A CENTRAL LIMIT THEOREM FOR STATIONARY RANDOM FIELDS

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Abstract

This paper establishes a central limit theorem and an invariance principle for a wide class of stationary random fields under natural and easily verifiable conditions. More precisely, we deal with random fields of the form $X_k = g\left(\varepsilon_{k-s}, s \in \mathbb{Z}^d\right)$, $k \in \mathbb{Z}^d$, where $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ are i.i.d random variables and g is a measurable function. Such kind of spatial processes provides a general framework for stationary ergodic random fields. Under a short-range dependence condition, we show that the central limit theorem holds without any assumption on the underlying domain on which the process is observed. A limit theorem for the sample auto-covariance function is also established.

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1 Introduction

Central limit theory plays a fundamental role in statistical inference of random fields. There have been a substantial literature for central limit theorems

of random fields under various dependence conditions. See [1], [2], [3], [4], [6], [7], [14], [16], [21], [22], [23], [24], [25], [26], among others. However, many of them require that the underlying random fields have very special structures such as Gaussian, linear, Markovian or strong mixing of various types. In applications those structural assumptions can be violated, or not easily verifiable.

In this paper we consider stationary random fields which are viewed as nonlinear transforms of independent and identically distributed (iid) random variables. Based on that representation we introduce dependence measures and establish a central limit theorem and an invariance principle. We assume that the random field $(X_i)_{i\in\mathbb{Z}^d}$ has the form

$$X_i = g\left(\varepsilon_{i-s};\, s\in\mathbb{Z}^d\right), \quad i\in\mathbb{Z}^d, \tag{1}$$

where $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ are iid random variables and g is a measurable function. In the one-dimensional case ($d=1$) the model (1) is well known and includes linear as well as many widely used nonlinear time series models as special cases. In Section 2 based on (1) we shall introduce dependence measures. It turns out that, with our dependence measure, central limit theorems and moment inequalities can be established in a very elegant and natural way.

The rest of the paper is organized as follows. In Section 3 we present a central limit theorem and an invariance principle for

$$S_\Gamma = \sum_{i\in\Gamma} X_i,$$

where Γ is a finite subset of $\mathbb{Z}^d$ which grows to infinity. The proof of our Theorem 1 is based on a central limit theorem for m_n-dependent random fields established by Heinrich [15]. Unlike most existing results on central limit theorems for random fields which require certain regularity conditions on the boundary of Γ, Heinrich's central limit theorem (and consequently our Theorem 1) has the very interesting property that no condition on the boundary

of Γ is needed, and the central limit theorem holds under the minimal condition that $|\Gamma| \to \infty$, where $|\Gamma|$ the cardinal of Γ. This is a very attractive property in spatial applications in which the underlying observation domains can be quite irregular. As an application, we establish a central limit theorem for sample auto-covariances. Section 3 also present an invariance principle. Proofs are provided in Section 4.

2 Examples and Dependence Measures

In (1), we can interpret $(\varepsilon_s)_{s\in\mathbb{Z}^d}$ as the input random field, g is a transform or map and $(X_i)_{i\in\mathbb{Z}^d}$ as the output random field. Based on this interpretation, we define dependence measure as follows: let $(\varepsilon_j^{'})_{j\in\mathbb{Z}^d}$ be an iid copy of $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ and consider for any positive integer n the coupled version X_i^* of X_i defined by

$$X_i^* = g\left(\varepsilon_{i-s}^*\,;\, s\in\mathbb{Z}^d\right),$$

where for any j in $\mathbb{Z}^d$,

$$\varepsilon_j^* = \begin{cases} \varepsilon_j & \text{if } j\neq 0 \\ \varepsilon_0^{'} & \text{if } j = 0. \end{cases}$$

Recall that a Young function ψ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies $\lim_{t\to\infty}\psi(t) = \infty$ and $\psi(0) = 0$. We define the Orlicz space $\mathbb{L}_\psi$ as the space of real random variables Z defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[\psi(|Z|/c)] < +\infty$ for some $c > 0$. The Orlicz space $\mathbb{L}_\psi$ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable Z by

$$\|Z\|_\psi = \inf\{\, c > 0\,;\, \mathbb{E}[\psi(|Z|/c)] \leq 1\,\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [17].

Following Wu [31], we introduce the following dependence measures which are directly related to the underlying processes.

Definition 1 (Physical dependence measure). *Let ψ be a Young function and i in $\mathbb{Z}^d$ be fixed. If X_i belongs to $\mathbb{L}_\psi$, we define the physical dependence measure $\delta_{i,\psi}$ by*

$$\delta_{i,\psi} = \|X_i - X_i^*\|_\psi.$$

If $p \in]0, +\infty]$ and X_i belongs to $\mathbb{L}^p$, we denote $\delta_{i,p} = \|X_i - X_i^*\|_p$.

Physical dependence measure should be seen as a measure of the dependence of the function g (defined by (1)) in the coordinate zero. In mathematical physics, various versions of similar ideas (local perturbation of a configuration) appear. One can refer for example to Liggett [20] or Stroock and Zegarlinski [28].

Definition 2 (Stability). *We say that the random field X defined by* (1) *is p-stable if*

$$\Delta_p := \sum_{i \in \mathbb{Z}^d} \delta_{i,p} < \infty.$$

As an illustration, we give some examples of p-stable spatial processes.

Example 1. (Linear random fields) Let $(\varepsilon_i)_{i \in \mathbb{Z}^d}$ be i.i.d random variables with ε_i in $\mathbb{L}^p$, $p \geq 2$. The linear random field X defined for any k in $\mathbb{Z}^d$ by

$$X_k = \sum_{s \in \mathbb{Z}^d} a_s \varepsilon_{k-s}$$

is of the form (1) with a linear functional g. For any i in $\mathbb{Z}^d$, $\delta_{i,p} = |a_i| \|\varepsilon_0 - \varepsilon_0'\|_p$. So, X is p-stable if

$$\sum_{i \in \mathbb{Z}^d} |a_i| < \infty.$$

Clearly, if K is a Lipschitz continuous function, under the above condition, the subordinated process $Y_i = K(X_i)$ is also p-stable since $\delta_{i,p} = O(|a_i|)$.

