Invariance principles for standard-normalized and self-normalized random fields

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Abstract

We investigate the invariance principle for set-indexed partial sums of a stationary field $(X_k)_{k\in\mathbb{Z}^d}$ of martingale-difference or independent random variables under standard-normalization or self-normalization respectively.

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1 Introduction

Let $(X_k)_{k\in\mathbb{Z}^d}$ be a stationary field of real-valued random variables defined on a given probability space $(\Omega,\mathcal{F},\mathbb{P})$. If $\mathcal{A}$ is a collection of Borel subsets of $[0,1]^d$, define the smoothed partial sum process $\{S_n(A)\,;\,A\in\mathcal{A}\}$ by

$$S_n(A)=\sum_{i\in\{1,\dots,n\}^d}\lambda(nA\cap R_i)X_i \tag{1}$$

where $R_i=]i_1-1,i_1]\times\ldots\times]i_d-1,i_d]$ is the unit cube with upper corner at i and λ is the Lebesgue measure on $\mathbb{R}^d$. We equip the collection $\mathcal{A}$ with the pseudo-metric ρ defined for any A,B in $\mathcal{A}$ by $\rho(A,B)=\sqrt{\lambda(A\Delta B)}$. To measure the size of $\mathcal{A}$ one considers the metric entropy: denote by $H(\mathcal{A},\rho,\varepsilon)$

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the logarithm of the smallest number $N(\mathcal{A},\rho,\varepsilon)$ of open balls of radius ε with respect to ρ which form a covering of $\mathcal{A}$. The function $H(\mathcal{A},\rho,.)$ is the entropy of the class $\mathcal{A}$. A more strict tool is the metric entropy with inclusion: assume that $\mathcal{A}$ is totally bounded with inclusion i.e. for each positive ε there exists a finite collection $\mathcal{A}(\varepsilon)$ of Borel subsets of $[0,1]^d$ such that for any $A\in\mathcal{A}$, there exist A^- and A^+ in $\mathcal{A}(\varepsilon)$ with $A^-\subseteq A\subseteq A^+$ and $\rho(A^-,A^+)\leq\varepsilon$. Denote by $\mathbb{H}(\mathcal{A},\rho,\varepsilon)$ the logarithm of the cardinality of the smallest collection $\mathcal{A}(\varepsilon)$. The function $\mathbb{H}(\mathcal{A},\rho,.)$ is the entropy with inclusion (or bracketing entropy) of the class $\mathcal{A}$. Let $C(\mathcal{A})$ be the space of continuous real functions on $\mathcal{A}$, equipped with the norm $\|.\|_{\mathcal{A}}$ defined by

$$\|f\|_{\mathcal{A}}=\sup_{A\in\mathcal{A}}|f(A)|.$$

A standard Brownian motion indexed by $\mathcal{A}$ is a mean zero Gaussian process W with sample paths in $C(\mathcal{A})$ and Cov(W(A),W(B))= $\lambda(A\cap B)$. From Dudley [8] we know that such a process exists if

$$\int_0^1\sqrt{H(\mathcal{A},\rho,\varepsilon)}\,d\varepsilon<+\infty. \tag{2}$$

Since $H(\mathcal{A},\rho,.)\leq\mathbb{H}(\mathcal{A},\rho,.)$, the standard Brownian motion W is well defined if

$$\int_0^1\sqrt{\mathbb{H}(\mathcal{A},\rho,\varepsilon)}\,d\varepsilon<+\infty. \tag{3}$$

For any probability measure m defined on $[0,1]^d$ equipped with its Borel σ-algebra, we define the pseudo-metric ρ_m by $\rho_m=\sqrt{m(A\Delta B)}$ for any A and B in $\mathcal{A}$. For any positive $\varepsilon>0$, we denote $N(\mathcal{A},\varepsilon)=\sup_m N(\mathcal{A},\rho_m,\varepsilon)$ and we say that the collection $\mathcal{A}$ has uniformly integrable entropy if

$$\int_0^1\sqrt{\log N(\mathcal{A},\varepsilon)}d\varepsilon<+\infty. \tag{4}$$

We say that the (classical) invariance principle or functional central limit theorem (FCLT) holds if the sequence $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ converges in distribution to an $\mathcal{A}$-indexed Brownian motion in the space $C(\mathcal{A})$. The first weak convergence results for $\mathcal{Q}_d$-indexed partial sum processes were established for i.i.d. random fields and for the collection $\mathcal{Q}_d$ of lower-left quadrants in $[0,1]^d$, that is to say the collection $\{[0,t_1]\times\ldots\times[0,t_d]\,;\,(t_1,\ldots,t_d)\in[0,1]^d\}$. They were proved by Wichura [25] under a finite variance condition and earlier by Kuelbs [17] under additional moment restrictions. When the dimension d is reduced to one, these results coincide with the original invariance principle of Donsker [7]. In 1983, Pyke [21] derived a weak convergence result for the process $\{S_n(A)\,;\, A \in \mathcal{A}\}$ for i.i.d. random fields provided that the collection $\mathcal{A}$ satisfies the bracketing entropy condition (3). However, his result required moment conditions which depend on the size of the collection $\mathcal{A}$. Bass [3] and simultaneously Alexander and Pyke [1] extended Pyke's result to i.i.d. random fields with finite variance. More precisely, the following result is proved.

Theorem A (Bass (1985), Alexander and Pyke (1986)) *Let $(X_k)_{k\in\mathbb{Z}^d}$ be a stationary field of independent real random variables with zero mean and finite variance. If $\mathcal{A}$ is a collection of regular Borel subsets of $[0,1]^d$ which satisfies Assumption (3) then the sequence of processes $\{n^{-d/2}S_n(A); A \in \mathcal{A}\}$ converge in distribution to $\sqrt{E(X_0^2)}W$ where W is a standard Brownian motion indexed by $\mathcal{A}$.*

Unfortunately, the bracketing condition (3) is not automatically fulfilled in the important case of $\mathcal{A}$ being a Vapnik-Chervonenkis class of sets. Ziegler [26] has covered this case by proving (among other results) that the FCLT of Bass, Alexander and Pyke (i.e. Theorem A) still holds for classes of sets which satisfy the uniformly integrable entropy condition (4). Recently, Dedecker [6] gave an L^∞-projective criterion for the process $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ to converge to a mixture of $\mathcal{A}$-indexed Brownian motions when the collection $\mathcal{A}$ satisfies only the entropy condition (2) of Dudley. This new criterion is valid for martingale-difference bounded random fields and provides a new criterion for non-uniform ϕ-mixing bounded random fields. In the unbounded case, using the chaining method of Bass [3] and establishing Bernstein type inequalities, Dedecker proved also the FCLT for the partial sum $\{S_n(A)\,;\, A \in \mathcal{A}\}$ of non-uniform ϕ-mixing random fields provided that the collection $\mathcal{A}$ satisfies the more strict entropy condition with inclusion (3) and under both finite fourth moments and a polynomial decay of the mixing coefficients. In a previous work (see [12]), it is shown that the FCLT may be not valid for p-integrable ($0 \leq p < +\infty$) martingale-difference random fields. More precisely, the following result is established.

