Nonparametric regression estimation for random fields in a fixed-design

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Abstract

We investigate the nonparametric estimation for regression in a fixed-design setting when the errors are given by a field of dependent random variables. Sufficient conditions for kernel estimators to converge uniformly are obtained. These estimators can attain the optimal rates of uniform convergence and the results apply to a large class of random fields which contains martingale-difference random fields and mixing random fields.

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1 Introduction

Over the last few years nonparametric estimation for random fields (or spatial processes) was given increasing attention stimulated by a growing demand from applied research areas (see Guyon [18]). In fact, spatial data arise in various areas of research including econometrics, image analysis, meterology, geostatistics... Our aim in this paper is to investigate uniform strong convergence rates of a regression estimator in a fixed design setting when the errors are given by a stationary field of dependent random variables which show spatial interaction. We are most interested in conditions which ensure convergence rates to be identical to those in the case of independent errors (see Stone [33]). Currently the author is working on extensions of the present results to the random design framework. Let $\mathbb{Z}^d$, $d \geq 1$ denote the integer lattice points in the d-dimensional Euclidean space. By a stationary real random field we mean any family $(\varepsilon_k)_{k\in\mathbb{Z}^d}$ of real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $(k,n) \in \mathbb{Z}^d \times \mathbb{N}^*$ and any $(i_1, ..., i_n) \in (\mathbb{Z}^d)^n$, the random vectors $(\varepsilon_{i_1}, ..., \varepsilon_{i_n})$ and $(\varepsilon_{i_1+k}, ..., \varepsilon_{i_n+k})$ have the same law. The regression model which we are interested in is

$$Y_i = g(i/n) + \varepsilon_i, \quad i \in \Lambda_n = \{1, ..., n\}^d \tag{1}$$

where g is an unknown smooth function and $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ is a zero mean stationary real random field. Note that this model was considered also by Bosq [8] and Hall et Hart [19] for time series ($d = 1$). Let K be a probability kernel defined on $\mathbb{R}^d$ and $(h_n)_{n\geq 1}$ a sequence of positive numbers which converges to zero and which satisfies $(nh_n)_{n\geq 1}$ goes to infinity. We estimate the function g by the kernel-type estimator g_n defined for any x in $[0,1]^d$ by

$$g_n(x) = \frac{\sum_{i\in\Lambda_n} Y_i\, K\left(\frac{x-i/n}{h_n}\right)}{\sum_{i\in\Lambda_n} K\left(\frac{x-i/n}{h_n}\right)}. \tag{2}$$

Note that Assumption **A1**) in section 2 ensures that g_n is well defined. Until now, most of existing theoretical nonparametric results of dependent random variables pertain to time series (see Bosq [9]) and relatively few generalizations to the spatial domain are available. Key references on this topic are Biau [5], Carbon et al. [10], Carbon et al. [11], Hallin et al. [20], [21], Tran [34], Tran and Yakowitz [35] and Yao [36] who have investigated nonparametric density estimation for random fields and Altman [2], Biau and Cadre [6], Hallin et al. [22] and Lu and Chen [25], [26] who have studied spatial prediction and spatial regression estimation. The classical asymptotic theory in statistics is built upon central limit theorems, law of large numbers and

large deviations inequalities for the sequences of random variables. These classical limit theorems have been extended to the setting of spatial processes. In particular, some key results on the central limit theorem and its functional versions are Alexander and Pyke [1], Bass [3], Basu and Dorea [4], Bolthausen [7] and more recently Dedecker [12], [13], El Machkouri [16] and El Machkouri and Volný [17]. For a survey on limit theorems for spatial processes and some applications in statistical physics, one can refer to Nahapetian [28]. Note also that the main results (section 3) of this work are obtained via exponential inequalities for random fields discovered by El Machkouri [16].
The paper is organized as follows. The next section sets up the notations and the assumptions which will be considered in the sequel. In section 3, we present our main results on both weak and strong consistencies rates of the estimator g_n. The last section is devoted to the proofs.

2 Notations and Assumptions

In the sequel we denote $\|x\| = \max_{1\leq k\leq d}|x_k|$ for any $x = (x_1, ..., x_d) \in [0, 1]^d$. With a view to obtain optimal convergence rates for the estimator g_n defined by (2), we have to make the following assumptions on the regression function g and the probability kernel K:

A1) The probability kernel K is symmetric, nonnegative, supported by $[-1, 1]^d$ and satisfies a Lipschitz condition $|K(x) - K(y)| \leq \eta\|x - y\|$ for any $x, y \in [-1, 1]^d$ and some $\eta > 0$. In addition there exists $c, C > 0$ such that $c \leq K(x) \leq C$ for any $x \in [-1, 1]^d$.

A2) There exists a constant $B > 0$ such that $|g(x) - g(y)| \leq B\|x - y\|$ for any $x, y \in [0, 1]^d$, that is g is B-Lipschitz.