Example 2. (Volterra field) Another class of nonlinear random field is the Volterra process which plays an important role in the nonlinear system theory

(Casti [5], Rugh [27]): consider the second order Volterra process

$$X_k = \sum_{s_1,s_2 \in \mathbb{Z}^d} a_{s_1,s_2} \varepsilon_{k-s_1} \varepsilon_{k-s_2},$$

where a_{s_1,s_2} are real coefficients with $a_{s_1,s_2} = 0$ if $s_1 = s_2$ and ε_i in $\mathbb{L}^p$, $p \geq 2$. Let

$$A_k = \sum_{s_1,s_2 \in \mathbb{Z}^d} (a_{s_1,k}^2 + a_{k,s_2}^2) \text{ and } B_k = \sum_{s_1,s_2 \in \mathbb{Z}^d} (|a_{s_1,k}|^p + |a_{k,s_2}|^p).$$

By the Rosenthal inequality, there exists a constant $C_p > 0$ such that

$$\delta_{k,p} = \|X_k - X_k^*\|_p \leq C_p A_k^{1/2} \|\varepsilon_0\|_2 \|\varepsilon_0\|_p + C_p B_k^{1/p} \|\varepsilon_0\|_p^2.$$

3 Main Results

To establish a central limit theorem for S_Γ we need the following moment inequality. With the physical dependence measure, it turns out that the moment bound can have an elegant and concise form.

Proposition 1. *Let Γ be a finite subset of $\mathbb{Z}^d$ and $(a_i)_{i \in \Gamma}$ be a family of real numbers. For any $p \geq 2$, we have*

$$\left\| \sum_{i \in \Gamma} a_i X_i \right\|_p \leq \left(2p \sum_{i \in \Gamma} a_i^2 \right)^{\frac{1}{2}} \Delta_p$$

where $\Delta_p = \sum_{i \in \mathbb{Z}^d} \delta_{i,p}$.

In the sequel, for any i in $\mathbb{Z}^d$, we denote δ_i in place of $\delta_{i,2}$.

Proposition 2. *If $\Delta_2 := \sum_{i \in \mathbb{Z}^d} \delta_i < \infty$ then $\sum_{k \in \mathbb{Z}^d} |\mathbb{E}(X_0 X_k)| < \infty$. Moreover, if $(\Gamma_n)_{n \geq 1}$ is a sequence of finite subsets of $\mathbb{Z}^d$ such that $|\Gamma_n|$ goes to infinity and $|\partial \Gamma_n|/|\Gamma_n|$ goes to zero then*

$$\lim_{n \to +\infty} |\Gamma_n|^{-1} \mathbb{E}(S_{\Gamma_n}^2) = \sum_{k \in \mathbb{Z}^d} \mathbb{E}(X_0 X_k). \tag{2}$$

3.1 Central Limit Theorem

Our first main result is the following central limit theorem.

Theorem 1. *Let $(X_i)_{i\in\mathbb{Z}^d}$ be the stationary centered random field defined by (1) satisfying $\Delta_2 := \sum_{i\in\mathbb{Z}^d}\delta_i < \infty$. Assume that $\sigma_n^2 := \mathbb{E}\left(S_{\Gamma_n}^2\right) \to \infty$. Let $(\Gamma_n)_{n\geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^d$ such that $|\Gamma_n| \to \infty$, then the Levy distance*

$$L[S_{\Gamma_n}/\sqrt{|\Gamma_n|},\, N(0,\sigma_n^2/|\Gamma_n|)] \to 0 \text{ as } n\to\infty. \tag{3}$$

We emphasize that in Theorem 1 no condition on the domains Γ_n is imposed other than the natural one $|\Gamma_n| \to \infty$. Applying Proposition 2, if $|\partial\Gamma_n|/|\Gamma_n|$ goes to zero and $\sigma^2 := \sum_{k\in\mathbb{Z}^d}\mathbb{E}(X_0X_k) > 0$ then

$$\frac{S_{\Gamma_n}}{\sqrt{|\Gamma_n|}} \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0,\sigma^2).$$

Theorem 1 can be applied to the mean estimation problem: suppose that a stationary random field X_i with unknown mean $\mu = \mathbb{E}X_i$ is observed on the domain Γ. Then μ can be estimated by the sample mean $\hat{\mu} = S_\Gamma/|\Gamma|$ and a confidence interval for $\hat{\mu}$ can be constructed if there is a consistent estimate for $\mathrm{var}(S_\Gamma)/|\Gamma|$.

Interestingly, the Theorem can also be applied to the estimation of auto-covariance functions. For $k\in\mathbb{Z}^d$ let

$$\gamma_k = \mathrm{cov}(X_0,X_k) = \mathbb{E}(X_0X_k) - \mu^2. \tag{4}$$

Assume X_i is observed over $i\in\Gamma$ and let $\Xi = \{i\in\Gamma :\, i+k\in\Gamma\}$. Then γ_k can be estimated by

$$\hat{\gamma}_k = \frac{1}{|\Xi|}\sum_{i\in\Xi} X_iX_{i+k} - \hat{\mu}^2. \tag{5}$$