Theorem B (El Machkouri, Volný, 2002) *Let $(\Omega, \mathcal{F}, \mu, T)$ be an ergodic dynamical system with positive entropy where Ω is a Lebesgue space, μ is a probability measure and T is a $\mathbb{Z}^d$-action. For any nonnegative real p, there exist a real function $f \in L^p(\Omega)$ and a collection $\mathcal{A}$ of regular Borel subsets of $[0,1]^d$ such that*

- *For any k in $\mathbb{Z}^d$, $E\left(f\circ T^k | \sigma(f\circ T^i\,;\, i \neq k)\right) = 0$. We say that the random field $(f\circ T^k)_{k\in\mathbb{Z}^d}$ is a strong martingale-difference random field.*
- *The collection $\mathcal{A}$ satisfies the entropy condition with inclusion (3).*

- *The partial sum process* $\{n^{-d/2}S_n(f,A)\,;\,A\in\mathcal{A}\}$ *is not tight in the space* $C(\mathcal{A})$

where

$$S_n(f,A) := \sum_{i\in\{1,\dots,n\}^d} \lambda(nA\cap R_i) f\circ T^i.$$

The above theorem shows that not only Dedecker's FCLT for bounded random fields (see [6]) cannot be extended to p-integrable ($0\leq p<+\infty$) random fields but also it lays emphasis on that Bass, Alexander and Pyke's result for i.i.d. random fields (Theorem A) cannot hold for martingale-difference random fields without additional assumptions. Recently, El Machkouri [11] has shown that the FCLT still holds for unbounded random fields which satisfy both a finite exponential moment condition and a projective criterion similar to Dedecker's one. All these results put on light that the moment assumption on the random field is very primordial in the FCLT question for random fields indexed by large classes of sets.
In the present work, we give a positive answer to the validity of the FCLT for square-integrable martingale-difference random fields which conditional variances are bounded almost surely (cf. Theorem 1). Next, we consider self-normalized i.i.d. random fields, more precisely, we investigate the validity of the FCLT when the stationary random field $(X_k)_{k\in\mathbb{Z}^d}$ is assumed to be independent and the classical normalization $n^{d/2}$ is replaced by U_n defined by (5) (cf. Theorem 2). From a statistical point of view, the self-normalization is natural and several articles in the literature are devoted to limit theorems for self-normalized sequences $(X_k)_{k\in\mathbb{Z}}$ of independent random variables with statistical applications. Logan et al. [19] investigate the various possible limit distributions of self-normalized sums. Giné et al. [13] prove that $\sum_{i=1}^n X_i/\sqrt{\sum_{i=1}^n X_i^2}$ converges to the Gaussian standard distribution if and only if X_1 is in the domain of attraction of the normal distribution (the symmetric case was previously treated by Griffin and Mason [14]). Egorov [10] investigates the non identically distributed case. Large deviations are investigated in Shao [23] without moment conditions. Račkausksas and Suquet [22] gives invariance principles for various partial sums processes under self-normalization in $\mathcal{C}([0,1])$ and in the stronger topological framework of Hölder spaces. Our Theorem 2 below improves on Račkauskas and Suquet's result in $\mathcal{C}([0,1])$.

2 Main results

By a stationary real random field we mean any family $(X_k)_{k\in\mathbb{Z}^d}$ of real-valued random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that for any

$(k,n) \in \mathbb{Z}^d \times \mathbb{N}^*$ and any $(i_1, ..., i_n) \in (\mathbb{Z}^d)^n$, the random vectors $(X_{i_1}, ..., X_{i_n})$ and $(X_{i_1+k}, ..., X_{i_n+k})$ have the same law.
On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i = (i_1, ..., i_d)$ and $j = (j_1, ..., j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{lex} j$ means that either $i_1 < j_1$ or for some p in $\{2, 3, ..., d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \le q < p$. A real random field $(X_k)_{k\in\mathbb{Z}^d}$ is said to be a martingale-difference random field if it satisfies the following condition: for any m in $\mathbb{Z}^d$, $E\left(X_m|\mathcal{F}_m\right) = 0$ a.s. where $\mathcal{F}_m$ is the σ-algebra generated by the random variables X_k, $k <_{lex} m$. Our first result is the following.

Theorem 1 *Let $(X_k)_{k\in\mathbb{Z}^d}$ be a stationary field of martingale-difference random variables with finite variance such that $E(X_0^2|\mathcal{F}_0)$ is bounded almost surely and let $\mathcal{A}$ be a collection of regular Borel subsets of $[0,1]^d$ satisfying the condition (3). Then the sequence $\{n^{-d/2}S_n(A);\ A \in \mathcal{A}\}$ converges weakly in $C(\mathcal{A})$ to $\sqrt{E(X_0^2)}W$ where W is the standard Brownian motion indexed by $\mathcal{A}$.*

Comparing Theorem 1 and Theorem B in section 1, one can notice that the conditional variance $E\left(X_0^2|\mathcal{F}_0\right)$ is primordial in the invariance principle problem for martingale-difference random fields. More generally, the conditional variance for martingales is known to play an important role in modern martingale limit theory (see Hall and Heyde [15]).
For any integer $n \ge 1$, we define

$$U_n^2 = \sum_{i\in\Lambda_n} X_i^2 \tag{5}$$

where $\Lambda_n = \{1, ..., n\}^d$. We say that X_0 belongs to the domain of attraction of the normal distribution (and we denote $X_0 \in DAN$) if there exists a norming sequence b_n of real numbers such that $b_n^{-1}S_{\Lambda_n}$ converges in distribution to a standard normal law. We should recall that if $X_0 \in DAN$ then $\|X_0\|_p < \infty$ for any $0 < p < 2$ and that constants b_n have the form $b_n = n^{d/2}l(n)$ for some function l slowly varying at infinity. Moreover, for each $\tau > 0$, we have

$$\lim_{n\to\infty} n^d E X_{0,n} = 0,\ \lim_{n\to\infty} n^d \mathbb{P}(|X_0| \ge \tau b_n) = 0 \text{ and } \lim_{n\to\infty} b_n^{-2} n^d E(X_{0,n}^2) = 1 \tag{6}$$

where $X_{0,n} = X_0 \, \mathbb{1}_{|X_0|<\tau b_n}$ (see for instance Araujo and Giné [2]). Note also that $X_0 \in DAN$ implies (Raikov's theorem) that

$$\frac{1}{b_n^2} \sum_{i\in\Lambda_n} X_i^2 \xrightarrow[n\to\infty]{\mathbb{P}} 1. \tag{7}$$

