JOURNAL OF ALGEBRA 177, 847–869 (1995)

Isoclinism Classes and Commutativity Degrees of Finite Groups

Paul Lescot

8 Passage Montbrun, 75 014 Paris, France

Communicated by Walter Feit

Received October 27, 1994

INTRODUCTION

For G a finite group, let the *commutativity degree* $d(G)$ of G be defined by

$$d(G) = \frac{1}{|G|^2} |\{(x, y) \in G \times G \mid xy = yx\}|.$$

Obviously, G is abelian if and only if $d(G) = 1$; furthermore, the following results are known:

(1) If G is not abelian, then $d(G) \leq \frac{5}{8}$ [3, p. 1032; 6, p. 447; 7, Théorème 2].

(2) If $d(G) > \frac{1}{2}$, then G is nilpotent [7, Théorème 3].

(3) If $d(G) = \frac{1}{2}$ and G is not nilpotent, then $G/Z(G) \simeq \Sigma_3$ and $G' \simeq Z_3$ [8, Théorème 5].

(4) If $\frac{1}{2} < d(G) < 1$, then $d(G) \in \{\frac{1}{2}(1 + 1/4^n) \mid n \in \mathbf{N}, n \geq 1\}$ [11, p. 246].

All these results can be interpreted and even slightly improved upon by means of the concept of *isoclinism* first introduced by Philip Hall [4]. We shall begin by proving the most elementary properties of $d(G)$, then remind the reader of this notion. In the third section, we shall determine, up to isoclinism, all finite groups with $d(G) \geq \frac{1}{2}$; the above-mentioned results will follow. Then the computation of the "n-uple" commutativity degree will be reduced to the case $n = 2$; finally, we shall reprove a recent result of Leavitt, Sherman, and Walker [6].

0021-8693/95 $12.00

Let G be a finite group; we denote by $k(G)$ the number of conjugacy classes in G, by G' its derived group, by $Z(G)$ its center, and by a_G: $(G/Z(G))^2 \to G'$ the map defined by

$$\forall (x, y) \in G^2 \qquad a_G(xZ(G), yZ(G)) = [x, y] = x^{-1}y^{-1}xy$$

(it is easy to see that this definition makes sense).

1. PRELIMINARY RESULTS ON $d(G)$

LEMMA 1.1 [3, p. 1032; 6, p. 447; 7, Lemme 1]. $d(G) = k(G)/|G|$.

Proof. Let $C_1, \ldots, C_r$ be the conjugacy classes of G, and for $i \in \{1, \ldots, r\}$, let $x_i \in C_i$. For $y \in C_i$, $y = x_i^g$ for some $g \in G$, whence $C_G(y) = C_G(x_i^g) = C_G(x_i)^g$ and $|C_G(y)| = |C_G(x_i)|$. Therefore

$$\begin{aligned}
|G|^2 d(G) &= |\{(x, y) \in G \times G \mid xy = yx\}| \\
&= \sum_{x \in G} |C_G(x)| \\
&= \sum_{i=1}^{r} \sum_{x \in C_i} |C_G(x)| \\
&= \sum_{i=1}^{r} |G : C_G(x_i)|\,|C_G(x_i)| \\
&= \sum_{i=1}^{r} |G| \\
&= |G| r \\
&= |G| k(G).
\end{aligned}$$

The result follows. ∎

LEMMA 1.2 [7, Lemme 2; 11, p. 244]. *Whenever $d(G) > \frac{1}{4}$, one has*

$$|G'| \leq \frac{3}{4d(G) - 1}.$$

Proof. For $n \in \mathbf{N}$, let ρ_n be the number of irreducible characters of G that have degree n. The total number of irreducible characters of G is

$k(G)$ [5, p. 16, Corollary 2.5], and the sum of the squares of their degrees is $|G|$ [5, p. 16, Corollary 2.7], whence

$$k(G) = \sum_{i \geq 1} \rho_i$$
$$|G| = \sum_{i \geq 1} i^2 \rho_i.$$

Thus

$$\begin{aligned} |G| - \rho_1 &= \sum_{i \geq 2} i^2 \rho_i \\ &\geq 4 \sum_{i \geq 2} \rho_i \\ &= 4(k(G) - \rho_1) \\ &= 4(|G| d(G) - \rho_1) \end{aligned}$$

by Lemma 1.1. It follows that

$$d(G) \leq \frac{1}{4} + \frac{3}{4} \frac{\rho_1}{|G|}. \qquad (*)$$

But ρ_1 is the number of linear characters of G, i.e., the number of linear characters of the abelian group G/G'. Thus $\rho_1 = |G : G'|$; $(*)$ now becomes

$$d(G) \leq \frac{1}{4} + \frac{3}{4|G'|},$$

from which the assertion of the lemma follows. ∎

LEMMA 1.3. *Let G be a nonabelian p-group; then*

$$d(G) \leq \frac{p^2 + p - 1}{p^3}.$$

Proof. Let $|G| = p^n$, and $|Z(G)| = p^m$; then $m \leq n - 2$, else $G/Z(G)$ would have order 1 or p, hence be cyclic, and G would be abelian. One may write

$$\begin{aligned}
p^{2n}d(G) &= |G|^2 d(G) \\
&= \sum_{x \in G} |C_G(x)| \\
&= \sum_{x \in Z(G)} |C_G(x)| + \sum_{x \in G \setminus Z(G)} |C_G(x)| \\
&= p^m p^n + \sum_{x \in G \setminus Z(G)} |C_G(x)| \\
&\leq p^{m+n} + p^{n-1}(|G| - |Z(G)|) \\
&= p^{m+n} + p^{n-1}(p^n - p^m) \\
&= p^{2n-1} + p^{m+n-1}(p-1) \\
&\leq p^{2n-1} + p^{2n-3}(p-1) \\
&= p^{2n-3}(p^2 + p - 1),
\end{aligned}$$

whence the result. ∎

LEMMA 1.4 [2, pp. 175–176]. *Let G be a finite group, and N a normal subgroup of G, then*

$$d(G) \leq d(N) d\left(\frac{G}{N}\right).$$

If N is abelian and one has equality, then $N \subseteq Z(G)$.