To apply Theorem 1, we need to compute the physical dependence measure for the process $Y_i := X_iX_{i+k}, i\in\mathbb{Z}^d$. It turns out that the dependence for Y_i

can be easily obtained from that of X_i. Note that

$$\begin{aligned}\delta_{i,p/2}(Y) &= \|X_iX_{i+k} - X_i^*X_{i+k}^*\|_{p/2} \\ &\leq \|X_iX_{i+k} - X_iX_{i+k}^*\|_{p/2} + \|X_iX_{i+k}^* - X_i^*X_{i+k}^*\|_{p/2} \\ &\leq \|X_i\|_p\delta_{i+k,p} + \delta_{i,p}\|X_{i+k}^*\|_p = \|X_0\|_p(\delta_{i+k,p} + \delta_{i,p}).\end{aligned}$$

Hence, if $\Delta_4 = \sum_{i\in\mathbb{Z}^d}\delta_{i,4} < \infty$, we have $\sum_{i\in\mathbb{Z}^d}\delta_{i,2}(Y) < \infty$ and the central limit theorem for $\sum_{i\in\Xi}X_iX_{i+k}/|\Xi|$ holds if $|\Xi| \to \infty$.

3.2 Invariance Principles

Now, we are going to see that an invariance principle holds too. If $\mathcal{A}$ is a collection of Borel subsets of $[0,1]^d$, define the smoothed partial sum process $\{S_n(A)\,;\, A \in \mathcal{A}\}$ by

$$S_n(A) = \sum_{i\in\{1,\ldots,n\}^d} \lambda(nA \cap R_i)X_i \tag{6}$$

where $R_i =]i_1 - 1, i_1] \times \ldots \times]i_d - 1, i_d]$ is the unit cube with upper corner at i, λ is the Lebesgue measure on $\mathbb{R}^d$ and X_i is defined by (1). We equip the collection $\mathcal{A}$ with the pseudo-metric ρ defined for any A, B in $\mathcal{A}$ by $\rho(A,B) = \sqrt{\lambda(A\Delta B)}$. To measure the size of $\mathcal{A}$ one considers the metric entropy: denote by $H(\mathcal{A},\rho,\varepsilon)$ the logarithm of the smallest number $N(\mathcal{A},\rho,\varepsilon)$ of open balls of radius ε with respect to ρ which form a covering of $\mathcal{A}$. The function $H(\mathcal{A},\rho,.)$ is the entropy of the class $\mathcal{A}$. Let $\mathcal{C}(\mathcal{A})$ be the space of continuous real functions on $\mathcal{A}$, equipped with the norm $\|.\|_{\mathcal{A}}$ defined by $\|f\|_{\mathcal{A}} = \sup_{A\in\mathcal{A}}|f(A)|$.

A standard Brownian motion indexed by $\mathcal{A}$ is a mean zero Gaussian process W with sample paths in $\mathcal{C}(\mathcal{A})$ and $\mathrm{Cov}(W(A),W(B)) = \lambda(A\cap B)$. From Dudley [10] we know that such a process exists if

$$\int_0^1 \sqrt{H(\mathcal{A},\rho,\varepsilon)}\,d\varepsilon < +\infty. \tag{7}$$

We say that the invariance principle or functional central limit theorem (FCLT) holds if the sequence $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ converges in distribution to an $\mathcal{A}$-indexed Brownian motion in the space $(\mathcal{A})$. The first weak convergence results for $\mathcal{Q}_d$-indexed partial sum processes were established for i.i.d. random fields and for the collection $\mathcal{Q}_d$ of lower-left quadrants in $[0,1]^d$, that is to say the collection $\{[0,t_1]\times\ldots\times[0,t_d]\,;\,(t_1,\ldots,t_d)\in[0,1]^d\}$. They were proved by Wichura [30] under a finite variance condition and earlier by Kuelbs [18] under additional moment restrictions. When the dimension d is reduced to one, these results coincide with the original invariance principle of Donsker [9]. Dedecker [8] gave an $\mathbb{L}^\infty$-projective criterion for the process $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ to converge in the space $\mathcal{C}(\mathcal{A})$ to a mixture of $\mathcal{A}$-indexed Brownian motions when the collection $\mathcal{A}$ satisfies only the entropy condition (7). This projective criterion is valid for martingale-difference bounded random fields and provides a sufficient condition for ϕ-mixing bounded random fields. For unbounded random fields, the result still holds provided that the metric entropy condition on the class $\mathcal{A}$ is reinforced (see [11]). It is shown in [13] that the FCLT may be not valid for p-integrable martingale-difference random fields ($0\leq p<+\infty$) but it still holds if the conditional variances of the martingale-difference random field are assumed to be bounded a.s. (see [12]). In this paper, we are going to establish the FCLT for random fields of the form (1) (see Theorem 2).

Following [29], we recall the definition of Vapnik-Chervonenkis classes (VC-classes) of sets: let $\mathcal{C}$ be a collection of subsets of a set $\mathcal{X}$. An arbitrary set of n points $F_n:=\{x_1,...,x_n\}$ possesses 2^n subsets. Say that $\mathcal{C}$ *picks out* a certain subset from F_n if this can be formed as a set of the form $C\cap F_n$ for a C in $\mathcal{C}$. The collection $\mathcal{C}$ is said to *shatter* F_n if each of its 2^n subsets can be picked out in this manner. The *VC-index* $V(\mathcal{C})$ of the class $\mathcal{C}$ is the smallest n for which no set of size n is shattered by $\mathcal{C}$. Clearly, the more refined $\mathcal{C}$ is, the larger is its index. Formally, we have

$$V(\mathcal{C})=\inf\left\{n\,;\,\max_{x_1,\ldots,x_n}\Delta_n(\mathcal{C},x_1,...,x_n)<2^n\right\}$$

where $\Delta_n(\mathcal{C}, x_1, ..., x_n) = \#\{C \cap \{x_1, ..., x_n\}\,;\, C \in \mathcal{C}\}$. Two classical examples of VC-classes are the collection $\mathcal{Q}_d = \left\{[0,t]\,;\, t \in [0,1]^d\right\}$ and $\mathcal{Q}_d' = \left\{[s,t]\,;\, s,t \in [0,1]^d,\, s \leq t\right\}$ with index $d+1$ and $2d+1$ respectively (where $s \leq t$ means $s_i \leq t_i$ for any $1 \leq i \leq d$). Fore more about Vapnik-Chervonenkis classes of sets, one can refer to [29].