Theorem 2 *Let $(X_k)_{k\in\mathbb{Z}^d}$ be a field of i.i.d. centered random variables and let $\mathcal{A}$ be a collection of regular Borel subsets of $[0,1]^d$ satisfying the condition* (3)*. Then $X_0 \in DAN$ if and only if the sequence $\{U_n^{-1}S_n(A);\, A \in \mathcal{A}\}$ converges weakly in $C(\mathcal{A})$ to the standard Brownian motion W.*

Let us remark that the necessity of $X_0 \in DAN$ in Theorem 2 follows from Giné et al. ([13], Theorem 3.3). Our result contrasts with the invariance principle established by Bass and Alexander and Pyke (cf. Theorem A in section 1) where square integrable random variables are required. We do not know if Theorem 2 still hold if one replace the condition (3) by condition (2). However, our next result is a counter-example which shows that Theorem A in section 1 does not hold when the condition (3) is replaced by condition (2).

Theorem 3 *For any positive real number p, there exist a stationary field $(X_k)_{k\in\mathbb{Z}^d}$ of independent, symmetric and p-integrable real random variables and a collection $\mathcal{A}$ of regular Borel subsets of $[0,1]^d$ which satisfies the condition* (2) *such that the partial sum process $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ do not be tight in the space $C(\mathcal{A})$.*

Note that Dudley and Strassen [9] have built a sequence of i.i.d. random variables X_n with values in the space of continuous functions on $[0,1]$ such that $E(X_1(t)) = 0$ and the finite dimensional marginals of $Z_n(t) = n^{-1/2}\sum_{i=1}^n X_i(t)$ converge to that of a Gaussian process Z. It was shown that this process Z has a version with almost sure continuous sample paths and that the process $Z_n(t)$ is not tight for the topology of the uniform metric. However, contrary to our example, one can check that the limiting process Z does not satisfy the Dudley's entropy condition (2) for the intrinsic distance $\rho(s,t) = \|Z(s) - Z(t)\|_2$. In fact, it is well known that the condition (2) is sufficient for Gaussian processes to have a version with almost sure continuous sample paths but it falls to be necessary (see van der Vaart and Wellner [24], p. 445).

3 Proofs

Recall that a Young function ψ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies $\psi(0) = 0$. We define the Orlicz space L_ψ as the space of real random variables Z defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[\psi(|Z|/c)] < +\infty$ for some $c > 0$. The Orlicz space L_ψ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable Z by

$$\|Z\|_\psi = \inf\{\, c > 0\,;\, E[\psi(|Z|/c)] \le 1\,\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [16]. Let $\psi_1, \psi_2 : \mathbb{R}^+ \to \mathbb{R}$ be the Young functions defined by $\psi_1(x) = \exp(x) - 1$ and $\psi_2(x) = \exp(x^2) - 1$ for any $x \in \mathbb{R}^+$. We need the following lemma which is of independent interest.

Lemma 1 *Let $(\theta_i)_{i\in\mathbb{Z}^d}$ be an arbitrary field of random variables and let $\mathcal{H}_i$ denote the σ-algebra generated by the random variables θ_j, $j <_{lex} i$, $i \in \mathbb{Z}^d$. Let also $0 \leq \alpha \leq \beta \leq 1$ and $0 < \tau \leq 1$ be fixed and let $(c_n)_{n\geq 1}$ be a sequence of real numbers. For any integer $n \geq 1$ and any Borel subset A of $[0,1]^d$, denote*

$$\theta_i(n,\alpha,\beta) = \theta_i \, \mathbb{1}_{\alpha\tau c_n \leq |\theta_i| < \beta\tau c_n}$$

and

$$\Theta_n(A,\alpha,\beta) = \frac{1}{c_n}\sum_{i\in\Lambda_n} \lambda(nA\cap R_i)[\theta_i(n,\alpha,\beta) - E\left(\theta_i(n,\alpha,\beta)|\mathcal{H}_i\right)].$$

Assume also that there exists $C > 0$ such that for any integer $n \geq 1$ and any i in $\mathbb{Z}^d$,

$$\frac{n^d}{c_n^2} E\left(\theta_i^2 \, \mathbb{1}_{|\theta_i|<c_n} |\mathcal{H}_i\right) \leq C. \tag{8}$$

If $\mathcal{G}_1, \mathcal{G}_2$ are finite collections of Borel subsets of $[0,1]^d$ then

$$\left\| \max_{(A,B)\in\mathcal{G}} \left|\Theta_n(A,\alpha,\beta) - \Theta_n(B,\alpha,\beta)\right| \right\|_{\psi_1} \leq K[\beta\,\tau\,\psi_1^{-1}(|\mathcal{G}|) + \max_{(A,B)\in\mathcal{G}} \rho(A,B)\,\psi_2^{-1}(|\mathcal{G}|)]$$

where $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, $|\mathcal{G}|$ is the cardinal of $\mathcal{G}$ and $K > 0$ is a universal constant.