Proof. Let us assume $xy = yx$; then

$$(xN)(yN) = (xy)N = (yx)N = (yN)(xN).$$

Thus

$$\forall x \in G \ \frac{C_G(x)N}{N} \subseteq C_{G/N}(xN).$$

Therefore

$$\begin{aligned}
|G|^2 d(G) &= \sum_{x \in G} |C_G(x)| \\
&= \sum_{S \in G/N} \sum_{x \in S} |C_G(x)| \\
&= \sum_{S \in G/N} \sum_{x \in S} \left| \frac{C_G(x)N}{N} \right| |C_N(x)| \\
&\leq \sum_{S \in G/N} \sum_{x \in S} |C_{G/N}(S)| \, |C_N(x)| \\
&= \sum_{S \in G/N} |C_{G/N}(S)| \sum_{x \in S} |\{ y \in N \mid xy = yx \}| \\
&= \sum_{S \in G/N} |C_{G/N}(S)| \sum_{y \in N} |C_G(y) \cap S|.
\end{aligned}$$

Let us suppose $S \cap C_G(y) \neq \varnothing$, and let $x_0 \in S \cap C_G(y)$; then $S = Nx_0$, whence

$$\begin{aligned}
S \cap C_G(y) &= Nx_0 \cap C_G(y) \\
&= (N \cap C_G(y))x_0 \\
&= C_N(y)x_0.
\end{aligned}$$

Therefore $S \cap C_G(y)$ is either empty, or a left coset of $C_N(y)$; in both cases one has

$$|S \cap C_G(y)| \leq |C_N(y)|$$

and

$$\begin{aligned}
|G|^2 d(G) &\leq \sum_{S \in G/N} |C_{G/N}(S)| \sum_{y \in N} |C_N(y)| \\
&= \left(\left| \frac{G}{N} \right|^2 d\left(\frac{G}{N} \right) \right) \left(|N|^2 d(N) \right) \\
&= |G|^2 d(N) d\left(\frac{G}{N} \right).
\end{aligned}$$

Hence $d(G) \leq d(N)d(G/N)$, and equality implies

$$\forall S \in \frac{G}{N} \; \forall y \in N \qquad S \cap C_G(y) \neq \varnothing,$$

i.e.,

$$\forall y \in N \qquad G = NC_G(y).$$

If N is abelian, this yields

$$\forall y \in N \qquad G = C_G(y),$$

i.e.,

$$N \subseteq Z(G). \quad \blacksquare$$

2. ISOCLINISMS BETWEEN GROUPS

We shall generally follow Hall [4], albeit using a more "categorical" language.

LEMMA 2.1. *Let G and H be two groups, and let φ: $G \to H$ be an isomorphism. Then φ induces isomorphisms φ_1: $G/Z(G) \to H/Z(H)$ (given by $\forall g \in G$ $\varphi_1(gZ(G)) = \varphi(g)Z(H)$) and φ_2: $G' \to H'$ (given by $\forall g \in G'$ $\varphi_2(g) = \varphi(g)$) such that the following diagram commutes:*

$$\begin{array}{ccc} \frac{G}{Z(G)} \times \frac{G}{Z(G)} & \xrightarrow{\varphi_1 \times \varphi_1} & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \\ \downarrow a_G & & \downarrow a_H \\ G' & \xrightarrow{\varphi_2} & H' \end{array} \qquad (**)$$

Proof. Let $(\alpha, \beta) \in (G/Z(G))^2$; one may write $\alpha = g_1Z(G)$ $(g_1 \in G)$ and $\beta = g_2Z(G)$ $(g_2 \in G)$, whence

$$\begin{aligned} [a_H \circ (\varphi_1 \times \varphi_1)](\alpha, \beta) &= a_H(\varphi_1(\alpha), \varphi_1(\beta)) \\ &= a_H(\varphi(g_1)Z(H), \varphi(g_2)Z(H)) \\ &= [\varphi(g_1), \varphi(g_2)] \\ &= \varphi([g_1, g_2]) \\ &= \varphi_2([g_1, g_2]) \\ &= \varphi_2(a_G(\alpha, \beta)) \\ &= [\varphi_2 \circ a_G](\alpha, \beta). \end{aligned}$$

Therefore $a_H \circ (\varphi_1 \times \varphi_1) = \varphi_2 \circ a_G$, as claimed. $\blacksquare$

This result suggests:

DEFINITION 2.2. Let G and H be two groups; a pair (φ_1, φ_2) is termed an isoclinism from G to H if:

(1) φ_1 is an isomorphism from $G/Z(G)$ to $H/Z(H)$;

(2) φ_2 is an isomorphism from G' to H';

(3) the diagram $(**)$ is commutative.

If there is an isoclinism from G to H, we shall say that G and H are isoclinic.