Let $\beta > 0$ and $h_\beta = ((1-\beta)/\beta)^{\frac{1}{\beta}}\, \mathbb{1}_{\{0<\beta<1\}}$. We denote by ψ_β the Young function defined by $\psi_\beta(x) = e^{(x+h_\beta)^\beta} - e^{h_\beta^\beta}$ for any x in $\mathbb{R}^+$.

Theorem 2. *Let $(X_i)_{i\in\mathbb{Z}^d}$ be the stationary centered random field defined by (1) and let $\mathcal{A}$ be a collection of regular Borel subsets of $[0,1]^d$. Assume that one of the following condition holds:*

(i) The collection $\mathcal{A}$ is a Vapnik-Chervonenkis class with index V and there exists $p > 2(V-1)$ such that X_0 belongs to $\mathbb{L}^p$ and $\Delta_p := \sum_{i\in\mathbb{Z}^d} \delta_{i,p} < \infty$.

(ii) There exists $\theta > 0$ and $0 < q < 2$ such that $\mathbb{E}[\exp(\theta|X_0|^{\beta(q)})] < \infty$ where $\beta(q) = 2q/(2-q)$ and $\Delta_{\psi_{\beta(q)}} := \sum_{i\in\mathbb{Z}^d} \delta_{i,\psi_{\beta(q)}} < \infty$ and such that the class $\mathcal{A}$ satisfies the condition

$$\int_0^1 (H(\mathcal{A}, \rho, \varepsilon))^{1/q}\, d\varepsilon < +\infty. \tag{8}$$

(iii) X_0 belongs to $\mathbb{L}^\infty$, the class $\mathcal{A}$ satisfies the condition (7) and $\Delta_\infty := \sum_{i\in\mathbb{Z}^d} \delta_{i,\infty} < \infty$.

Then the sequence of processes $\{n^{-d/2} S_n(A)\,;\, A \in \mathcal{A}\}$ converges in distribution in $\mathcal{C}(\mathcal{A})$ to σW where W is a standard Brownian motion indexed by $\mathcal{A}$ and $\sigma^2 = \sum_{k\in\mathbb{Z}^d} \mathbb{E}(X_0 X_k)$.

4 Proofs

Proof of Proposition 1. Let $\tau : \mathbb{Z} \to \mathbb{Z}^d$ be a bijection. For any $i \in \mathbb{Z}$, for any $j \in \mathbb{Z}^d$,

$$P_i X_j := \mathbb{E}(X_j|\mathcal{F}_i) - \mathbb{E}(X_j|\mathcal{F}_{i-1}) \tag{9}$$

where $\mathcal{F}_i = \sigma\left(\varepsilon_{\tau(l)}; l \leq i\right)$.

Lemma 1. *For any i in $\mathbb{Z}$ and any j in $\mathbb{Z}^d$, we have $\|P_i X_j\|_p \leq \delta_{j-\tau(i),p}$.*

Proof of Lemma 1.

$$\begin{aligned}
\|P_i X_j\|_p &= \|\mathbb{E}(X_j|\mathcal{F}_i) - \mathbb{E}(X_j|\mathcal{F}_{i-1})\|_p \\
&= \left\|\mathbb{E}(X_0|T^j\mathcal{F}_i) - \mathbb{E}(X_0|T^j\mathcal{F}_{i-1})\right\|_p \quad \text{where } T^j\mathcal{F}_i = \sigma\left(\varepsilon_{\tau(l)-j}; l \leq i\right) \\
&= \left\|\mathbb{E}\left(g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d}\right)|T^j\mathcal{F}_i\right) - \mathbb{E}\left(g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\setminus\{j-\tau(i)\}}; \varepsilon'_{\tau(i)-j}\right)|T^j\mathcal{F}_i\right)\right\|_p \\
&\leq \left\|g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d}\right) - g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\setminus\{j-\tau(i)\}}; \varepsilon'_{\tau(i)-j}\right)\right\|_p \\
&= \left\|g\left((\varepsilon_{j-\tau(i)-s})_{s\in\mathbb{Z}^d}\right) - g\left((\varepsilon_{j-\tau(i)-s})_{s\in\mathbb{Z}^d\setminus\{j-\tau(i)\}}; \varepsilon'_0\right)\right\|_p \\
&= \left\|X_{j-\tau(i)} - X^*_{j-\tau(i)}\right\|_p \\
&= \delta_{j-\tau(i),p}.
\end{aligned}$$

The proof of Lemma 1 is complete.

For all j in $\mathbb{Z}^d$,

$$X_j = \sum_{i\in\mathbb{Z}} P_i X_j.$$

Consequently,

$$\left\|\sum_{j\in\Gamma} a_j X_j\right\|_p = \left\|\sum_{j\in\Gamma} a_j \sum_{i\in\mathbb{Z}} P_i X_j\right\|_p = \left\|\sum_{i\in\mathbb{Z}}\sum_{j\in\Gamma} a_j P_i X_j\right\|_p .$$