Proof of Lemma 1. Consider the field of martingale-difference random variables $Y_i(n,\alpha,\beta)$, $i \in \Lambda_n$ defined by

$$Y_i(n,\alpha,\beta) = \frac{1}{c_n}(\lambda(nA\cap R_i) - \lambda(nB\cap R_i))[\theta_i(n,\alpha,\beta) - E\left(\theta_i(n,\alpha,\beta)|\mathcal{H}_i\right)]$$

and note that $|Y_i(n,\alpha,\beta)| \leq 2\beta\tau$. Using (8) and keeping in mind that τ and β are less than 1, there exists a universal constant $C > 0$ such that

$$\sum_{i\in\Lambda_n} E\left(Y_i(n,\alpha,\beta)^2|\mathcal{H}_i\right) \leq 4C \max_{(A,B)\in\mathcal{G}} \rho^2(A,B).$$

Noting that $\Theta_n(A,\alpha,\beta) - \Theta_n(B,\alpha,\beta) = \sum_{i\in\Lambda_n} Y_i(n,\alpha,\beta)$ and applying Theorem 1.2A in de la Pena [4], we derive the following Bernstein inequality

$$\mathbb{P}\left(\left|\Theta_n(A,\alpha,\beta) - \Theta_n(B,\alpha,\beta)\right| > x\right) \leq 2\exp\left(\frac{-x^2}{8C\max_{(A,B)\in\mathcal{G}}\rho^2(A,B) + 4\beta\tau x}\right).$$

The proof is completed by using Lemma 2.2.10 in van der Vaart and Wellner [24].

3.1 Proof of Theorem 1

a) Tightness

It suffices to prove that for any $x > 0$

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}}\left|n^{-d/2}S_n(A)-n^{-d/2}S_n(B)\right|>x\right)=0. \tag{9}$$

In the sequel, we write $\mathbb{H}(x)$ for $\mathbb{H}(\mathcal{A},\rho,x)$. Let $\delta > 0$ be fixed, denote $\tau = \delta/\sqrt{\mathbb{H}(\delta/2)} > 0$ and assume (without loss of generality) that $\tau \leq 1$. Let $i \in \mathbb{Z}^d$, since X_i is a martingale-difference random variable, we have $X_i = X_{i,n} - E(X_{i,n}|\mathcal{F}_i) + \overline{X}_{i,n} - E(\overline{X}_{i,n}|\mathcal{F}_i)$ where $X_{i,n} = X_i\,\mathbb{1}_{|X_i|<\tau n^{d/2}}$ and $\overline{X}_{i,n} = X_i - X_{i,n}$, hence it follows

$$\mathbb{P}\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}}\left|n^{-d/2}S_n(A)-n^{-d/2}S_n(B)\right|>x\right)\leq E_1+E_2$$

where

$$E_1=\mathbb{P}\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}}\left|\sum_{i\in\Lambda_n}(\lambda(nA\cap R_i)-\lambda(nB\cap R_i))[X_{i,n}-E\left(X_{i,n}|\mathcal{F}_i\right)]\right|>xn^{d/2}/2\right)$$

$$E_2=n^d\mathbb{P}\left(|X_0|\geq\tau n^{d/2}\right)\xrightarrow[n\to+\infty]{}0\qquad(\text{since } X_0\in L^2).$$

We are going to control E_1. Now, for any constants $0 \leq \alpha \leq \beta \leq 1$ define $X_i(n,\alpha,\beta) = X_i\,\mathbb{1}_{\alpha\tau n^{d/2}\leq|X_i|<\beta\tau n^{d/2}}$ and

$$Z_n(A,\alpha,\beta)=\frac{1}{n^{d/2}}\sum_{i\in\Lambda_n}\lambda(nA\cap R_i)[X_i(n,\alpha,\beta)-E\left(X_i(n,\alpha,\beta)|\mathcal{F}_i\right)].$$

One can notice that

$$E_1\leq\frac{2}{x}E\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}}\left|Z_n(A,0,1)-Z_n(B,0,1)\right|\right).$$

Let $\delta_k = 2^{-k}\delta$. If A and B are any sets in $\mathcal{A}$, there exists sets A_k, A_k^+, B_k, B_k^+ in the finite class $\mathcal{A}(\delta_k)$ such that $A_k \subset A \subset A_k^+$ and $\rho(A_k, A_k^+) \leq \delta_k$, and similarly for B, B_k, B_k^+. Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers decreasing to zero sucht that $a_0 = 1$. Following the chaining method initiated by Bass [3], we write

$$Z_n(A,0,1) - Z_n(A_0,0,1) = \sum_{k=0}^{+\infty} Z_n(A_{k+1},0,a_k) - Z_n(A_k,0,a_k) \\ + \sum_{k=1}^{+\infty} Z_n(A,a_k,a_{k-1}) - Z_n(A_k,a_k,a_{k-1}).$$

So, we have $\frac{x}{2}E_1 \leq F_1 + F_2 + F_3$ where

$$F_1 = E\left(\max_{\substack{A_0,\,B_0\in\mathcal{A}(\delta_0)\\ \rho(A_0,B_0)\leq 3\delta_0}} \left|Z_n(A_0,0,1) - Z_n(B_0,0,1)\right|\right)$$

$$F_2 = 2\sum_{k=0}^{+\infty} E\left(\max_{\substack{A_k\in\mathcal{A}(\delta_k),\,A_{k+1}\in\mathcal{A}(\delta_{k+1})\\ \rho(A_k,A_{k+1})\leq 2\delta_k}} \left|Z_n(A_{k+1},0,a_k) - Z_n(A_k,0,a_k)\right|\right)$$

$$F_3 = 2\sum_{k=1}^{+\infty} E\left(\max_{\substack{A_k,\,A_k^+\in\mathcal{A}(\delta_k)\\ \rho(A_k,A_k^+)\leq\delta_k}} \sup_{A_k\subset A\subset A_k^+} \left|Z_n(A,a_k,a_{k-1}) - Z_n(A_k,a_k,a_{k-1})\right|\right)$$

In the sequel, we denote by K any universal positive constant. Applying Lemma 1 with $c_n = n^{d/2}$, we derive

$$F_1 \leq K\left(\tau\mathbb{H}(\delta_0) + \delta_0\sqrt{\mathbb{H}(\delta_0)}\right), \tag{10}$$

similarly

$$F_2 \leq K\sum_{k=0}^{+\infty}(a_k\tau\mathbb{H}(\delta_{k+1}) + \delta_k\sqrt{\mathbb{H}(\delta_{k+1})}). \tag{11}$$

Now, we are going to control the last term F_3. For any Borel subset A of $[0,1]^d$, we denote

$$\widetilde{Z}_n(A,a_k,a_{k-1}) = \frac{1}{n^{d/2}}\sum_{i\in\Lambda_n}\lambda(nA\cap R_i)[|X_i(n,a_k,a_{k-1})| - E\left(|X_i(n,a_k,a_{k-1})||\mathcal{F}_i\right)].$$