Clearly isoclinism is an equivalence relation between groups; we may restate Lemma 2.1 as:

LEMMA 2.3. *If G and H are isomorphic, then they are isoclinic.*

Basic for our purpose is

LEMMA 2.4. *Let G and H be two isoclinic finite groups; then $d(G) = d(H)$.*

Proof. Let (φ_1, φ_2) be an isoclinism from G to H; one has

$$\begin{aligned}\left|\frac{G}{Z(G)}\right|^2 d(G) &= \frac{1}{|Z(G)|^2}|G|^2 d(G)\\ &= \frac{1}{|Z(G)|^2}|\{(x,y) \in G \times G \mid xy = yx\}|\\ &= \frac{1}{|Z(G)|^2}|\{(x,y) \in G \times G \mid x^{-1}y^{-1}xy = 1\}|\\ &= \frac{1}{|Z(G)|^2}|\{(x,y) \in G \times G \mid a_G(xZ(G), yZ(G)) = 1\}|\\ &= \left|\left\{(\alpha,\beta) \in \left(\frac{G}{Z(G)}\right)^2 \middle| a_G(\alpha,\beta) = 1\right\}\right|.\end{aligned}$$

But this equals

$$\left|\left\{(\alpha,\beta) \in \left(\frac{G}{Z(G)}\right)^2 \middle| \varphi_2(a_G(\alpha,\beta)) = 1\right\}\right|$$

(because φ_2 is an isomorphism)

$$=\left|\left\{(\alpha,\beta)\in\left(\frac{G}{Z(G)}\right)^2\middle|a_H(\varphi_1(\alpha),\varphi_1(\beta))=1\right\}\right|$$

(because of condition (3) in the definition of isoclinism)

$$=\left|\left\{(\gamma,\delta)\in\left(\frac{H}{Z(H)}\right)^2\middle|a_H(\gamma,\delta)=1\right\}\right|$$

(because φ_1 is an isomorphism).

But, by the above reasoning applied to H in place of G, this expression equals $|H/Z(H)|^2 d(H)$, i.e.,

$$\left|\frac{G}{Z(G)}\right|^2 d(G)=\left|\frac{H}{Z(H)}\right|^2 d(H).$$

But $G/Z(G)$ and $H/Z(H)$ are isomorphic (via φ_1), hence $|G/Z(G)| = |H/Z(H)|$; the equality $d(G) = d(H)$ follows. ∎

PROPOSITION 2.5. *Let G be any group; then there is a group G_1 isoclinic to G and such that $Z(G_1)\subseteq G_1'$. If G is finite, so is any such G_1.*

Proof. Let F be a free group, and let $\pi\colon F\to G$ be a surjective homomorphism. We define

$$\phi\colon F\to G\times\frac{F}{F'}$$

by

$$\forall x\in F\qquad \phi(x)=(\pi(x),xF').$$

Let $T=\phi(F)$; it is clear that $Z(T)=\phi(A)$, where $A=\pi^{-1}(Z(G))$. $T/T'=\phi(F)/\phi(F)'$ is isomorphic to

$$\frac{F/\ker(\phi)}{(F/\ker(\phi))'}\simeq\frac{F}{F'\ker(\phi)};$$

but

$$\ker(\phi)=\ker(\pi)\cap F'\subseteq F';$$

thus

$$\frac{T}{T'} \simeq \frac{F}{F'}$$

is free abelian. $Z(T)/(Z(T) \cap T')$ is isomorphic to $Z(T)T'/T'$, hence to a subgroup of T/T'; therefore it is free abelian too. We can thus find a (free abelian) subgroup B of $Z(T)$ such that

$$Z(T) = B \times (Z(T) \cap T').$$

B, as a subgroup of $Z(T)$, is normal in T; we intend to show that

$$G_1 = \frac{T}{B}$$

satisfies our requirements.

Let us first define

$$\tau\colon \frac{G}{Z(G)} \to \frac{G_1}{Z(G_1)}$$

by

$$\tau(\alpha) = (\phi(f)B)Z(G_1),$$

where

$$\alpha = \pi(f)Z(G) \qquad (f \in F).$$

If $\alpha = \pi(f_1)Z(G) = \pi(f_2)Z(G)$, then $\pi(f_1^{-1}f_2) = \pi(f_1)^{-1}\pi(f_2) \in Z(G)$, hence $f_1^{-1}f_2 \in \pi^{-1}(Z(G)) = A$. Therefore

$$\phi(f_1)^{-1}\phi(f_2) = \phi(f_1^{-1}f_2) \in \phi(A) = Z(T),$$

from which it follows that

$$\begin{aligned}(\phi(f_1)B)^{-1}(\phi(f_2)B) &= \big(\phi(f_1)^{-1}\phi(f_2)\big)B \\ &\in \frac{Z(T)}{B} \\ &\subseteq Z\left(\frac{T}{B}\right) \\ &= Z(G_1)\end{aligned}$$

and $(\phi(f_1)B)Z(G_1) = (\phi(f_2)B)Z(G_1)$. We have proved that τ is well-defined; it is clearly a homomorphism of groups. Obviously,

$$\tau(G) = \frac{\phi(F)B/B}{Z(G_1)} = \frac{T/B}{Z(G_1)} = \frac{G_1}{Z(G_1)},$$

i.e., τ is surjective. Let $C/B = Z(G_1) = Z(T/B)$; then

$$[C,T] \subseteq T' \cap B = T' \cap Z(T) \cap B = \{1\},$$

i.e., $C \subseteq Z(T)$ and (the other inclusion being trivial)

$$Z\left(\frac{T}{B}\right) = \frac{Z(T)}{B}.$$

In particular

$$\begin{aligned} Z(G_1) &= \frac{Z(T)}{B} \\ &= \frac{B \times (Z(T) \cap T')}{B} \\ &\subseteq \frac{BT'}{B} \\ &= \left(\frac{T}{B}\right)' \\ &= G_1'. \end{aligned}$$

Let $\alpha = \pi(f)Z(G) \in \ker(\tau)$; then $\phi(f)B \in Z(G_1) = Z(T)/B$, i.e., $\phi(f) \in Z(T) = \phi(A)$, whence $f \in A \ker(\phi) \subseteq A \ker(\pi)$. It follows that

$$\pi(f) \in \pi(A) \subseteq Z(G)$$

and

$$\alpha = \pi(f)Z(G) = 1_{G/Z(G)},$$

i.e., τ is injective, therefore an isomorphism.