A Young function ψ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies $\lim_{t\to\infty}\psi(t) = +\infty$ and $\psi(0) = 0$. We define the Orlicz space L_ψ as the space of real random variables Z defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[\psi(|Z|/c)] < +\infty$ for some $c > 0$. The Orlicz space L_ψ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable Z by

$$\|Z\|_\psi = \inf\{\, c > 0\, ;\, E[\psi(|Z|/c)] \leq 1\,\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [24]. Let $\beta > 0$. We denote by ψ_β

the Young function defined for any $x \in \mathbb{R}^+$ by

$$\psi_\beta(x) = \exp((x+\xi_\beta)^\beta) - \exp(\xi_\beta^\beta) \quad \text{where} \quad \xi_\beta = ((1-\beta)/\beta)^{1/\beta}\, \mathbb{1}_{\{0<\beta<1\}}.$$

On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i = (i_1, ..., i_d)$ and $j = (j_1, ..., j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{lex} j$ means that either $i_1 < j_1$ or for some p in $\{2, 3, ..., d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \le q < p$. Let the sets $\{V_i^k\,;\, i \in \mathbb{Z}^d\,,\, k \in \mathbb{N}^*\}$ be defined as follows:

$$V_i^1 = \{j \in \mathbb{Z}^d\,;\, j <_{lex} i\},$$

and for $k \ge 2$

$$V_i^k = V_i^1 \cap \{j \in \mathbb{Z}^d\,;\, |i-j| \ge k\} \quad \text{where} \quad |i-j| = \max_{1\le l\le d} |i_l - j_l|.$$

For any subset Γ of $\mathbb{Z}^d$ define $\mathcal{F}_\Gamma = \sigma(\varepsilon_i\,;\, i \in \Gamma)$ and set

$$E_{|k|}(\varepsilon_i) = E(\varepsilon_i | \mathcal{F}_{V_i^{|k|}}), \quad k \in V_i^1.$$

Denote $\beta(q) = 2q/(2-q)$ for $0 < q < 2$ and consider the following conditions:

C1) $\varepsilon_0 \in L^\infty$ and

$$\sum_{k\in V_0^1} \|\varepsilon_k E_{|k|}(\varepsilon_0)\|_\infty < \infty.$$

C2) There exists $0 < q < 2$ such that $\varepsilon_0 \in L_{\psi_{\beta(q)}}$ and

$$\sum_{k\in V_0^1} \left\| \sqrt{|\varepsilon_k E_{|k|}(\varepsilon_0)|} \right\|_{\psi_{\beta(q)}}^2 < \infty.$$

C3) There exists $p > 2$ such that $\varepsilon_0 \in L^p$ and

$$\sum_{k\in V_0^1} \|\varepsilon_k E_{|k|}(\varepsilon_0)\|_{\frac{p}{2}} < \infty.$$

C4) $\varepsilon_0 \in L^2$ and $\sum_{k\in\mathbb{Z}^d} |E(\varepsilon_0\varepsilon_k)| < \infty$.

Remark 1 Note that Dedecker [12] established the central limit theorem for any stationary square-integrable random field $(\varepsilon_k)_{k\in\mathbb{Z}^d}$ which satisfies the condition $\sum_{k\in V_0^1} \|\varepsilon_k E_{|k|}(\varepsilon_0)\|_1 < \infty$.

In classical statistical physics, there exists spatial processes which satisfy conditions **C1**),...,**C4**). For example, Nahapetian and Petrosian [29] gave sufficient conditions for a Gibbs field $(\varepsilon_k)_{k\in\mathbb{Z}^d}$ to possess the following martingale difference property: for any i in $\mathbb{Z}^d$, $E(\varepsilon_i|\mathcal{F}_{V_i^1}) = 0$ a.s. Another examples of random fields which satisfy conditions **C1**),...,**C4**) can be found also among the class of mixing random fields. More precisely, given two sub-σ-algebras $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{F}$, different measures of their dependence have been considered in the literature. We are interested by two of them. The α-mixing and ϕ-mixing coefficients had been introduced by Rosenblatt [31] and Ibragimov [23] respectively and can be defined by

$$\alpha(\mathcal{U},\mathcal{V}) = \sup\{|\mathbb{P}(U\cap V)-\mathbb{P}(U)\mathbb{P}(V)|,\, U\in\mathcal{U},\, V\in\mathcal{V}\}$$
$$\phi(\mathcal{U},\mathcal{V}) = \sup\{\|\mathbb{P}(V|\mathcal{U})-\mathbb{P}(V)\|_\infty,\, V\in\mathcal{V}\}.$$

We have $2\alpha(\mathcal{U},\mathcal{V})\leq\phi(\mathcal{U},\mathcal{V})$ and these coefficients equal zero if and only if the σ-algebras $\mathcal{U}$ and $\mathcal{V}$ are independent. Denote by $\sharp\Gamma$ the cardinality of any subset Γ of $\mathbb{Z}^d$. In the sequel, we shall use the following non-uniform mixing coefficients defined for any (k,l,n) in $(\mathbb{N}^*\cup\{\infty\})^2\times\mathbb{N}$ by

$$\alpha_{k,l}(n) = \sup\{\alpha(\mathcal{F}_{\Gamma_1},\mathcal{F}_{\Gamma_2}),\, \sharp\Gamma_1\leq k,\, \sharp\Gamma_2\leq l,\, \rho(\Gamma_1,\Gamma_2)\geq n\},$$
$$\phi_{k,l}(n) = \sup\{\phi(\mathcal{F}_{\Gamma_1},\mathcal{F}_{\Gamma_2}),\, \sharp\Gamma_1\leq k,\, \sharp\Gamma_2\leq l,\, \rho(\Gamma_1,\Gamma_2)\geq n\},$$

where the distance ρ is defined by $\rho(\Gamma_1,\Gamma_2)=\min\{|i-j|,\, i\in\Gamma_1,\, j\in\Gamma_2\}$. We say that the random field $(\varepsilon_k)_{k\in\mathbb{Z}^d}$ is α-mixing or ϕ-mixing if there exists a pair (k,l) in $(\mathbb{N}^*\cup\{\infty\})^2$ such that $\lim_{n\to\infty}\alpha_{k,l}(n)=0$ or $\lim_{n\to\infty}\phi_{k,l}(n)=0$ respectively. For more about mixing coefficients one can refer to Doukhan [15]. We consider the following mixing conditions:

C$'$1) $\varepsilon_0\in L^\infty$ and

$$\sum_{k\in\mathbb{Z}^d}\phi_{\infty,1}(|k|)<\infty.$$

C$'$2) There exists $0<q<2$ such that $\varepsilon_0\in L_{\psi_{\beta(q)}}$ and

$$\sum_{k\in\mathbb{Z}^d}\sqrt{\phi_{\infty,1}(|k|)}<\infty$$

or

$$\sum_{k\in\mathbb{Z}^d}c_k^2(\beta(q))<\infty$$

where for any $\beta>0$

$$c_k(\beta)=\inf\left\{c>0\,\Big|\,\int_0^{\alpha_{1,\infty}(|k|)}\psi_\beta\left(\frac{Q_{\varepsilon_0}(u)}{c}\right)du\leq 1\right\}. \tag{3}$$

$\mathbf{C}'\mathbf{3}$) There exists $p > 2$ such that $\varepsilon_0 \in L^p$ and

$$\sum_{k\in\mathbb{Z}^d} \left(\int_0^{\alpha_{1,\infty}(|k|)} Q^p_{\varepsilon_0}(u)\, du \right)^{2/p} < \infty \tag{4}$$

where Q_{ε_0} is the inverse cadlag of the tail function $t \to \mathbb{P}(|\varepsilon_0| > t)$ (i.e. for any $u \geq 0$, $Q_{\varepsilon_0}(u) = \inf\{t > 0 \,|\, \mathbb{P}(|\varepsilon_0| > t) \leq u\}$).

Remark 2 Let us note that if $p = 2 + \delta$ for some $\delta > 0$ then the condition

$$\sum_{m=1}^{\infty} m^{d-1} \alpha_{1,\infty}^{\frac{\delta}{2+\delta} - \varepsilon}(m) < \infty \quad \text{for some } \varepsilon > 0$$

is more restrictive than condition (4) and is known to be sufficient for the random field $(\varepsilon_k)_{k\in\mathbb{Z}^d}$ to satisfy a functional central limit theorem (cf. Dedecker [13]).

In statistical physics, using the Dobrushin's uniqueness condition (cf. [14]), one can construct Gibbs fields satisfying a uniform exponential mixing condition which is more restrictive than conditions $\mathbf{C}'\mathbf{1}$), $\mathbf{C}'\mathbf{2}$) and $\mathbf{C}'\mathbf{3}$) (see Guyon [18], theorem 2.1.3, p. 52).

3 Main results

Let $(Z_n)_{n\geq 1}$ be a sequence of real random variables and $(v_n)_{n\geq 1}$ be a sequence of positive numbers. We say that

$$Z_n = O_{a.s.}\left[v_n\right]$$

if there exists $\lambda > 0$ such that

$$\limsup_{n\to\infty} \frac{|Z_n|}{v_n} \leq \lambda \quad \text{a.s.}$$

Our main result is the following.

Theorem 1 *Assume that the assumption* **A1)** *holds.*

1) *If* **C1**) *holds then*

$$\sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| = O_{a.s.}\left[\frac{(\log n)^{1/2}}{(nh_n)^{d/2}}\right]. \tag{5}$$

2) *If* **C2**) *holds for some* $0 < q < 2$ *then*

$$\sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| = O_{a.s.}\left[\frac{(\log n)^{1/q}}{(nh_n)^{d/2}}\right]. \quad (6)$$

3) *Assume that* **C3**) *holds for some* $p > 2$ *and* $h_n = n^{-\theta_2}(\log n)^{\theta_1}$ *for some* $\theta_1, \theta_2 > 0$. *Let* $a, b \geq 0$ *be fixed and denote*

$$v_n = \frac{n^a(\log n)^b}{(nh_n)^{d/2}} \quad \text{and} \quad \theta = \frac{2a(d+p) - d^2 - 2}{d(3d+2)}.$$

If $\theta \geq \theta_2$ *and* $d(3d+2)\theta_1 + 2(d+p)b > 2$ *then*

$$\sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| = O_{a.s.}[v_n]. \quad (7)$$

Remark 3 Theorem 1 shows that the optimal uniform convergence rate is obtained for bounded errors (cf. estimation (5)) and that it is "almost" optimal if one considers errors with only finite exponential moments (cf. estimation (6)).

Theorem 2 *Assume that the assumption* **A1)** *holds.*

1) *Assume that* **C3**) *holds for some* $p > 2$. *Let* $a > 0$ *be fixed and denote*

$$v_n = \frac{n^a}{(nh_n)^{d/2}} \quad \text{and} \quad \theta = \frac{2a(d+p) - d^2}{d(3d+2)}.$$

If $\theta > 0$ *and* $h_n \geq n^{-\theta}$ *then*

$$\left\| \sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| \right\|_p = O[v_n]. \quad (8)$$

2) *If* **C4**) *holds then*

$$\sup_{x\in[0,1]^d} \|g_n(x) - Eg_n(x)\|_2 = O\left[(nh_n)^{-d/2}\right]. \quad (9)$$

In the sequel, we denote by $\text{Lip}(B)$ the set of B-Lipschitz functions. The following proposition gives the convergence of $Eg_n(x)$ to $g(x)$.

Proposition 1 *Assume that the assumption* **A2)** *holds then*

$$\sup_{x\in[0,1]^d} \sup_{g\in\text{Lip}(B)} |Eg_n(x) - g(x)| = O[h_n].$$

From Proposition 1 and Theorem 1 we derive the following corollary.

Corollary 1 *Assume that* **A1)** *and* **A2)** *hold and let* $h_n = \left(n^{-d} \log n\right)^{1/(2+d)}$.