One can check that

$$\begin{aligned}
&\sup_{A_k\subset A\subset A_k^+}|Z_n(A,a_k,a_{k-1})-Z_n(A_k,a_k,a_{k-1})| \\
&\quad\le \frac{1}{n^{d/2}}\sum_{i\in\Lambda_n}(\lambda(nA_k^+\cap R_i)-\lambda(nA_k\cap R_i))[|X_i(n,a_k,a_{k-1})|-E\left(|X_i(n,a_k,a_{k-1})||\mathcal{F}_i\right)] \\
&\qquad+\frac{2}{n^{d/2}}\sum_{i\in\Lambda_n}(\lambda(nA_k^+\cap R_i)-\lambda(nA_k\cap R_i))E\left(|X_i(n,a_k,a_{k-1})||\mathcal{F}_i\right) \\
&\quad=\widetilde{Z}_n(A_k^+,a_k,a_{k-1})-\widetilde{Z}_n(A_k,a_k,a_{k-1}) \\
&\qquad+\frac{2}{n^{d/2}}\sum_{i\in\Lambda_n}\lambda(n\left(A_k^+\backslash A_k\right)\cap R_i)E\left(|X_i(n,a_k,a_{k-1})||\mathcal{F}_i\right)
\end{aligned}$$

Recall that by assumption we have $E(X_i^2|\mathcal{F}_i)\le C$ for some $C>0$. So, using Lemma 1, it follows

$$\left\|\max_{A_k,A_k^+\in\mathcal{A}(\delta_k)}\left|\widetilde{Z}_n(A_k^+,a_k,a_{k-1})-\widetilde{Z}_n(A_k,a_k,a_{k-1})\right|\right\|_{\psi_1}\le K(a_{k-1}\tau\mathbb{H}(\delta_k)+\delta_k\sqrt{\mathbb{H}(\delta_k)}).$$

Moreover, one can check that

$$E\left(|X_i(n,a_k,a_{k-1})||\mathcal{F}_i\right)\le\frac{E\left(X_i^2|\mathcal{F}_i\right)}{a_k\tau n^{d/2}}\le\frac{C}{a_k\tau n^{d/2}}.$$

Consequently, we obtain

$$F_3\le K\left(\sum_{k=1}^{+\infty}a_{k-1}\tau\mathbb{H}(\delta_k)+\delta_k\sqrt{\mathbb{H}(\delta_k)}+\frac{\delta_k^2}{\tau a_k}\right)\tag{12}$$

Now, we choose $a_k=\delta_k/(\tau\sqrt{\mathbb{H}(\delta_{k+1})})$ for all $k\in\mathbb{N}$ (note that $a_0=1$), hence, we obtain the following estimations:

$$\begin{aligned}
F_1&\le K\,\delta\sqrt{\mathbb{H}(\delta/2)} \\
F_2&\le K\sum_{k=0}^{+\infty}\delta_k\sqrt{\mathbb{H}(\delta_{k+1})} \\
F_3&\le K\sum_{k=1}^{+\infty}\delta_{k-1}\sqrt{\mathbb{H}(\delta_{k+1})}
\end{aligned}$$

Now, recall that $\frac{2}{x}E_1\le F_1+F_2+F_3$ and keep in mind that the entropy condition (3) holds then

$$\limsup_{n\to\infty}\frac{2}{x}E_1\le K\sum_{k=1}^{+\infty}\delta_{k+1}\sqrt{\mathbb{H}(\delta_k)}\le K\int_0^\delta\sqrt{\mathbb{H}(x)}dx\xrightarrow[\delta\to0]{}0.$$

Finally, the condition (9) holds and the sequence $\{n^{-d/2}S_n(A)\,;\,A \in \mathcal{A}\}$ is tight in the space $C(\mathcal{A})$.

b) Finite dimensional convergence

The convergence of the finite-dimensional laws is a simple consequence of both the central limit theorem for random fields ([5], Theorem 2.2) and the following lemma (see [6]). For any subset Γ of $\mathbb{Z}^d$ we consider

$$\partial\Gamma = \big\{ i \in \Gamma\,;\, \exists j \notin \Gamma \text{ such that } |i-j| = 1 \big\}.$$

For any Borel set A of $[0,1]^d$, we denote by $\Gamma_n(A)$ the finite subset of $\mathbb{Z}^d$ defined by $\Gamma_n(A) = nA \cap \mathbb{Z}^d$.

Lemma 2 (Dedecker, 2001) *Let A be a regular Borel set of $[0,1]^d$ with $\lambda(A) > 0$. We have*

$$(i)\ \lim_{n\to+\infty} \frac{|\Gamma_n(A)|}{n^d} = \lambda(A) \qquad (ii)\ \lim_{n\to+\infty} \frac{|\partial\Gamma_n(A)|}{|\Gamma_n(A)|} = 0.$$

Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary random field with mean zero and finite variance. Assume that $\sum_{k\in\mathbb{Z}^d} |E(X_0X_k)| < +\infty$. Then

$$\lim_{n\to+\infty} \quad n^{-d/2} \left\| S_n(A) - \sum_{k\in\Gamma_n(A)} X_k \right\|_2 = 0.$$

3.2 Proof of Theorem 2

Similarly, we are going to prove both the convergence of the finite-dimensional laws and the tightness of the sequence of processes $\{U_n^{-1}S_n(A)\,;\,A \in \mathcal{A}\}$ in the space $C(\mathcal{A})$.

a) Tightness

It suffices to establish that for any $x > 0$

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} \mathbb{P}\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}} \left| U_n^{-1}S_n(A) - U_n^{-1}S_n(B) \right| > x \right) = 0. \qquad (13)$$

Let $\delta > 0$ and $0 < \tau \le 1$ defined as in the proof of theorem 1. In the sequel, we denote $(b_n)_{n\ge1}$ the sequence which satisfies condition (6) and we define $X_{i,n} = X_i\, 1\!\!1_{|X_i|<\tau b_n}$. One can check that

$$\mathbb{P}\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}} \left| U_n^{-1}S_n(A) - U_n^{-1}S_n(B) \right| > x \right) \le E_1 + E_2 + E_3 + E_4$$

where

$$E_1 = \mathbb{P}\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}}\left|\sum_{i\in\Lambda n}(\lambda(nA\cap R_i)-\lambda(nB\cap R_i))[X_{i,n}-EX_{i,n}]\right|>xb_n/2\right)$$

$$E_2 = \mathbb{P}\left(U_n \le b_n/2\right) \xrightarrow[n\to+\infty]{} 0 \qquad \text{(by Raikov's theorem)}$$

$$E_3 = n^d\mathbb{P}\left(|X_0|\ge \tau b_n\right) \xrightarrow[n\to+\infty]{} 0 \qquad \text{(by (6))}$$

$$E_4 = x^{-1}b_n^{-1}n^d|EX_{0,n}| \xrightarrow[n\to+\infty]{} 0 \qquad \text{(by (6))}.$$