Obviously,

$$\begin{aligned} G_1' &= \left(\frac{T}{B}\right)' \\ &= \frac{T'B}{B} \\ &\simeq \frac{T'}{T' \cap B} \\ &\simeq T' \\ &= \phi(F)' \\ &= \pi(F)' \times \{1\} \\ &= G' \times \{1\} \\ &\simeq G', \end{aligned}$$

all the isomorphisms being *canonical*. We have in fact shown that

$$\sigma : G' \to G_1'$$

defined by

$$\forall x \in G' \qquad \sigma(x) = (x, 1)B$$

is an isomorphism. We shall establish that (τ, σ) is an isoclinism from G to G_1, thereby completing the proof of the proposition. It is now clearly enough to check condition (3) in Definition 2.2. Then let $(\alpha, \beta) \in (G/Z(G))^2$, with $\alpha = \pi(f_1)Z(G)(f_1 \in F)$ and $\beta = \pi(f_2)Z(G)(f_2 \in F)$. One has

$$\begin{aligned} \left[a_{G_1} \circ (\tau \times \tau)\right](\alpha, \beta) &= a_{G_1}(\tau(\alpha), \tau(\beta)) \\ &= a_{G_1}((\phi(f_1)B)Z(G_1), (\phi(f_2)B)Z(G_1)) \\ &= [\phi(f_1)B, \phi(f_2)B] \\ &= [\phi(f_1), \phi(f_2)]B \\ &= ([\pi(f_1), \pi(f_2)], 1)B \\ &= \sigma([\pi(f_1), \pi(f_2)]) \\ &= \sigma(a_G(\alpha, \beta)) \\ &= [\sigma \circ a_G](\alpha, \beta). \end{aligned}$$

Thus $a_{G_1} \circ (\tau \times \tau) = \sigma \circ a_G$, as claimed.

If G is finite, so are $G/Z(G)$ and hence $G_1/Z(G_1) \simeq G/Z(G)$; but $Z(G_1) \subseteq G_1'$, so $|G_1 : G_1'|$ is finite. But $G_1' \simeq G'$ is finite, so G_1 is too. ∎

LEMMA 2.6. *Let S be a non-abelian simple group; then any group G isoclinic to S is isomorphic to $S \times A$ for some abelian group A.*

Proof. One has $G' \simeq S' \simeq S$, whence $G' \cap Z(G) \subseteq Z(G') = \{1\}$. But $G/Z(G) \simeq S/Z(S) \simeq S$ is perfect, hence $G = G'Z(G) = G' \times Z(G) \simeq S \times Z(G)$. The result follows, with $A = Z(G)$. ∎

LEMMA 2.7. *Let G be a group, and H be a subgroup of G such that $G = HZ(G)$; then G and H are isoclinic. The converse is true if H is finite.*

Proof. If $G = HZ(G)$, then $Z(H)$ centralizes H and $Z(G)$, hence centralizes G, thus $Z(H) \subseteq H \cap Z(G) \subseteq Z(H)$ and

$$\frac{H}{Z(H)} = \frac{H}{Z(G) \cap H} \simeq \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)},$$

the isomorphism i_1: $H/Z(H) \to G/Z(G)$ being induced by the inclusion i: $H \to G$.

Furthermore, let $(x, y) \in G^2$; then $x = h_1 z_1$ $(h_1 \in H,\ z_1 \in Z(G))$ and $y = h_2 z_2$ $(h_2 \in H,\ z_2 \in Z(G))$, whence $[x, y] = [h_1, h_2] \in H'$ and $G' = H'$. We have actually proved that $(i_1, \mathbf{1}_{G'})$ is an isoclinism from H to G.

Conversely, if H is isoclinic to G and is finite, then $G/Z(G) \simeq H/Z(H)$ is also finite. But

$$\left|\frac{G}{Z(G)}\right| \geq \left|\frac{HZ(G)}{Z(G)}\right| = \left|\frac{H}{H \cap Z(G)}\right| = \left|\frac{H}{Z(H)}\right| \left|\frac{Z(H)}{H \cap Z(G)}\right| \geq \left|\frac{H}{Z(H)}\right| = \left|\frac{G}{Z(G)}\right|.$$

Thus one has equality all along, and so $G = HZ(G)$. ∎

LEMMA 2.8. *Let G and H be two isoclinic finite groups; then*

$$|G' \cap Z(G)| = |H' \cap Z(H)|.$$

Proof. $G/Z(G)$ and $H/Z(H)$ are isomorphic, hence so are $(G/Z(G))'$ and $(H/Z(H))'$; but

$$\left(\frac{G}{Z(G)}\right)' = \frac{G'Z(G)}{Z(G)} \simeq \frac{G'}{G' \cap Z(G)}$$

and similarly

$$\left(\frac{H}{Z(H)}\right)' \simeq \frac{H'}{H' \cap Z(H)}$$

Therefore

$$\begin{aligned}\frac{|G'|}{|G' \cap Z(G)|} &= \left|\frac{G'}{G' \cap Z(G)}\right| \\ &= \left|\left(\frac{G}{Z(G)}\right)'\right| \\ &= \left|\left(\frac{H}{Z(H)}\right)'\right| \\ &= \left|\frac{H'}{H' \cap Z(H)}\right| \\ &= \frac{|H'|}{|H' \cap Z(H)|}.\end{aligned}$$

But G' and H' are isomorphic, whence $|G'| = |H'|$ and the result follows. ∎

We can use the previous lemma to give a new proof of one of Rusin's results:

COROLLARY 2.9 [11, p. 242]. *Let G be a finite group such that $G' \cap Z(G) = \{1\}$; then there is a finite group K such that $d(K) = d(G)$, $K' \simeq G'$, and $Z(K) = \{1\}$.*

Proof. By Proposition 2.5, there is a finite group K isoclinic to G and such that $Z(K) \subseteq K'$. Then

$$|Z(K)| = |Z(K) \cap K'| = |Z(G) \cap G'| = |\{1\}| = 1,$$

by Lemma 2.8 and the hypothesis on G, i.e., $Z(K) = \{1\}$. Now the isoclinism between K and G implies $K' \simeq G'$ by definition, and $d(K) = d(G)$ by Lemma 2.4. ∎