1) *If* **C1**) *holds then*

$$\sup_{x \in [0,1]^d} \sup_{g \in \mathrm{Lip}(B)} |g_n(x) - g(x)| = O_{a.s.} \left[\left(\frac{\log n}{n^d} \right)^{\frac{1}{2+d}} \right]. \tag{10}$$

2) *If* **C2**) *holds for some* $0 < q < 2$ *then*

$$\sup_{x \in [0,1]^d} \sup_{g \in \mathrm{Lip}(B)} |g_n(x) - g(x)| = O_{a.s.} \left[u(n) \left(\frac{\log n}{n^d} \right)^{\frac{1}{2+d}} \right] \tag{11}$$

where $u(n) = (\log n)^{(2-q)/2q}$.

3) *Let* $\varepsilon > 0$ *be fixed. If* **C3**) *holds for some* $p > 2$ *satisfying*

$$p \geq \frac{4d^3 + (4 - 2\varepsilon)d^2 + (2 - 4\varepsilon)d + 4}{2\varepsilon(2 + d)} \tag{12}$$

then

$$\sup_{x \in [0,1]^d} \sup_{g \in \mathrm{Lip}(B)} |g_n(x) - g(x)| = O_{a.s.} \left[u(n) \left(\frac{\log n}{n^d} \right)^{\frac{1}{2+d}} \right] \tag{13}$$

where $u(n) = n^{\varepsilon}$.

Remark 4 Note that the consistency rate $(n^{-d} \log n)^{1/(2+d)}$ is known to be the optimal one (see Stone [33]).

From Proposition 1 and Theorem 2 we derive the following corollary.

Corollary 2 *Assume that* **A1)** *and* **A2)** *hold and let* $h_n = n^{-d/(2+d)}$.

1) *Let* $\varepsilon > 0$ *be fixed. If* **C3**) *holds for some* $p > 2$ *satisfying*

$$p \geq \frac{4d^3 + (4 - 2\varepsilon)d^2 - 4\varepsilon d}{2\varepsilon(2 + d)} \tag{14}$$

then

$$\left\| \sup_{x \in [0,1]^d} \sup_{g \in \mathrm{Lip}(B)} |g_n(x) - g(x)| \right\|_p = O\left[n^{-\frac{d}{2+d} + \varepsilon} \right]. \tag{15}$$

2) *If* **C4**) *holds then*

$$\sup_{x\in[0,1]^d}\left\|\sup_{g\in\mathrm{Lip}(B)}|g_n(x)-g(x)|\right\|_2 = O\left[n^{-\frac{d}{2+d}}\right]. \tag{16}$$

Finally the rates of convergence obtained above are valid when the errors are given by a mixing random field. More precisely, we have the following corollary.

Corollary 3 *Theorems* 1 *and* 2 *and Corollaries* 1 *and* 2 *still hold if one replace conditions* **C1**), **C2**) *and* **C3**) *by conditions* **C′1**), **C′2**) *and* **C′3**) *respectively.*

4 Proofs

For any x in $[0,1]^d$ and any integer $n \geq 1$ we define $B_n(x) = Eg_n(x) - g(x)$ and $V_n(x) = g_n(x) - Eg_n(x)$. More precisely

$$B_n(x) = \frac{\sum_{i\in\Lambda_n} a_i(x)g(i/n)}{\sum_{i\in\Lambda_n} a_i(x)} - g(x)$$
$$V_n(x) = \frac{\sum_{i\in\Lambda_n} a_i(x)\varepsilon_i}{\sum_{i\in\Lambda_n} a_i(x)}$$

where $a_i(x) = K\left(\frac{x-i/n}{h_n}\right)$. In the sequel, we denote also $S_n(x) = \sum_{i\in\Lambda_n} a_i(x)\varepsilon_i$ for any $x \in [0,1]^d$. We start with the following lemma.

Lemma 1 *There exists constants $c, C > 0$ such that for any $x \in [0,1]^d$ and any $n \in \mathbb{N}^*$,*

$$c\prod_{k=1}^{d}[n(x_k+h_n)] \leq \sum_{i\in\Lambda_n} a_i(x) \leq C\prod_{k=1}^{d}[n(x_k+h_n)] \tag{17}$$

where $[\,.\,]$ *denote the integer part function.*

Proof of Lemma 1. Since the kernel K is supported by $[-1,1]^d$, we have

$$\sum_{i\in\Lambda_n} a_i(x) = \sum_{i_1=1}^{[n(x_1+h_n)]}\ldots\sum_{i_d=1}^{[n(x_d+h_n)]} a_i(x).$$

By assumption, there exists constants $c, C > 0$ such that $c \leq K(y) \leq C$ for any $y \in [-1,1]^d$. The proof of Lemma 1 is complete.

4.1 Proof of Theorem 1

Let $(\upsilon_n)_{n\geq 1}$ be a sequence of positive numbers going to zero. Following Carbon and al. [11] the compact set $[0,1]^d$ can be covered by r_n cubes I_k having sides of length $l_n = \upsilon_n h_n^{2d+1}$ and center at c_k. Clearly there exists $c > 0$ such that $r_n \leq c/l_n^d$. Define

$$\begin{aligned} A_{1,n}(g) &= \max_{1\leq k\leq r_n} \sup_{x\in I_k} |g_n(x) - g_n(c_k)| \\ A_{2,n}(g) &= \max_{1\leq k\leq r_n} \sup_{x\in I_k} |Eg_n(x) - Eg_n(c_k)| \\ A_{3,n} &= \max_{1\leq k\leq r_n} |g_n(c_k) - Eg_n(c_k)| \end{aligned}$$

then

$$\sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| \leq \sup_{g\in Lip(B)} [A_{1,n}(g) + A_{2,n}(g)] + A_{3,n}. \qquad (18)$$