So, it suffices to control E_1. As in the proof of Theorem 1, we apply the chaining method by Bass [3] with the following notations: for any constants $0\le\alpha\le\beta\le 1$, we define $X_i(n,\alpha,\beta)=X_i\,\mathbb{1}_{\alpha\tau b_n\le|X_0|<\beta\tau b_n}$ and

$$Z_n(A,\alpha,\beta)=\frac{1}{b_n}\sum_{i\in\Lambda_n}\lambda(nA\cap R_i)[X_i(n,\alpha,\beta)-EX_i(n,\alpha,\beta)].$$

So, we obtain

$$E_1 \le \frac{2}{x}E\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}}\left|Z_n(A,0,1)-Z_n(B,0,1)\right|\right)\le\frac{2}{x}\left(F_1+F_2+F_3\right)$$

where F_1, F_2 and F_3 are defined in the proof of Theorem 1. Applying Lemma 1 with $c_n=b_n$, the estimations (10) and (11) still hold for F_1 and F_2 respectively. In order to control the last term F_3, for any Borel subset A of $[0,1]^d$, we denote

$$\widetilde{Z}_n(A,a_k,a_{k-1})=\frac{1}{b_n}\sum_{i\in\Lambda_n}\lambda(nA\cap R_i)[|X_i(n,a_k,a_{k-1})|-E|X_i(n,a_k,a_{k-1})|].$$

We have

$$\begin{aligned}
&\sup_{A_k\subset A\subset A_k^+}|Z_n(A,a_k,a_{k-1})-Z_n(A_k,a_k,a_{k-1})|\\
&\quad\le\frac{1}{b_n}\sum_{i\in\Lambda_n}(\lambda(nA_k^+\cap R_i)-\lambda(nA_k\cap R_i))[|X_i(n,a_k,a_{k-1})|-E|X_i(n,a_k,a_{k-1})|]\\
&\qquad+2\frac{n^d}{b_n}E|X_0(n,a_k,a_{k-1})|\,\delta_k^2\\
&\quad=\widetilde{Z}_n(A_k^+,a_k,a_{k-1})-\widetilde{Z}_n(A_k,a_k,a_{k-1})+2\frac{n^d}{b_n}E|X_0(n,a_k,a_{k-1})|\,\delta_k^2
\end{aligned}$$

Using Lemma 1, we derive

$$\left\| \max_{A_k, A_k^+ \in \mathcal{A}(\delta_k)} \left| \widetilde{Z}_n(A_k^+, a_k, a_{k-1}) - \widetilde{Z}_n(A_k, a_k, a_{k-1}) \right| \right\|_{\psi_1} \leq K(a_{k-1}\tau \mathbb{H}(\delta_k) + \delta_k \sqrt{\mathbb{H}(\delta_k)}).$$

In the other hand

$$\frac{n^d}{b_n} E|X_0(n, a_k, a_{k-1})|\, \delta_k^2 \leq \frac{\delta_k^2}{a_k \tau} \frac{n^d}{b_n^2} E X_0^2\, \mathbb{1}_{|X_0|<b_n}.$$

So, the estimation (12) still hold for F_3 and choosing again $a_k = \delta_k / (\tau \sqrt{\mathbb{H}(\delta_{k+1})})$, we derive

$$\limsup_{n\to\infty} \frac{2}{x} E_1 \leq K \sum_{k=1}^{+\infty} \delta_{k+1} \sqrt{\mathbb{H}(\delta_k)} \leq K \int_0^\delta \sqrt{\mathbb{H}(x)} dx \xrightarrow[\delta\to 0]{} 0.$$

Finally, the condition (13) holds and the sequence $\{U_n^{-1} S_n(A)\,;\, A \in \mathcal{A}\}$ is tight in the space $C(\mathcal{A})$.

b) Finite dimensional convergence

For any Borel set A of $[0,1]^d$ recall that $\Gamma_n(A)$ is the finite set defined by $\Gamma_n(A) = nA \cap \mathbb{Z}^d$ and denote $S_{\Gamma_n(A)} = \sum_{i\in\Gamma_n(A)} X_i$.

Lemma 3 *Let A be a regular Borel set of $[0,1]^d$ with $\lambda(A) > 0$. For any $x > 0$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(U_n^{-1} |S_n(A) - S_{\Gamma_n(A)}| > x\right) = 0.$$

Proof of Lemma 3. Consider the subsets of $\mathbb{Z}^d$

$$A_1 = \{i\,;\, R_i \subset nA\}, \quad A_2 = \{i\,;\, R_i \cap nA \neq \emptyset\}, \quad A_3 = A_2 \cap \{i\,;\, R_i \cap (nA)^c \neq \emptyset\}$$

and set $a_i = \lambda(nA \cap R_i) - \mathbb{1}_{i\in\Gamma_n(A)}$. Since a_i equals zero if i belongs to A_1, we have

$$S_n(A) - S_{\Gamma_n(A)} = \sum_{i\in A_3} a_i X_i.$$

Let $\tau > 0$ and recall that $X_{i,n} = X_i\, \mathbb{1}_{|X_i|<\tau b_n}$. We have

$$\mathbb{P}\left(U_n^{-1} |S_n(A) - S_{\Gamma_n(A)}| > x\right) \leq P_1 + P_2 + P_3$$

where

$$P_1 = \mathbb{P}\left(\left| \sum_{i\in A_3} a_i X_{i,n} \right| > x b_n / 2 \right)$$

$$P_2 = \mathbb{P}\left(U_n \leq b_n/2\right) \xrightarrow[n\to+\infty]{} 0 \quad \text{(by (7))}$$

$$P_3 = n^d \mathbb{P}\left(|X_0| \geq \tau b_n\right) \xrightarrow[n\to+\infty]{} 0 \quad \text{(by (6))}.$$

Moreover

$$P_1 \leq \frac{4|A_3|}{x^2 b_n^2} E X_{0,n}^2 = \frac{4|A_3|}{x^2 n^d} \times \frac{n^d}{b_n^2} E X_{0,n}^2.$$

Keeping in mind that $n^{-d}|A_3|$ tends to zero as n goes to infinity (cf. Dedecker [6]) and using (6) then the proof of Lemma 3 is complete.