3. THE CASE $d(G) \geq \frac{1}{2}$

THEOREM 3.1. *Let G be a finite group such that $d(G) \geq \frac{1}{2}$; then G is isoclinic to $\{1\}$, to an extraspecial 2-group, or to Σ_3.*

Proof. By Proposition 2.5, there is a group H isoclinic to G such that $Z(H) \subseteq H'$; Lemma 2.4 now implies $d(H) = d(G) \geq \frac{1}{2}$. We may therefore assume that $Z(G) \subseteq G'$. Lemma 1.2 yields the bound

$$|G'| \leq \frac{3}{4d(G) - 1} \leq \frac{3}{4 \cdot \frac{1}{2} - 1} = 3.$$

Let us now consider two cases:

(1) $Z(G) \subset G'$; then $Z(G) = \{1\}$ (because $|G'| \in \{1, 2, 3\}$). Let

$$E = \{g \in G \mid |G : C_G(g)| = 2\},$$

let $n = |G|$, and let $m = |E|$. Clearly there are exactly $n - m - 1$ elements g of G such that $|G : C_G(g)| \geq 3$, whence

$$\begin{aligned} \frac{n^2}{2} &\leq |G|^2 d(G) = \sum_{x \in G} |C_G(x)| \\ &= n + m \cdot \frac{n}{2} + \sum_{x \in G; |G:C_G(x)| \geq 3} |C_G(x)| \\ &\leq n + \frac{mn}{2} + (n - m - 1)\frac{n}{3} \\ &= \frac{2n}{3} + \frac{mn}{6} + \frac{n^2}{3}, \end{aligned}$$

i.e., $m \geq n - 4$. If $n < 10$, the condition $\{1\} = Z(G) \subset G'$ forces $G \simeq \Sigma_3$, and we are done. We may therefore assume $n \geq 10$, and thus $m \geq n - 4 \geq n/2 + 1$. Now let $g \in E$; by definition of E, $|G : C_G(g)| = 2$, therefore, as is well known, $C_G(g) \lhd G$ and $G/C_G(g)$ has order 2, thus is abelian, i.e., $G' \subseteq C_G(g)$, and $g \in C_G(G')$. We have shown that

$$E \subseteq C_G(G').$$

If follows that $|C_G(G')| \geq |E| = m > n/2$, i.e., $|G : C_G(G')| < 2$, that is, $G = C_G(G')$, and $G' \subseteq Z(G)$, a contradiction.

(2) $Z(G) = G'$; then G is nilpotent. For each prime p, let G_p be the Sylow p-subgroup of G. If G_p is non-abelian, one has

$$d(G) \leq d(G_p) d\left(\frac{G}{G_p}\right) \leq d(G_p)$$

(by Lemma 1.4), whence

$$\frac{1}{2} \leq d(G) \leq d(G_p) \leq \frac{p^2 + p - 1}{p^3}$$

(by Lemma 1.3), that is, $p^3 \leq 2p^2 + 2p - 2 < 2p(p+1)$ and $p^2 < 2p + 2$, hence $(p-1)^2 < 3 < 4$, i.e., $p < 3$, thus $p = 2$. Therefore G is the direct product of its Sylow 2-subgroup G_2 and an abelian 2-group $G_{2'}$. Then $Z(G) = Z(G_2) \times G_{2'}$ and $G' = G_2'$, thus $G_{2'} = \{1\}$, i.e., G is a 2-group. Now $|G'| \leq 3$ forces $Z(G) = G' = \{1\}$ or $|G'| = 2$; in the first case $G = \{1\}$, and in the second case $Z(G) = G'$ has order 2, i.e., G is an extraspecial 2-group. ∎

COROLLARY 3.2. *If $d(G) \geq \frac{1}{2}$, one of the following holds*:

(1) *G is abelian and $d(G) = 1$.*

(2) *$G/Z(G)$ is elementary abelian of order 2^{2m} for some $m \geq 1$, $|G'| = 2$, and $d(G) = \frac{1}{2}(1 + 1/4^m) \leq \frac{5}{8}$.*

(3) *G is isoclinic to Σ_3 and $d(G) = \frac{1}{2}$.*

Remark. The results (1) to (4) in the Introduction follow immediately.

Proof. By Theorem 3.1, G is isoclinic to $\{1\}$, an extraspecial 2-group, or Σ_3; the first case leads to (1), and the third one to (3). We may therefore restrict ourselves to the case in which G is isoclinic to an extraspecial 2-group, and even assume that it *is* one. In this case it is well known that, after one identifies G' with the additive group of the field $\mathbf{F}_2$, a_G becomes a non-degenerate alternating bilinear form on $G/Z(G) \simeq \mathbf{F}_2^n$. Therefore $n = 2m$ for m the dimension of a maximal totally isotropic subspace of $G/Z(G)$ for a_G; furthermore, for $x \in G \setminus Z(G)$, $C_G(x)/Z(G)$ is the kernel of the nonzero linear form $a_G(xZ(G), \cdot\,)$, hence has order $\frac{1}{2} \cdot 2^n$, and $|C_G(x)| = 2^n = 2^{2m}$. It appears that

$$\begin{aligned} 2^{4m+2}d(G) &= |G|^2 d(G) \\ &= \sum_{x \in G} |C_G(x)| \\ &= |Z(G)|\,|G| + (|G| - |Z(G)|)2^{2m} \\ &= 2 \cdot 2^{2m+1} + (2^{2m+1} - 2)2^{2m} \\ &= 2^{4m+1} + 2^{2m+1} \\ &= \tfrac{1}{2} \cdot 2^{4m+2}\left(1 + \frac{1}{4^m}\right). \end{aligned}$$

Therefore $d(G) = \frac{1}{2}(1 + 1/4^m)$. ∎

Let us define, for $r \in \mathbf{N}$, $r \geq 1$, a group G_r by generators and relations:

$$G_r = \langle \sigma, \tau \mid \sigma^3 = \tau^{2^r} = 1, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle.$$

It is easily seen that G_r is isoclinic to Σ_3 (in particular $d(G_r) = \frac{1}{2}$), that $Z(G_r) = \langle \tau^2 \rangle$, and that $G'_r = \langle \sigma \rangle$; clearly, $G_1 \simeq \Sigma_3$.