Lemma 2 *For $i = 1, 2$ we have*

$$\sup_{g\in \mathrm{Lip}(B)} A_{i,n}(g) = O_{a.s.}[\upsilon_n].$$

Proof of Lemma 2. Since $g \in \mathrm{Lip}(B)$, we can assume without loss of generality that g is bounded by B on the set $[0,1]^d$. For any $x \in I_k$, we have

$$g_n(x) - g_n(c_k) = \sigma_1 + \sigma_2$$

where

$$\sigma_1 = \frac{\sum_{i\in\Lambda_n} Y_i(a_i(x) - a_i(c_k))}{\sum_{i\in\Lambda_n} a_i(x)}$$

and

$$\sigma_2 = \frac{\sum_{i\in\Lambda_n}(a_i(c_k) - a_i(x))}{\sum_{i\in\Lambda_n} a_i(x) \times \sum_{i\in\Lambda_n} a_i(c_k)} \sum_{i\in\Lambda_n} Y_i a_i(c_k).$$

Now, by Lemma 1 and Assumption **A1**), we derive that there exists constants $c, \eta > 0$ such that for any n sufficiently large

$$|\sigma_1| \leq \frac{2^d \eta l_n/h_n}{c(nh_n)^d} \sum_{i\in\Lambda_n} |Y_i| \leq \frac{\eta \upsilon_n h_n^d}{c} \left(B + \frac{1}{n^d} \sum_{i\in\Lambda_n} |\varepsilon_i| \right)$$

and

$$|\sigma_2| \leq \frac{4^d \eta n^d l_n/h_n}{c^2(nh_n)^{2d}} \sum_{i\in\Lambda_n} |Y_i| \leq \frac{\eta \upsilon_n}{c^2} \left(B + \frac{1}{n^d} \sum_{i\in\Lambda_n} |\varepsilon_i| \right)$$

Since (ε_i) is a stationary ergodic random field the lemma easily follows from the last inequalities and the Birkhoff ergodic theorem. The proof of Lemma 2 is complete.

Lemma 3 *Assume that either* **C1**) *holds and* $v_n = (\log n)^{1/2}/(nh_n)^{d/2}$ *or* **C2**) *holds for some* $0 < q < 2$ *and* $v_n = (\log n)^{1/q}/(nh_n)^{d/2}$ *then*

$$A_{3,n} = O_{a.s.}[v_n]$$

Proof of Lemma 3. Let $0 < q \leq 2$ be fixed. We consider the exponential Young function define for any $x \in \mathbb{R}^+$ by $\psi_q(x) = \exp((x+\xi_q)^q) - \exp(\xi_q^q)$ where $\xi_q = ((1-q)/q)^{1/q} \mathbb{1}_{\{0<q<1\}}$. Let $\lambda > 0$ and $x \in [0,1]^d$ be fixed

$$\begin{aligned}\mathbb{P}\left(|V_n(x)| > \lambda v_n\right) &= \mathbb{P}\left(\left|S_n(x)\right| > \lambda v_n \sum_{i\in\Lambda_n} a_i(x)\right)\\ &\leq (1+e^{\xi_q^q})\,\exp\left[-\left(\frac{\lambda v_n \sum_{i\in\Lambda_n} a_i(x)}{\|\sum_{i\in\Lambda_n} a_i(x)\varepsilon_i\|_{\psi_q}} + \xi_q\right)^q\right].\end{aligned}$$

For any $i \in \Lambda_n$ and any $0 < q < 2$ denote

$$b_{i,q}(a(x)\varepsilon) = \left\|a_i(x)\varepsilon_i\right\|^2_{\psi_{\beta(q)}} + \sum_{k\in V_i^1}\left\|\sqrt{\left|a_k(x)\varepsilon_k E_{|k-i|}(a_i(x)\varepsilon_i)\right|}\right\|^2_{\psi_{\beta(q)}} \tag{19}$$

and

$$b_{i,2}(a(x)\varepsilon) = \left\|a_i(x)\varepsilon_i\right\|^2_{\infty} + \sum_{k\in V_i^1}\left\|a_k(x)\varepsilon_k E_{|k-i|}(a_i(x)\varepsilon_i)\right\|_{\infty} \tag{20}$$

where $V_i^1 = \{j \in \mathbb{Z}^d\,;\, j <_{lex} i\}$. Using Kahane-Khintchine inequalities (cf. El Machkouri [16], Theorem 1) we derive that if Condition **C2**) holds for some $0 < q < 2$ then

$$\mathbb{P}\left(|V_n(x)| > \lambda v_n\right) \leq (1+e^{\xi_q^q})\,\exp\left[-\left(\frac{\lambda\, v_n \sum_{i\in\Lambda_n} a_i(x)}{M(\sum_{i\in\Lambda_n} b_{i,q}(a(x)\varepsilon))^{1/2}} + \xi_q\right)^q\right] \tag{21}$$

where M is a positive constant depending only on q and on the probability kernel K. Now using the definition (19) and Lemma 1 there exist constants $c, M > 0$ such that

$$\begin{aligned}\sup_{x\in[0,1]^d}\mathbb{P}\left(|V_n(x)| > \lambda v_n\right) &\leq (1+e^{\xi_q^q})\,\exp\left[-\left(\frac{\lambda\, v_n\, (\sum_{i\in\Lambda_n} a_i(x))^{1/2}}{M} + \xi_q\right)^q\right]\\ &\leq (1+e^{\xi_q^q})\,\exp\left[-\frac{c^q\,\lambda^q\, v_n^q\,([nh_n])^{dq/2}}{M^q}\right]\end{aligned}$$