Lemma 4 *For any regular Borel set A in $\mathcal{A}$, the sequence $\left(U_n^{-1} S_{\Gamma_n(A)}\right)_{n \geq 1}$ converge in distribution to $\sqrt{\lambda(A)}\,\varepsilon$ where ε has the standard normal law.*

Proof of Lemma 4. Let $x > 0$, $n \in \mathbb{N}^*$ and $A \in \mathcal{A}$ be fixed. We have

$$U_n^{-1} S_{\Gamma_n(A)} = \underbrace{\frac{\sum_{i \in \Gamma_n(A)} X_i}{\sqrt{\sum_{i \in \Gamma_n(A)} X_i^2}}}_{T_{n,1}(A)} \times \underbrace{\sqrt{\frac{\sum_{i \in \Gamma_n(A)} X_i^2}{\sum_{i \in \Lambda_n} X_i^2}}}_{T_{n,2}(A)}.$$

Using Theorem 3.3 in [13], we derive that $T_{n,1}(A)$ converges in distribution to the standard normal law. So, it suffices to prove that $T_{n,2}^2(A)$ converges in probability to $\lambda(A)$. Let $\tau > 0$ be fixed. Denoting $X_{i,n} = X_i \, \mathbb{1}_{|X_i| < \tau b_n}$ and $\overline{X}_{i,n} = X_i - X_{i,n}$, we have

$$|T_{n,2}^2(A) - \lambda(A)| \leq \underbrace{\left| T_{n,2}^2(A) - \frac{\sum_{i \in \Gamma_n(A)} X_{i,n}^2}{\sum_{i \in \Lambda_n} X_{i,n}^2} \right|}_{(*)} + \underbrace{\left| \frac{\sum_{i \in \Gamma_n(A)} X_{i,n}^2}{\sum_{i \in \Lambda_n} X_{i,n}^2} - \lambda(A) \right|}_{(**)}. \tag{14}$$

Now, noting that $X_i^2 = X_{i,n}^2 + \overline{X}_{i,n}^2$, we derive

$$\begin{aligned} (*) &= \left| \frac{\sum_{i \in \Lambda_n} X_{i,n}^2 \sum_{i \in \Gamma_n(A)} X_i^2 - \sum_{i \in \Lambda_n} X_i^2 \sum_{i \in \Gamma_n(A)} X_{i,n}^2}{\sum_{i \in \Lambda_n} X_i^2 \sum_{i \in \Lambda_n} X_{i,n}^2} \right| \\ &= \left| \frac{\sum_{i \in \Lambda_n} X_{i,n}^2 \sum_{i \in \Gamma_n(A)} \overline{X}_{i,n}^2 - \sum_{i \in \Lambda_n} \overline{X}_{i,n}^2 \sum_{i \in \Gamma_n(A)} X_{i,n}^2}{\sum_{i \in \Lambda_n} X_i^2 \sum_{i \in \Lambda_n} X_{i,n}^2} \right| \\ &\leq 2 \, \frac{\sum_{i \in \Lambda_n} \overline{X}_{i,n}^2}{\sum_{i \in \Lambda_n} X_i^2} \\ &= 2 \, (1 - R_n) \end{aligned}$$

where

$$R_n = \frac{\sum_{i \in \Lambda_n} X_{i,n}^2}{\sum_{i \in \Lambda_n} X_i^2} \leq 1 \quad \text{a.s.}$$

Let $x > 0$ be fixed. Using (6) we derive that

$$\mathbb{P}((*) > 3x) \leq \mathbb{P}((*) > 0) \leq \mathbb{P}(R_n < 1) \leq n^d \mathbb{P}(|X_0| \geq \tau b_n) \xrightarrow[n\to+\infty]{} 0. \quad (15)$$

In the other hand,

$$\begin{aligned}
(**) &\leq \left| \frac{\sum_{i\in\Gamma_n(A)} X_{i,n}^2}{\sum_{i\in\Lambda_n} X_{i,n}^2} - \frac{1}{b_n^2} \sum_{i\in\Gamma_n(A)} X_{i,n}^2 \right| + \left| \frac{1}{b_n^2} \sum_{i\in\Gamma_n(A)} X_{i,n}^2 - \lambda(A) \right| \\
&\leq \left| 1 - \frac{1}{b_n^2} \sum_{i\in\Lambda_n} X_{i,n}^2 \right| + \left| \frac{1}{b_n^2} \sum_{i\in\Gamma_n(A)} X_{i,n}^2 - \lambda(A) \right| \\
&\leq \underbrace{\left| 1 - \frac{1}{b_n^2} \sum_{i\in\Lambda_n} X_{i,n}^2 \right|}_{\gamma_{n,1}} + \underbrace{\left| \frac{1}{b_n^2} \sum_{i\in\Gamma_n(A)} \left(X_{i,n}^2 - EX_{i,n}^2 \right) \right|}_{\gamma_{n,2}} + \underbrace{\left| \frac{|\Gamma_n(A)|}{b_n^2} EX_{0,n}^2 - \lambda(A) \right|}_{\gamma_{n,3}}.
\end{aligned}$$

By (6) and the point (i) of Lemma 2, it is clear that

$$\gamma_{n,3} \xrightarrow[n\to\infty]{} 0. \quad (16)$$

Noting that

$$b_n^{-2} \sum_{i\in\Lambda_n} X_{i,n}^2 = \frac{\sum_{i\in\Lambda_n} X_i^2}{b_n^2} \times R_n \quad \text{a.s.}$$

we have

$$\begin{aligned}
\mathbb{P}(\gamma_{n,1} > x) &\leq \mathbb{P}\left(\left| 1 - R_n \right| > x/2 \right) + \mathbb{P}\left(\left| 1 - \frac{\sum_{i\in\Lambda_n} X_i^2}{b_n^2} \right| > x/2 \right) \\
&\leq \mathbb{P}(R_n < 1) + \mathbb{P}\left(\left| 1 - \frac{\sum_{i\in\Lambda_n} X_i^2}{b_n^2} \right| > x/2 \right) \\
&\leq n^d \mathbb{P}(|X_0| \geq \tau b_n) + \mathbb{P}\left(\left| 1 - \frac{\sum_{i\in\Lambda_n} X_i^2}{b_n^2} \right| > x/2 \right).
\end{aligned}$$

