$ conditioned by $w(\alpha^{-1}) = 0$, often denoted by $w_{0,0}^{0,\alpha^{-1}}(t)$.

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Stochastic Methods in Financial Models