THEOREM 3.3. *Let G be a (nonnilpotent) finite group with $d(G) = \frac{1}{2}$ ("groupe non-nilpotent minimal" in the terminology of* [8, 9]); *then G has a subgroup H isomorphic to some G_r ($r \geq 1$) such that $G = HZ(G)$.*

Proof. By Corollary 3.2, $|G'| = 3$; let $G' = \langle \sigma \rangle$. There is an element x of G that does not centralize G' (else G' would be central in G and G nilpotent, contradicting Corollary 3.2(3)); necessarily $x^{-1}\sigma x = \sigma^{-1}$. Let $\omega(x) = 2^r(2n + 1)$; $z = x^{2n+1}$ has order 2^r and inverts σ, therefore $H = \langle z, \sigma \rangle$ is isomorphic to G_r. But G is isoclinic to Σ_3, which is isoclinic to G_r, itself isomorphic (hence, by Lemma 2.3, isoclinic) to H. Lemma 2.7 now yields $G = HZ(G)$. ∎

If G contains $H \simeq G_r$, then $3 = |H'| \leq |G'| = 3$, whence $G' = H'$. Combined with the proof of Theorem 3.3, this makes clear that the smallest r such that the conclusion of the said theorem be true equals the smallest r such that G have a subgroup isomorphic to G_r, and equals the smallest r such that some element of G with order 2^r not centralize G'; we shall denote this integer by $r_0(G)$.

COROLLARY 3.4. *In the context of Theorem 3.3, let us suppose that $r_0(G) \leq 2$; then $G \simeq G_{r_0(G)} \times A$ for some abelian group A.*

Proof. Let x be a 2-element of G of minimal order among those inverting σ (where, as above, σ is a generator of G'); then $xZ(G)$ is a 2-element in $G/Z(G) \simeq \Sigma_3$, whence $u = x^2 \in Z(G)$. Clearly $\omega(x) = 2^r \leq 4$, hence $\omega(u) \leq 2$. If $u = 1$ then $\langle x, \sigma \rangle \simeq \Sigma_3$ and $\langle x, \sigma \rangle \cap Z(G) \subseteq Z(\langle x, \sigma \rangle) = \{1\}$, whence

$$G \simeq \langle x, \sigma \rangle \times Z(G) \simeq \Sigma_3 \times Z(G) \simeq G_1 \times Z(G)$$

and we are done. Then let u have order 2, and thus x order 4; if u were the square of a (2-)element y of $Z(G)$, we might replace x by xy^{-1} of order 2, a contradiction. Let $Z(G) = A \times B$, where A is a 2-group and B a $2'$-group; then u is an element of A that is not the square of any element of A, i.e., $u \in A \setminus \operatorname{Frat}(A)$. Thus there exists a subgroup M of A, of index 2, such that $u \notin M$, hence $A = M\langle u \rangle$ and $A = M \times \langle u \rangle$ because $|\langle u \rangle| = \omega(u) = 2$. Thus

$$Z(G) = A \times B = \langle u \rangle \times (M \times B)$$

and

$$\begin{aligned} G &= \langle x, \sigma, Z(G) \rangle = \langle x, \sigma, u, M \times B \rangle = \langle x, \sigma, M \times B \rangle \\ &= \langle x, \sigma \rangle \times (M \times B) \simeq G_r \times (M \times B). \end{aligned}$$ ∎

Remark. In [9, Théorème 1]. I somewhat rashly asserted that the conclusion of Corollary 3.4 always holds; unfortunately there was a flaw in my proof: on page 200, right-hand column, line 2 from below, "donc" is correct only when u has order 2.

4. MULTIPLE COMMUTATIVITY DEGREE

As in [3], we define the "nth-commutativity degree" of a finite group G ($n \in \mathbf{N}$) by

$$d_n(G) = \frac{1}{|G|^{n+1}} \left| \left\{ (x_1, \ldots, x_{n+1}) \in G^{n+1} \,\middle|\, \forall (i,j) \in \{1, \ldots, n+1\}^2, \; x_i x_j = x_j x_i \right\} \right|.$$

Clearly $d_0(G) = 1$, $d_1(G) = d(G)$, and one has:

LEMMA 4.1. *Let $\{g_1, \ldots, g_{k(G)}\}$ be a system of representatives for the conjugacy classes of G; then*

$$\forall n \in \mathbf{N} \qquad d_{n+1}(G) = \frac{1}{|G|} \sum_{i=1}^{k(G)} \frac{1}{|\mathscr{C}\ell_G(g_i)|^n} d_n(C_G(g_i)).$$

Proof.

$$
\begin{aligned}
&|G|^{n+2}d_{n+1}(G)\\
&=\left|\left\{(x_1,\dots,x_{n+2})\in G^{n+2}\,|\,\forall(i,j)\in\{1,\dots,n+2\}^2, x_ix_j=x_jx_i\right\}\right|\\
&=\sum_{x\in G}\left|\left\{(x_1,\dots,x_{n+1})\in C_G(x)^{n+1}\,\middle|\,\forall(i,j)\in\{1,\dots,n+1\}^2, x_ix_j=x_jx_i\right\}\right|\\
&=\sum_{x\in G}|C_G(x)|^{n+1}d_n(C_G(x))\\
&=\sum_{i=1}^{k(G)}|G:C_G(g_i)|\,|C_G(g_i)|^{n+1}d_n(C_G(g_i))\\
&=\sum_{i=1}^{k(G)}|\mathscr{C}\ell_G(g_i)|\left(\frac{|G|}{|\mathscr{C}\ell_G(g_i)|}\right)^{n+1}d_n(C_G(g_i))\\
&=|G|^{n+1}\sum_{i=1}^{k(G)}\frac{1}{|\mathscr{C}\ell_G(g_i)|^n}d_n(C_G(g_i)).
\end{aligned}
$$