Since $\left(\sum_{j\in\Gamma} a_j P_i X_j\right)_{i\in\mathbb{Z}}$ is a martingale-difference sequence, by Burkholder inequality, we have

$$\left\|\sum_{j\in\Gamma} a_j X_j\right\|_p \leq \left(2p\sum_{i\in\mathbb{Z}}\left\|\sum_{j\in\Gamma} a_j P_i X_j\right\|_p^2\right)^{\frac{1}{2}} \leq \left(2p\sum_{i\in\mathbb{Z}}\left(\sum_{j\in\Gamma}|a_j|\,\|P_i X_j\|_p\right)^2\right)^{\frac{1}{2}} \tag{10}$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{j\in\Gamma}|a_j|\,\|P_i X_j\|_p\right)^2 \leq \left(\sum_{j\in\Gamma} a_j^2\,\|P_i X_j\|_p\right)\times\left(\sum_{j\in\Gamma}\|P_i X_j\|_p\right)$$

and by Lemma 1,

$$\sum_{j\in\mathbb{Z}^d}\|P_i X_j\|_p \leq \sum_{j\in\mathbb{Z}^d}\delta_{j-\tau(i),p} = \Delta_p.$$

So, we obtain

$$\left\|\sum_{j\in\Gamma} a_j X_j\right\|_p \leq \left(2p\Delta_p\sum_{j\in\Gamma} a_j^2\sum_{i\in\mathbb{Z}}\|P_i X_j\|_p\right)^{\frac{1}{2}}.$$

Applying again Lemma 1, for any j in $\mathbb{Z}^d$, we have

$$\sum_{i\in\mathbb{Z}}\|P_i X_j\|_p \leq \sum_{i\in\mathbb{Z}}\delta_{j-\tau(i),p} = \Delta_p,$$

Finally, we derive

$$\left\|\sum_{j\in\Gamma} a_j X_j\right\|_p \leq \left(2p\sum_{j\in\Gamma} a_j^2\right)^{\frac{1}{2}}\Delta_p.$$

The proof of Proposition 1 is complete.

Proof of Proposition 2. Let k in $\mathbb{Z}^d$ be fixed. Since $X_k = \sum_{i\in\mathbb{Z}} P_i X_k$ where P_i is defined by (9) and $\mathbb{E}((P_i X_0)(P_j X_k)) = 0$ if $i \neq j$, we have

$$\mathbb{E}(X_0 X_k) = \sum_{i\in\mathbb{Z}}\mathbb{E}((P_i X_0)(P_i X_k)).$$

Thus, we obtain

$$\sum_{k\in\mathbb{Z}^d} |\mathbb{E}(X_0X_k)| \le \sum_{i\in\mathbb{Z}} \|P_iX_0\|_2 \sum_{k\in\mathbb{Z}^d} \|P_iX_k\|_2.$$

Applying again Lemma 1, we derive $\sum_{k\in\mathbb{Z}^d} |\mathbb{E}(X_0X_k)| \le \Delta_2^2 < \infty$.

In the other part, since $(X_k)_{k\in\mathbb{Z}^d}$ is stationary, we have

$$|\Gamma_n|^{-1}\mathbb{E}(S_{\Gamma_n}^2) = \sum_{k\in\mathbb{Z}^d} |\Gamma_n|^{-1}|\Gamma_n \cap (\Gamma_n - k)|\mathbb{E}(X_0X_k)$$

where $\Gamma_n - k = \{i-k\,;\, i\in\Gamma_n\}$. Moreover

$$|\Gamma_n|^{-1}|\Gamma_n \cap (\Gamma_n - k)||\mathbb{E}(X_0X_k)| \le |\mathbb{E}(X_0X_k)| \quad \text{and} \quad \sum_{k\in\mathbb{Z}^d} |\mathbb{E}(X_0X_k)| < \infty.$$

Since $\lim_{n\to+\infty} |\Gamma_n|^{-1}|\Gamma_n \cap (\Gamma_n - k)| = 1$, applying the Lebesgue convergence theorem, we derive

$$\lim_{n\to+\infty} |\Gamma_n|^{-1}\mathbb{E}(S_{\Gamma_n}^2) = \sum_{k\in\mathbb{Z}^d} \mathbb{E}(X_0X_k).$$

The proof of Proposition 2 is complete.

Proof of Theorem 1. We first assume that $\liminf_n \sigma_n^2/|\Gamma_n| > 0$. Let $(m_n)_{n\ge1}$ be a sequence of positive integers going to infinity. In the sequel, we denote $\overline{X}_j = \mathbb{E}\left(X_j|\mathcal{F}_{m_n}(j)\right)$ where $\mathcal{F}_{m_n}(j) = \sigma(\varepsilon_{j-s}\,;\, |s|\le m_n)$. By factorization, there exists a measurable function h such that $\overline{X}_j = h(\varepsilon_{j-s}\,;\, |s|\le m_n)$. So, we have

$$\overline{X}_j^* = h(\varepsilon_{j-s}^*\,;\, |s|\le m_n) = \mathbb{E}\left(X_j^*|\mathcal{F}_{m_n}^*(j)\right) \tag{11}$$

where $\mathcal{F}_{m_n}^*(j) = \sigma(\varepsilon_{j-s}^*\,;\, |s|\le m_n)$. We denote also for any j in $\mathbb{Z}^d$,

$$\delta_{j,p}^{(m_n)} = \left\|(X_j - \overline{X}_j) - (X_j - \overline{X}_j)^*\right\|_p.$$

The following result is a direct consequence of Proposition 1.

Proposition 3. *Let Γ be a finite subset of $\mathbb{Z}^d$ and $(a_i)_{i\in\Gamma}$ be a family of real numbers. For any n in $\mathbb{N}^*$ and any $p \in [2, +\infty]$, we have*

$$\left\| \sum_{j\in\Gamma} a_j (X_j - \overline{X}_j) \right\|_p \leq \left(2p \sum_{i\in\Gamma} a_i^2 \right)^{\frac{1}{2}} \Delta_p^{(m_n)}$$

where $\Delta_p^{(m_n)} = \sum_{j\in\mathbb{Z}^d} \delta_{j,p}^{(m_n)}$.

We need also the following lemma.