So if $v_n = (\log n)^{1/q}/(nh_n)^{d/2}$ and n is sufficiently large then

$$\sup_{x\in[0,1]^d}\mathbb{P}\left(|V_n(x)| > \lambda v_n\right) \leq (1+e^{\xi_q^q})\,\exp\left[-\frac{c^q\,\lambda^q\,\log n}{2^{dq/2}M^q}\right]. \tag{22}$$

If Condition **C1**) holds then (21) still hold with $q = 2$ (cf. El Machkouri [16], Theorem 1). So if $v_n = (\log n)^{1/2}/(nh_n)^{d/2}$ and n is large it follows that

$$\sup_{x\in[0,1]^d} \mathbb{P}\left(|V_n(x)| > \lambda v_n\right) \leq 2\exp\left[-\frac{c^2\,\lambda^2\,\log n}{2^d M^2}\right]. \tag{23}$$

Since

$$\mathbb{P}\left(|A_{3,n}| > \lambda v_n\right) \leq r_n \sup_{x\in[0,1]^d} \mathbb{P}\left(|V_n(x)| > \lambda v_n\right),$$

using (22) and (23), choosing λ sufficiently large and applying Borel-Cantelli's lemma, we derive

$$\mathbb{P}\left(\limsup_{n\to\infty}\{|A_{3,n}| > \lambda v_n\}\right) = 0$$

and

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{|A_{3,n}|}{v_n} \leq \lambda\right) = 1.$$

The proof of points **1**) and **2**) of Theorem 1 are completed by combining Inequality (18) with Lemmas 2 and 3.

Lemma 4 *Assume that* **C3**) *holds for some* $p > 2$ *and* $h_n = n^{-\theta_2}(\log n)^{\theta_1}$ *for some* $\theta_1, \theta_2 > 0$. *Let* $a, b \geq 0$ *be fixed and denote*

$$v_n = \frac{n^a(\log n)^b}{(nh_n)^{d/2}} \quad \textit{and} \quad \theta = \frac{2a(d+p) - d^2 - 2}{d(3d+2)}.$$

If $\theta \geq \theta_2$ *and* $d(3d+2)\theta_1 + 2(d+p)b > 2$ *then*

$$\lim_{n\to+\infty} \frac{|A_{3,n}|}{v_n} = 0 \quad a.s.$$

Proof of Lemma 4. Let $p > 2$ be fixed. For any $\lambda > 0$

$$\begin{aligned}
\mathbb{P}\left(|V_n(x)| > \lambda v_n\right) &= \mathbb{P}\left(|S_n(x)| > \lambda v_n \sum_{i\in\Lambda_n} a_i(x)\right) \\
&\leq \frac{\lambda^{-p} E|S_n(x)|^p}{v_n^p\left(\sum_{i\in\Lambda_n} a_i(x)\right)^p} \\
&\leq \frac{\lambda^{-p}}{v_n^p\left(\sum_{i\in\Lambda_n} a_i(x)\right)^p}\left(2p\sum_{i\in\Lambda_n} c_i(x)\right)^{p/2}
\end{aligned}$$

where $c_i(x) = a_i(x)^2\|\varepsilon_i\|_p^2 + a_i(x)\sum_{k\in V_i^1} a_k(x)\|\varepsilon_k E_{|k-i|}(\varepsilon_i)\|_{\frac{p}{2}}$. The last estimate follows from a Marcinkiewicz-Zygmund type inequality by Dedecker (see [13]) for real random fields. Noting that there exists $\gamma > 0$ such that $c_i(x) \leq \gamma a_i(x)$, $x \in [0,1]^d$ and using Lemma 1, we derive that there exists $\gamma^{'} > 0$ such that

$$\mathbb{P}\left(|A_{3,n}| > \lambda v_n\right) \leq r_n \sup_{x\in[0,1]^d} \mathbb{P}\left(|V_n(x)| > \lambda v_n\right) \leq \frac{\gamma^{'}}{\tau_n \lambda^p}$$

where $\tau_n = l_n^d v_n^p([nh_n])^{dp/2}$. Since $v_n = n^a(\log n)^b/(nh_n)^{d/2}$ and $l_n = v_n h_n^{2d+1}$ it follows

$$\frac{1}{\tau_n} = \frac{(nh_n)^{d(d+p)/2}}{h_n^{d(2d+1)} n^{a(d+p)} (\log n)^{b(d+p)} ([nh_n])^{dp/2}}.$$

If n is sufficiently large, we derive

$$\begin{aligned}
\frac{1}{\tau_n} &\leq \frac{2^{dp/2}(nh_n)^{d(d+p)/2}}{h_n^{d(2d+1)} n^{a(d+p)} (\log n)^{b(d+p)} (nh_n)^{dp/2}} \\
&= \frac{2^{dp/2}}{h_n^{d(3d+2)/2} n^{a(d+p)-d^2/2} (\log n)^{b(d+p)}} \\
&\leq \frac{2^{dp/2}}{n(\log n)^{b(d+p)+\theta_1 d(3d+2)/2}} \quad \text{since } \theta \geq \theta_2.
\end{aligned}$$

Now $b(d+p) + \theta_1 d(3d+2)/2 > 1$ implies $\sum_{n\geq 1} \tau_n^{-1} < \infty$. Applying Borel-Cantelli's lemma, it follows that for any $\lambda > 0$

$$\mathbb{P}\left(\limsup_{n\to\infty}\{|A_{3,n}| > \lambda v_n\}\right) = 0,$$

that is for any $\lambda > 0$

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{|A_{3,n}|}{v_n} \leq \lambda\right) = 1.$$

The proof of Lemma 4 is complete and the point **3**) of Theorem 1 is obtained by combining Inequality (18) with Lemmas 2 and 4. The proof of Theorem 1 is complete.