Whence the result. ▮

LEMMA 4.2. *If G and H are isoclinic, then*

$$\forall n\in\mathbf{N}\qquad d_n(G)=d_n(H).$$

Proof. By the same reasoning as that in Lemma 2, one has

$$\left|\frac{G}{Z(G)}\right|^{n+1}d_n(G)=\left|\left\{(\alpha_1,\dots,\alpha_{n+1})\in\left(\frac{G}{Z(G)}\right)^{n+1}\,\middle|\,\forall(i,j)\in\{1,\dots,n\}^2, a_G(\alpha_i,\alpha_j)=1\right\}\right|$$

and one concludes as above. ▮

THEOREM 4.3 [9, p. 202]. *Let G be a non-abelian group; then*

$$\forall n \in \mathbf{N} \qquad d_n(G) \le \frac{3 \cdot 2^n - 1}{2^{2n+1}}. \qquad (***)$$

For a given $n \ge 1$, equality in $(***)$ holds if and only if G is isoclinic to $\mathscr{Q}_8$.

Proof. We use induction on n: for $n = 0$ the inequality is obvious. Let us assume $d_n(G) \le (3 \cdot 2^n - 1)/2^{2n+1}$ and then use Lemma 4.1, choosing the g_i in such a way that $Z(G) = \{g_1, \dots, g_{|Z(G)|}\}$. We get

$$\begin{aligned} d_{n+1}(G) &= \frac{1}{|G|} \sum_{i=1}^{k(G)} \frac{1}{|\mathscr{C}\ell_G(g_i)|^n} d_n(C_G(g_i)) \\ &= \frac{|Z(G)|}{|G|} d_n(G) + \frac{1}{|G|} \sum_{i=|Z(G)|+1}^{k(G)} \frac{1}{|\mathscr{C}\ell_G(g_i)|^n} d_n(C_G(g_i)) \\ &\le \frac{1}{|G:Z(G)|} \frac{3 \cdot 2^n - 1}{2^{2n+1}} + \frac{1}{|G|} \frac{1}{2^n} (k(G) - |Z(G)|). \end{aligned}$$

But $k(G) = |G| d(G) \le \frac{5}{8}|G|$ by Corollary 3.2, and $|G : Z(G)| \ge 4$ because $G/Z(G)$ is not cyclic. Therefore

$$\begin{aligned} d_{n+1}(G) &\le \frac{1}{|G:Z(G)|} \frac{3 \cdot 2^n - 1}{2^{2n+1}} + \frac{1}{2^n}\left(\frac{5}{8} - \frac{1}{|G:Z(G)|}\right) \\ &= \frac{5}{8 \cdot 2^n} + \frac{1}{|G:Z(G)|} \frac{1}{2^{2n+3}} (3 \cdot 2^{n+2} - 4 - 2^{n+3}) \\ &= \frac{5}{2^{n+3}} + \frac{2^{n+2} - 4}{2^{2n+3}|G:Z(G)|} \\ &\le \frac{5}{2^{n+3}} + \frac{2^{n+2} - 4}{4 \cdot 2^{2n+3}} \\ &= \frac{5 \cdot 2^n + 2^n - 1}{2^{2n+3}} \\ &= \frac{3 \cdot 2^{n+1} - 1}{2^{2(n+1)+1}} \end{aligned}$$

and the inequality is proved at rank $n + 1$. This computation also makes clear that, for a given $n \ge 1$, equality at rank $n + 1$ implies equality at rank n. Therefore, if, for a given $n \ge 1$,

$$d_n(G) = \frac{3 \cdot 2^n - 1}{2^{2n+1}},$$

then $d(G) = d_1(G) = \frac{5}{8}$. Therefore, by Corollary 3.2, G is isoclinic to an extraspecial group of order 2^{2m+1}, where $\frac{1}{2}(1 + 1/4^m) = \frac{5}{8}$, that is, $m = 1$. G is therefore isoclinic either to the dihedral group of order 8 $\mathcal{D}_8$, or to the quaternion group $\mathcal{Q}_8$. But these two groups are isoclinic, hence G is isoclinic to $\mathcal{Q}_8$ in any case.

Conversely, if G is isoclinic to $\mathcal{Q}_8$, one may assume, thanks to Lemma 4.2, that $G = \mathcal{Q}_8$. Then $|G : Z(G)| = 4$, $d_0(G) = 1$, and $C_G(x)$ is abelian and of index 2 in G for all $x \in G \setminus Z(G)$; therefore, in the previous computation, equality holds all along, which permits us to prove by induction on n that

$$\forall n \in \mathbf{N} \qquad d_n(\mathcal{Q}_8) = \frac{3 \cdot 2^n - 1}{2^{2n+1}}. \qquad \blacksquare$$

5. A THEOREM OF LEAVITT, SHERMAN, AND WALKER

In [6], using a result of Curzio, Longobardo, and Maj [1, Theorem 3], Leavitt, Sherman, and Walker have established:

THEOREM 5.1. *The following conditions on a finite group G are equivalent*:

(1) $|G'| \leq 2$.

(2) $\forall C \in \mathcal{C}\ell(G)$, $|C| \in \{1, 2\}$.

(3) $\forall x \in G$, $|C_G(x)| \in \{|G|, |G|/2\}$.

(4) $\forall (x, y, z) \in G^3$, $xyz \in \{yxz, yzx, xzy, zxy, zyx\}$ (*i.e.*, *G is "3-rewriteable"*).

(5) $d(G) > \frac{1}{2}$.

We intend to give a new proof of this result, using the tools from our first two paragraphs.