Lemma 2. *Let $p \in]0, +\infty]$ be fixed. If $\Delta_p < \infty$ then $\Delta_p^{(m_n)} \to 0$ as $n \to \infty$.*

Proof of Lemma 2. Let j in $\mathbb{Z}^d$ be fixed. Since $(X_j - \overline{X}_j)^* = X_j^* - \overline{X}_j^*$, we have

$$\begin{aligned}
\delta_{j,p}^{(m_n)} &= \left\| (X_j - \overline{X}_j) - (X_j - \overline{X}_j)^* \right\|_p \leq \|X_j - X_j^*\|_p + \|\overline{X}_j - \overline{X}_j^*\|_p \\
&= \delta_{j,p} + \|\mathbb{E}(X_j | \mathcal{F}_{m_n}(j) \vee \mathcal{F}_{m_n}^*(j)) - \mathbb{E}(X_j^* | \mathcal{F}_{m_n}^*(j) \vee \mathcal{F}_{m_n}(j))\|_p \\
&\leq 2\delta_{j,p}.
\end{aligned}$$

Moreover, $\lim_{n\to+\infty} \delta_{j,p}^{(m_n)} = 0$. Finally, applying the Lebesgue convergence theorem, we obtain $\lim_{n\to+\infty} \Delta_p^{(m_n)} = 0$. The proof of Lemma 2 is complete.

Let $(\Gamma_n)_{n\geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^d$ such that $\lim_{n\to+\infty} |\Gamma_n| = \infty$ and $\liminf_n \frac{\sigma_n^2}{|\Gamma_n|} > 0$ and recall that Δ_2 is assumed to be finite. Combining Proposition 3 and Lemma 2, we have

$$\limsup_{n\to+\infty} \frac{\left\| S_n - \overline{S}_n \right\|_2}{\sigma_n} = 0. \tag{12}$$

We are going to apply the following central limit theorem due to Heinrich ([15], Theorem 2).

Theorem 3 (Heinrich (1988))**.** *Let $(\Gamma_n)_{n\geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^d$ with $|\Gamma_n| \to \infty$ as $n \to \infty$ and let $(m_n)_{n\geq 1}$ be a sequence of positive integers. For each $n \geq 1$, let $\{U_n(j), j \in \mathbb{Z}^d\}$ be an m_n-dependent random*

field with $\mathbb{E}U_n(j) = 0$ *for all* j *in* $\mathbb{Z}^d$. *Assume that* $\mathbb{E}\left(\sum_{j\in\Gamma_n} U_n(j)\right)^2 \to \sigma^2$ *as* $n \to \infty$ *with* $\sigma^2 < \infty$. *Then* $\sum_{j\in\Gamma_n} U_n(j)$ *converges in distribution to a Gaussian random variable with mean zero and variance* σ^2 *if there exists a finite constant* $c > 0$ *such that for any* $n \geq 1$,

$$\sum_{j\in\Gamma_n} \mathbb{E}U_n^2(j) \leq c$$

and for any $\varepsilon > 0$ *it holds that*

$$\lim_{n\to+\infty} L_n(\varepsilon) := m_n^{2d} \sum_{j\in\Gamma_n} \mathbb{E}\left(U_n^2(j)\, \mathbb{1}_{|U_n(j)|\geq \varepsilon m_n^{-2d}}\right) = 0.$$

Since $\liminf_n \frac{\sigma_n^2}{|\Gamma_n|} > 0$, there exists $c_0 > 0$ and $n_0 \in \mathbb{N}$ such that $\frac{|\Gamma_n|}{\sigma_n^2} \leq c_0$ for any $n \geq n_0$. Consider $S_n = \sum_{i\in\Gamma_n} X_i$, $\overline{S}_n = \sum_{i\in\Gamma_n} \overline{X}_i$ and $U_n(j) := \frac{\overline{X}_j}{\sigma_n}$. We have

$$\mathbb{E}\left(\sum_{j\in\Gamma_n} U_n(j)\right)^2 = \frac{\mathbb{E}(\overline{S}_n^2) - \sigma_n^2}{\sigma_n^2} + 1.$$

So, for any $n \geq n_0$ we derive

$$\begin{aligned}
\frac{\left|\sigma_n^2 - \mathbb{E}(\overline{S}_n^2)\right|}{\sigma_n^2} &= \frac{1}{\sigma_n^2}\left|\mathbb{E}\left(\left(\sum_{j\in\Gamma_n}(\overline{X}_j - X_j)\right)\left(\sum_{j\in\Gamma_n}(\overline{X}_j + X_j)\right)\right)\right| \\
&\leq \frac{1}{\sigma_n^2}\left\|\sum_{j\in\Gamma_n}(\overline{X}_j - X_j)\right\|_2 \left\|\sum_{j\in\Gamma_n}(\overline{X}_j + X_j)\right\|_2 \\
&\leq \frac{2|\Gamma_n|\Delta_2^{(m_n)}}{\sigma_n^2}\left(4\Delta_2 + 2\Delta_2^{(m_n)}\right) \\
&\leq 4c_0\Delta_2^{(m_n)}\left(2\Delta_2 + \Delta_2^{(m_n)}\right) \xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

Consequently,

$$\lim_{n\to+\infty} \mathbb{E}\left(\sum_{j\in\Gamma_n} U_n(j)\right)^2 = 1.$$

Moreover, for any $n \geq n_0$,

$$\sum_{j\in\Gamma_n} \mathbb{E}U_n^2(j) = \frac{|\Gamma_n|\mathbb{E}(\overline{X}_0^2)}{\sigma_n^2} \leq c_0\mathbb{E}(X_0^2) < \infty.$$