4.2 Proof of Theorem 2

We follow the first part of the proof of Theorem 1 and we consider the estimation (18).

Lemma 5 *Assume that* **C3**) *holds for some* $p > 2$*. Let* $a > 0$ *be fixed and denote*

$$v_n = \frac{n^a}{(nh_n)^{d/2}} \quad \textit{and} \quad \theta = \frac{2a(d+p) - d^2}{d(3d+2)}.$$

If $\theta > 0$ *and* $h_n \geq n^{-\theta}$ *then*

$$\|A_{3,n}\|_p = O\left[v_n\right].$$

Proof of Lemma 5. Let $p > 2$ and $x \in [0,1]^d$ be fixed. Using the Marcinkiewicz-Zygmund type inequality by Dedecker (see [13]) as in the proof of Lemma 4 there exist $\gamma^{''}, c > 0$ such that

$$\begin{aligned} \|V_n(x)\|_p &= \left(\frac{E|S_n(x)|^p}{\left(\sum_{i\in\Lambda_n} a_i(x)\right)^p} \right)^{1/p} \\ &\leq \gamma^{''} \left(\sum_{i\in\Lambda_n} a_i(x) \right)^{-1/2} \\ &\leq \frac{\gamma^{''}}{\sqrt{c}}([nh_n])^{-d/2} \qquad \text{by Lemma 1.} \end{aligned}$$

It follows that

$$r_n^{1/p} \sup_{x\in[0,1]^d} \|V_n(x)\|_p = O\left[\frac{v_n}{\tau_n}\right]$$

where $\tau_n = l_n^{d/p} v_n([nh_n])^{d/2}$. If n is sufficiently large then $\tau_n \geq 2^{-d/2} l_n^{d/p} v_n (nh_n)^{d/2}$, hence using $h_n \geq n^{-\theta}$ we obtain $\tau_n \geq 2^{-d/2}$. Finally, we derive

$$\|A_{3,n}\|_p = \| \max_{1\leq k\leq r_n} |V_n(x_k)| \|_p \leq r_n^{1/p} \sup_{x\in[0,1]^d} \|V_n(x)\|_p = O\left[v_n\right].$$

The proof of Lemma 5 is complete. The point **1**) of Theorem 2 is obtained by combining inequality (18) and lemmas 2 and 5.

Now, we are going to prove the point **2**) of Theorem 2. We have

$$\begin{aligned} E(S_n(x)^2) &= \sum_{k,l\in\Lambda_n} a_k(x)\, a_l(x) E(\varepsilon_k \varepsilon_l) \\ &= \sum_{k\in\Lambda_n} a_k(x)^2 E(\varepsilon_k^2) + \sum_{k\neq l} a_k(x) a_l(x) E(\varepsilon_k \varepsilon_l) \\ &= E(\varepsilon_0^2) \sum_{k\in\Lambda_n} a_k(x)^2 + \sum_{k\in\Lambda_n} a_k(x) \sum_{l\in\Lambda_n\backslash\{k\}} a_l(x) E(\varepsilon_k \varepsilon_l) \\ &\leq \sum_{l\in\mathbb{Z}^d} |E(\varepsilon_0 \varepsilon_l)| \times \sum_{k\in\Lambda_n} a_k(x). \end{aligned}$$

If Condition **C4**) holds then using Lemma 1 there exists $\gamma > 0$ such that for any $x \in [0,1]^d$ we have $E(S_n(x)^2) \leq \gamma \prod_{k=1}^d [n(x_k + h_n)]$. Let $x \in [0,1]^d$ be fixed, using Lemma 1, there exists $c > 0$ such that

$$\begin{aligned}
\|V_n(x)\|_2 &= \frac{\|S_n(x)\|_2}{\sum_{i\in\Lambda_n} a_i(x)} \\
&\leq \frac{\sqrt{\gamma}}{c}\left(\prod_{k=1}^d [n(x_k+h_n)]\right)^{-1/2} \\
&\leq \frac{\sqrt{\gamma}}{c\,([nh_n])^{d/2}} \\
&\leq \frac{2^{d/2}\sqrt{\gamma}}{c\,(nh_n)^{d/2}} \qquad \text{for } n \text{ sufficiently large.}
\end{aligned}$$

The proof of Theorem 2 is complete.

4.3 Proof of Proposition 1

Since $g \in \mathrm{Lip}(B)$, it follows that

$$\begin{aligned}
|B_n(x)| &= \left|\frac{\sum_{i\in\Lambda_n}(g(i/n)-g(x))a_i(x)}{\sum_{i\in\Lambda_n} a_i(x)}\right| \\
&\leq Bh_n \frac{\sum_{i\in\Lambda_n}\|(i/n-x)/h_n\| a_i(x)}{\sum_{i\in\Lambda_n} a_i(x)} \\
&\leq Bh_n.
\end{aligned}$$

The proof of Proposition 1 is complete.

4.4 Proof of Corollary 1

Let $h_n = (n^{-d}\log n)^{1/(2+d)}$ then Proposition 1 gives

$$\sup_{x\in[0,1]^d}\ \sup_{g\in\mathrm{Lip}(B)} |Eg_n(x) - g(x)| = O\left[\left(\frac{\log n}{n^d}\right)^{\frac{1}{2+d}}\right]. \tag{24}$$

Assume that **C1**) holds. Noting that

$$\frac{(\log n)^{1/2}}{(nh_n)^{d/2}} = \left(\frac{\log n}{n^d}\right)^{\frac{1}{2+d}}$$

and using (5) we obtain

$$\sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| = O_{a.s.}\left[\left(\frac{\log n}{n^d}\right)^{\frac{1}{2+d}}\right]. \tag{25}$$

Combining (24) and (25) we derive (10).