Proof. (1) $\Rightarrow$ (2) Let $C = \mathcal{C}\ell_G(x)$; then each $c \in C$ can be written $c = y^{-1}xy$, whence $c = x(x^{-1}y^{-1}xy) \in xG'$ and $C \subseteq xG'$. Therefore

$$|C| \leq |xG'| = |G'| \leq 2.$$

(2) $\Rightarrow$ (3) Clear because

$$\forall x \in G \qquad |\mathcal{C}\ell_G(x)| = |G : C_G(x)|.$$

(3) $\Rightarrow$ (4) Let $(x, y, z) \in G^3$; then $C_G(y)$ has at most two right cosets in G. If $x \in C_G(y)$ then $xyz = yxz$; if $z \in C_G(y)$ then $xyz = xzy$. We

may therefore assume that $xC_G(y) = G \setminus C_G(y) = zC_G(y)$, whence $z^{-1}x \in C_G(y)$. If now $xyz \neq yzx$ then $yzC_G(x) = G \setminus C_G(x)$, whence $y \in yzC_G(x)$, i.e., $z \in C_G(x)$. It follows that

$$z^{-1}xy = yz^{-1}x = yxz^{-1},$$

that is, $xyz = zyx$.

(4) $\Rightarrow$ (5) Let us apply the hypothesis to a triple (x, y, x^2); we get

$$\forall (x, y) \in G \times G \qquad xyx^2 \in \{yxx^2, yx^2x, xx^2y, x^2xy, x^2yx\},$$

whence $xy = yx$ or $yx^2 = x^2y$, thus in any case $x^2 \in Z(G)$. Therefore $G/Z(G)$ is a group in which every element has square 1, i.e., an elementary abelian 2-group. By Proposition 2.5, there is a group H isoclinic to G and such that $Z(H) \subseteq H'$.

$$\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$$

is then elementary abelian, whence $H' \subseteq Z(H)$ and

$$H' = Z(H).$$

For each $h \in H$, ϕ_h: $H/Z(H) \to H'$ defined by

$$\forall u \in H \qquad \phi_h(uZ(H)) = [h, u]$$

(i.e., $\phi_h = a_H(hZ(H), \cdot)$) is a morphism of groups from $H/Z(H)$ to H', therefore, for $u \in H$, $[h, u] = \phi_h(uZ(H))$ has order 2. H', being abelian and generated by elements of order 2, is therefore an elementary abelian 2-group.

We now note that condition (4) can be written

$$\forall (x, y, z) \in G^3 \qquad \begin{aligned} &[x, y] = 1 \quad \text{or} \\ &[y, z] = 1 \quad \text{or} \\ &[x, yz] = 1 \quad \text{or} \\ &[xy, z] = 1 \quad \text{or} \\ &xyz = zyx, \end{aligned}$$

i.e.,

$$\forall (x, y, z) \in G^3 \qquad \begin{aligned} &[x, y] = 1 \quad \text{or} \\ &[y, z] = 1 \quad \text{or} \\ &[x, z][x, y]^z = 1 \quad \text{or} \\ &[x, z]^y[y, z] = 1 \quad \text{or} \\ &[x, y][x, z]^y[y, z] = 1. \end{aligned}$$

It follows that, because $G' \subseteq Z(G)$ (and consequently $H' \subseteq Z(H)$), condition (4) holds for H too. Now, the reasoning of [1, pp. 141–142], gives us

$$|H'| \leq 2.$$

Thence $Z(H) = H'$ has order 1 or 2, i.e., $H = \{1\}$ or H is an extraspecial 2-group. The computation made during the proof of Corollary 3.2 now proves that

$$d(H) \in \{1\} \cup \left\{ \frac{1}{2}\left(1 + \frac{1}{4^n}\right) \middle| n \in \mathbf{N}, n \geq 1 \right\}$$

and thus $d(G) = d(H) > \frac{1}{2}$.

(5) $\Rightarrow$ (1) By Lemma 1.2,

$$|G'| \leq \frac{3}{4d(G) - 1} < \frac{3}{\frac{4}{2} - 1} = 3,$$

thus $|G'| \leq 2$. ∎

ACKNOWLEDGMENTS

It is an agreeable duty to thank Jean-Louis Nicolas for sending me a copy of his seminar report [10]; my notes ([7–9] and the present paper all stem from his questions 7° and 8°) [10, p. 5]. I am also under a heavy debt to John D. Dixon for many stimulating discussions during the 1994 ICM.

REFERENCES

1. M. Curzio, P. Longobardi, and M. Maj, Su di un problema combinatorio in teoria dei gruppi, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **74** (1983), 136–142.
2. P. X. Gallagher, The number of conjugacy classes in a finite group, *Math. Z.* **118** (1970), 175–179.
3. W. H. Gustafson, What is the probability that two group elements commute? *Amer. Math. Monthly* **80** (1973), 1031–1034.
4. P. Hall, The classification of prime-power groups, *J. Reine Angew. Math.* **182** (1940), 130–141.
5. I. M. Isaacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.
6. J. L. Leavitt, G. J. Sherman, and M. E. Walker, Rewriteability in finite groups, *Amer. Math. Monthly* **99** (1992), 446–452.
7. P. Lescot, Sur certains groupes finis, *Rev. Math. Spéciales* **8** (1987), 276–277.
8. P. Lescot, Degré de commutativité et structure d'un groupe fini, *Rev. Math. Spéciales* **8** (1988), 276–279.

9. P. Lescot, Degré de commutativité et structure d'un groupe fini (2), *Rev. Math. Spéciales* **4** (1989), 200–202.
10. J. L. Nicolas, Statistiques sur le groupe symétrique, *Sém. Delange-Pisot-Poitou* **13** (1971–1972), G2-01–G2-06.
11. D. J. Rusin, What is the probability that two elements of a finite group commute? *Pacific J. Math.* **82**, No. 1 (1979), 237–247.