Let $\varepsilon > 0$ be fixed. We have

$$\begin{aligned} L_n(\varepsilon) &\leq c_0 m_n^{2d}\mathbb{E}\left(\overline{X}_0^2\, \mathbb{1}_{\left\{|\overline{X}_0| \geq \frac{\varepsilon\sigma_n}{m_n^{2d}}\right\}}\right) \leq c_0 m_n^{2d}\mathbb{E}\left(X_0^2\, \mathbb{1}_{\left\{|\overline{X}_0| \geq \frac{\varepsilon\sigma_n}{m_n^{2d}}\right\}}\right) \\ &\leq c_0 m_n^{2d}\sigma_n \mathbb{P}\left(|\overline{X}_0| \geq \frac{\varepsilon\sigma_n}{m_n^{2d}}\right) + c_0 m_n^{2d}\mathbb{E}\left(X_0^2\, \mathbb{1}_{\{|X_0|\geq\sqrt{\sigma_n}\}}\right) \\ &\leq \frac{c_0\mathbb{E}(X_0^2)m_n^{6d}}{\varepsilon^2\sigma_n} + c_0 m_n^{2d}\psi(\sqrt{\sigma_n}) \end{aligned}$$

where $\psi(x) = \mathbb{E}\left(X_0^2\, \mathbb{1}_{\{|X_0|\geq x\}}\right)$.

Lemma 3. *If the sequence $(m_n)_{n\geq 1}$ is defined for any integer $n \geq 1$ by $m_n = \min\left\{\left[\psi\left(\sqrt{\sigma_n}\right)^{\frac{-1}{4d}}\right], \left[\sigma_n^{\frac{1}{12d}}\right]\right\}$ if $\psi(\sqrt{\sigma_n}) \neq 0$ and by $m_n = \left[\sigma_n^{\frac{1}{12d}}\right]$ if $\psi(\sqrt{\sigma_n}) = 0$ where $[\,.\,]$ is the integer part function then*

$$m_n \to \infty, \quad \frac{m_n^{6d}}{\sigma_n} \to 0 \quad \textit{and} \quad m_n^{2d}\psi\left(\sqrt{\sigma_n}\right) \to 0.$$

Proof of Lemma 3. Since $\sigma_n \to \infty$ and $\psi(\sqrt{\sigma_n}) \to 0$, we derive $m_n \to \infty$. Moreover,

$$\frac{m_n^{6d}}{\sigma_n} \leq \frac{1}{\sqrt{\sigma_n}} \to 0 \quad \text{and} \quad m_n^{2d}\psi\left(\sqrt{\sigma_n}\right) \leq \sqrt{\psi\left(\sqrt{\sigma_n}\right)} \to 0.$$

The proof of Lemma 3 is complete.

Consequently, we obtain $\lim_{n\to\infty} L_n(\varepsilon) = 0$. So, applying Theorem 3, we derive that

$$\frac{\overline{S}_n}{\sigma_n} \xrightarrow[n\to+\infty]{\text{Law}} \mathcal{N}(0,1). \tag{13}$$

Combining (12) and (13), we deduce

$$\frac{S_n}{\sigma_n} \xrightarrow[n\to+\infty]{\text{Law}} \mathcal{N}(0,1).$$

Hence (3) holds if $\liminf_n \sigma_n^2/|\Gamma_n| > 0$. In the general case, we argue as follows: If (3) does not hold then there exists a subsequence $n' \to \infty$ such that

$$L\left[\frac{S_{n'}}{\sqrt{|\Gamma_{n'}|}},\, N\left(0, \frac{\sigma_{n'}^2}{|\Gamma_{n'}|}\right)\right] \quad \text{converges to some } l \text{ in }]0, +\infty]. \tag{14}$$

Assume that $\frac{\sigma_{n'}^2}{|\Gamma_{n'}|}$ does not converge to zero. Then there exists a subsequence n'' such that $\liminf_n \frac{\sigma_{n''}^2}{|\Gamma_{n''}|} > 0$. By the first part of the proof of Theorem 1, we obtain

$$L\left[\frac{S_{n''}}{\sqrt{|\Gamma_{n''}|}},\, N\left(0, \frac{\sigma_{n''}^2}{|\Gamma_{n''}|}\right)\right] \quad \text{converges to 0.} \tag{15}$$

Since (15) contradicts (14), we have $\frac{\sigma_{n'}^2}{|\Gamma_{n'}|}$ converges to zero. Consequently $S_{n'}/\sqrt{|\Gamma_{n'}|}$ converges to zero in probability and $L\left[\frac{S_{n'}}{\sqrt{|\Gamma_{n'}|}},\, N\left(0, \frac{\sigma_{n'}^2}{|\Gamma_{n'}|}\right)\right]$ converges to 0 which contradicts again (14). Consequently, (3) holds. The proof of Theorem 1 is then complete.

Proof of Theorem 2. As usual, we have to prove the convergence of the finite-dimensional laws and the tightness of the partial sum process $\{n^{-d/2} S_n(A)\,;\, A \in \mathcal{A}\}$ in $\mathcal{C}(\mathcal{A})$. For any Borel subset A of $[0,1]^d$, we denote by $\Gamma_n(A)$ the finite subset of $\mathbb{Z}^d$ defined by $\Gamma_n(A) = nA \cap \mathbb{Z}^d$. We say that A is a regular Borel set if $\lambda(\partial A) = 0$.

Proposition 4. *Let A be a regular Borel subset of $[0,1]^d$ with $\lambda(A) > 0$. We have*

$$\lim_{n\to+\infty} \frac{|\Gamma_n(A)|}{n^d} = \lambda(A) \quad \textit{and} \quad \lim_{n\to+\infty} \frac{|\partial\Gamma_n(A)|}{|\Gamma_n(A)|} = 0.$$

Moreover, if Δ_2 is finite then

$$\lim_{n\to+\infty} n^{-d/2} \|S_n(A) - S_{\Gamma_n(A)}\|_2 = 0 \tag{16}$$

where $S_{\Gamma_n(A)} = \sum_{i\in\Gamma_n(A)} X_i$.