Assume that **C2**) holds for some $0 < q < 2$. Noting that

$$\frac{(\log n)^{1/q}}{(nh_n)^{d/2}} = \left(\frac{\log n}{n^d}\right)^{\frac{1}{2+d}} \times (\log n)^{(2-q)/2q}$$

and using (6) we obtain

$$\sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| = O_{a.s.}\left[\left(\frac{\log n}{n^d}\right)^{\frac{1}{2+d}} \times (\log n)^{(2-q)/2q}\right]. \tag{26}$$

Combining (24) and (26) we derive (11).

Let $\varepsilon > 0$ be fixed and assume that **C3**) holds for some $p > 2$ which satisfies condition (12). Applying the point **3**) of Theorem 1 with $\theta_1 = 1/(2+d)$ and $\theta_2 = d/(2+d)$ and noting that

$$v_n = \frac{n^a(\log n)^b}{(nh_n)^{d/2}} = n^\varepsilon \left(\frac{\log n}{n^d}\right)^{\frac{1}{2+d}} \Longleftrightarrow \left\{a = \varepsilon \text{ and } b = \frac{1}{2}\right\}$$

it follows

$$\sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| = O_{a.s.}\left[n^\varepsilon \left(\frac{\log n}{n^d}\right)^{\frac{1}{2+d}}\right]. \tag{27}$$

Combining (24) and (27) we derive (13). The proof of Corollary 1 is complete.

4.5 Proof of Corollary 2

Let $h_n = n^{-d/(2+d)}$ then Proposition 1 gives

$$\sup_{x\in[0,1]^d} \sup_{g\in\mathrm{Lip}(B)} |Eg_n(x) - g(x)| = O\left[n^{-\frac{d}{2+d}}\right]. \tag{28}$$

Let $\varepsilon > 0$ be fixed and assume that **C3**) holds for some $p > 2$ which satisfies condition (14). Applying the point **1**) of Theorem 2 and noting that

$$v_n = \frac{n^a}{(nh_n)^{d/2}} = n^{-\frac{d}{2+d}+\varepsilon} \Longleftrightarrow a = \varepsilon$$

it follows that

$$\left\| \sup_{x\in[0,1]^d} |g_n(x) - Eg_n(x)| \right\|_p = O\left[n^{-\frac{d}{2+d}+\varepsilon}\right]. \tag{29}$$

Combining (28) and (29) we derive (15).

Since $h_n = n^{-d/(2+d)}$ then $(nh_n)^{-d/2} = h_n$. So, if **C4**) holds then combining (28) and (9) we derive (16). The proof of Corollary 2 is complete.

4.6 Proof of Corollary 3

Let $p > 2$ be fixed. Using Rio's inequality [30] (see also Dedecker [13]) we obtain the bound

$$\|\varepsilon_k E_{|k|}(\varepsilon_0)\|_{\frac{p}{2}} \le 4\left(\int_0^{\alpha_{1,\infty}(|k|)} Q^p_{\varepsilon_0}(u)\,du\right)^{2/p} \tag{30}$$

hence condition $\mathbf{C}'\mathbf{3}$) is more restrictive than condition **C3**).
By Serfling's inequality (see McLeish [27] or Serfling [32]) we know that

$$\|\varepsilon_k E_{|k|}(\varepsilon_0)\|_\infty \le 2\|\varepsilon_0\|^2_\infty \phi_{\infty,1}(|k|)$$

so condition $\mathbf{C}'\mathbf{1}$) is more restrictive than condition **C1**).
Now for $0 < q < 2$ there exists $C(q) > 0$ (cf. Inequality (17) in [16]) such that

$$\left\|\sqrt{|\varepsilon_k E_{|k|}(\varepsilon_0)|}\right\|^2_{\psi_{\beta(q)}} \le C(q)\sqrt{\phi_{\infty,1}(|k|)}. \tag{31}$$

In [16] we used the following lemma which can be obtain by the expansion of the exponential function.

Lemma 6 *Let β be a positive real number and Z be a real random variable. There exist positive universal constants A_β and B_β depending only on β such that*

$$A_\beta \sup_{p>2} \frac{\|Z\|_p}{p^{1/\beta}} \le \|Z\|_{\psi_\beta} \le B_\beta \sup_{p>2} \frac{\|Z\|_p}{p^{1/\beta}}.$$

Consider the coefficient $c_k(\beta)$ given by (3) and denote

$$d_k(p) = \left(\int_0^{\alpha_{1,\infty}(|k|)} Q^p_{\varepsilon_0}(u)\,du\right)^{1/p}$$

then the following version of lemma 6 holds.

Lemma 7 *Let β be a positive real number. There exist positive universal constants A_β and B_β depending only on β such that for any $k \in \mathbb{Z}^d$*

$$A_\beta \sup_{p>2} \frac{d_k(p)}{p^{1/\beta}} \leq c_k(\beta) \leq B_\beta \sup_{p>2} \frac{d_k(p)}{p^{1/\beta}}.$$

Now combining lemmas 6 and 7 and inequality (30) there exists $C^{'}(q) > 0$ such that

$$\left\| \sqrt{|\varepsilon_k E_{|k|}(\varepsilon_0)|} \right\|_{\psi_{\beta(q)}}^2 \leq C^{'}(q)\, c_k^2(\beta(q)). \tag{32}$$

Finally condition $\mathbf{C^{'}2}$) is more restrictive than condition **C2**) and the proof of Corollary 3 is complete.

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