Proof of Proposition 4. The first part of Proposition 4 is the first part of Lemma 2 in Dedecker [8]. So, we are going to prove only the second part. Let n be a positive integer. Arguing as in Dedecker [8], we have

$$S_n(A) - S_{\Gamma_n(A)} = \sum_{i \in W_n} a_i X_i \tag{17}$$

where $a_i = \lambda(nA \cap R_i) - \mathbb{1}_{i \in \Gamma_n(A)}$ and W_n is the set of all i in $\{1, .., n\}^d$ such that $R_i \cap (nA) \neq \emptyset$ and $R_i \cap (nA)^c \neq \emptyset$. Noting that $|a_i| \leq 1$ and applying Proposition 1, we obtain

$$\|S_n(A) - S_{\Gamma_n(A)}\|_2 \leq 2\Delta_2 \sqrt{\sum_{i \in W_n} a_i^2} \leq 2\Delta_2 \sqrt{|W_n|}. \tag{18}$$

Following the proof of Lemma 2 in [8], we have $|W_n| = o(n^d)$ and we derive (16). The proof of Proposition 4 is complete.
The convergence of the finite-dimensional laws follows from Proposition 4 and Theorem 1.

So, it suffices to establish the tightness property.

Proposition 5. *Assume that Assumption (i), (ii) or (iii) in Theorem* 2 *holds. Then for any* $x > 0$*, we have*

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} \mathbb{P}\left(\sup_{\substack{A,B \in \mathcal{A} \\ \rho(A,B) < \delta}} \left| n^{-d/2} S_n(A) - n^{-d/2} S_n(B) \right| > x \right) = 0. \tag{19}$$

Proof of Proposition 5. Let A and B be fixed in $\mathcal{A}$ and recall that $\rho(A, B) = \sqrt{\lambda(A \Delta B)}$. We have

$$S_n(A) - S_n(B) = \sum_{i \in \Lambda_n} a_i X_i$$

where $\Lambda_n = \{1, ..., n\}^d$ and $a_i = \lambda(nA \cap R_i) - \lambda(nB \cap R_i)$. Applying Propo-

sition 1, we have

$$n^{-d/2}\left\|S_n(A)-S_n(B)\right\|_p \leq \Delta_p\left(\frac{2p}{n^d}\sum_{i\in\Lambda_n}\lambda(n(A\Delta B)\cap R_i)\right)^{\frac{1}{2}} \leq \sqrt{2p}\Delta_p\rho(A,B). \tag{20}$$

Assume that Assumption (i) in Theorem 2 holds. Then there exists a positive constant K such that for any $0<\varepsilon<1$, we have (see Van der Vaart and Wellner [29], Theorem 2.6.4)

$$N(\mathcal{A},\rho,\varepsilon) \leq KV(4e)^V\left(\frac{1}{\varepsilon}\right)^{2(V-1)}$$

where $N(\mathcal{A},\rho,\varepsilon)$ is the smallest number of open balls of radius ϵ with respect to ρ which form a covering of $\mathcal{A}$. So, since $p>2(V-1)$, we have

$$\int_0^1 (N(\mathcal{A},\rho,\varepsilon))^{\frac{1}{p}}\,d\varepsilon < +\infty. \tag{21}$$

Combining (20) and (21) and applying Theorem 11.6 in Ledoux and Talagrand [19], we infer that the sequence $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ satisfies the following property: for each positive ϵ there exists a positive real δ, depending on ϵ and on the value of the entropy integral (21) but not on n, such that

$$\mathbb{E}\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}}|n^{-d/2}S_n(A)-n^{-d/2}S_n(B)|\right)<\epsilon. \tag{22}$$

The condition (19) is then satisfied under Assumption (i) in Theorem 2 and the sequence of processes $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ is tight in $\mathcal{C}(\mathcal{A})$.

Now, we assume that Assumption (ii) in Theorem 2 holds. The following technical lemma can be obtained using the expansion of the exponential function.

Lemma 4. *Let β be a positive real number and Z be a real random variable. There exist positive universal constants A_β and B_β depending only on β such*

that

$$A_\beta \sup_{p>2} \frac{\|Z\|_p}{p^{1/\beta}} \leq \|Z\|_{\psi_\beta} \leq B_\beta \sup_{p>2} \frac{\|Z\|_p}{p^{1/\beta}}.$$

Combining Lemma 4 with (20), for any $0 < q < 2$, there exists $C_q > 0$ such that

$$n^{-d/2} \left\|S_n(A) - S_n(B)\right\|_{\psi_q} \leq C_q \Delta_{\psi_{\beta(q)}} \rho(A, B) \tag{23}$$

where $\beta(q) = 2q/(2-q)$. Applying Theorem 11.6 in Ledoux and Talagrand [19], for each positive ϵ there exists a positive real δ, depending on ϵ and on the value of the entropy integral (8) but not on n, such that (22) holds. The condition (19) is then satisfied and the process $\{n^{-d/2} S_n(A)\,;\, A \in \mathcal{A}\}$ is tight in $\mathcal{C}(\mathcal{A})$.

Finally, if Assumption (*iii*) in Theorem 2 holds then combining Lemma 4 with (20), there exists $C > 0$ such that

$$\left\|n^{-d/2} S_n(A) - n^{-d/2} S_n(B)\right\|_{\psi_2} \leq C \Delta_\infty \rho(A, B). \tag{24}$$

Applying again Theorem 11.6 in Ledoux and Talagrand [19], we obtain the tightness of the process $\{n^{-d/2} S_n(A)\,;\, A \in \mathcal{A}\}$ in $\mathcal{C}(\mathcal{A})$. The proofs of Proposition 5 and Theorem 2 are complete. □

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