Using (6) and (7), we obtain

$$\mathbb{P}(\gamma_{n,1} > x) \xrightarrow[n\to\infty]{} 0. \quad (17)$$

We have also

$$\begin{aligned}
\mathbb{P}(\gamma_{n,2} > x) &\leq \frac{b_n^{-4}}{x^2} E\left(\sum_{i\in\Gamma_n(A)} X_{i,n}^2 - EX_{i,n}^2\right)^2 \\
&= \frac{b_n^{-4}}{x^2}|\Gamma_n(A)| E\left(X_{0,n}^2 - EX_{0,n}^2\right)^2 \\
&\leq \frac{4b_n^{-4}}{x^2}|\Gamma_n(A)| EX_{0,n}^4 \\
&\leq \frac{4\tau^2 b_n^{-2}}{x^2}|\Gamma_n(A)| EX_{0,n}^2 \\
&= \frac{4\tau^2|\Gamma_n(A)|}{n^d x^2} \times \frac{n^d}{b_n^2} EX_{0,n}^2.
\end{aligned}$$

Consequently, using (6) and the point (i) in Lemma 2, we derive

$$\lim_{n\to+\infty} \mathbb{P}(\gamma_{n,2} > x) \leq \frac{4\tau^2\lambda(A)}{x^2}. \tag{18}$$

Now, combining (16), (17) and (18), we obtain

$$\lim_{n\to+\infty} \mathbb{P}((**) > 3x) \leq \frac{4\tau^2\lambda(A)}{x^2}. \tag{19}$$

Combining (14), (15) and (19), it follows that

$$\lim_{n\to+\infty} \mathbb{P}\left(|T_{n,2}^2(A) - \lambda(A)| > 6x\right) \leq \frac{4\tau^2\lambda(A)}{x^2}.$$

Since $\tau > 0$ can be arbitrarily small, we obtain

$$\lim_{n\to+\infty} \mathbb{P}\left(|T_{n,2}^2(A) - \lambda(A)| > 6x\right) = 0.$$

Finally, $T_{n,2}^2(A)$ converges in probability to $\lambda(A)$ and the proof of Lemma 4 is complete. The convergence of the finite-dimensional laws of the sequence $\{U_n^{-1}S_n(A); A\in\mathcal{A}\}$ follows then from Lemmas 3 and 4. The proof of Theorem 2 is complete.

3.3 Proof of Theorem 3

Without loss of generality, we assume that p is a positive integer. Consider the field $X = (X_k)_{k\in\mathbb{Z}^d}$ of i.i.d. integer-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mu)$ by the following property: the random variable

X_0 is symmetric and satisfies $\mu(X_0 = 0) = 0$ and $\mu(|X_0| \geq k) = k^{-p-1}$ for any integer $k \geq 1$. The random field X is p-integrable since

$$\begin{aligned} E(|X_0|^p) &= \sum_{k\geq 1} \mu(|X_0| \geq k^{1/p}) \\ &= \sum_{k\geq 1} k^{-1-1/p} < +\infty. \end{aligned}$$

Let us fix an integer $r \geq 1$ and consider the following numbers:

$$n_r = 4^{rp},$$

$$\begin{aligned} \beta_r &= n_r^{d/2p} = 2^{rd}, \\ k_r &= n_r^d \mu(X_0 \geq \beta_r) = 2^{rd(p-1)}, \\ \varepsilon_r &= \left(\frac{k_r}{n_r^d}\right)^{1/2} = 2^{-rd(p+1)/2}. \end{aligned}$$

One can notice that $(n_r)_{r\geq 1}$, $(\beta_r)_{r\geq 1}$ and $(k_r)_{r\geq 1}$ are increasing sequences of positive integers while $(\varepsilon_r)_{r\geq 1}$ is a decreasing sequence of positive real numbers which converges to zero. We define the class $\mathcal{A}_r$ as the collection of all Borel subsets A of $[0,1]^d$ with the following property: A is empty or there exist $i_l = (i_{l,1}, ..., i_{l,d})$ in $\{1, ..., n_r\}^d$, $1 \leq l \leq k_r$ such that

$$A = \bigcup_{l=1}^{k_r} \left] \frac{i_{l,1} - 1}{n_r}, \frac{i_{l,1}}{n_r} \right] \times ... \times \left] \frac{i_{l,d} - 1}{n_r}, \frac{i_{l,d}}{n_r} \right].$$

Now, denote

$$\mathcal{A} = \mathcal{B}_r \cup \mathcal{C}_r$$

where

$$\mathcal{B}_r = \bigcup_{j=1}^{r-1} \mathcal{A}_j \quad \text{and} \quad \mathcal{C}_r = \bigcup_{j=r}^{+\infty} \mathcal{A}_j.$$

For any integer $j \geq 1$, the cardinal $|\mathcal{A}_j|$ of $\mathcal{A}_j$ equals $1 + \left(\begin{array}{c} n_j^d \\ k_j \end{array}\right)$, hence

$$N(\mathcal{B}_r, \rho, \varepsilon_r) \leq \sum_{j=1}^{r-1} \left(1 + \left(\begin{array}{c} n_j^d \\ k_j \end{array}\right)\right) \leq 2r n_r^{dk_r}.$$

On the other hand, since each element of the class $\mathcal{C}_r$ belongs to the ball with center $\emptyset$ and radius ε_r, it follows that $N(\mathcal{C}_r, \rho, \varepsilon_r) = 1$. Noting that

$$N(\mathcal{A}, \rho, \varepsilon_r) \leq N(\mathcal{B}_r, \rho, \varepsilon_r) + N(\mathcal{C}_r, \rho, \varepsilon_r),$$

we obtain

$$N(\mathcal{A}, \rho, \varepsilon_r) \le 1 + 2rn_r^{dk_r}$$

and also

$$H(\mathcal{A}, \rho, \varepsilon_r) = \log\, N(\mathcal{A}, \rho, \varepsilon_r) \le 3dk_r \log\, n_r.$$

Finally, there exists $K > 0$ such that

$$\begin{aligned} \sum_{r=2}^{+\infty} \varepsilon_{r-1}\sqrt{H(\mathcal{A}, \rho, \varepsilon_r)} &\le \sum_{r=2}^{+\infty} \varepsilon_{r-1}\sqrt{3dk_r \log\, n_r} \\ &\le K \sum_{r=2}^{+\infty} \frac{2^{rd(p-1)/2}\sqrt{r}}{2^{rd(p+1)/2}} \\ &= K \sum_{r=2}^{+\infty} \frac{\sqrt{r}}{2^{rd}} < +\infty. \end{aligned}$$

Consequently, the class $\mathcal{A}$ satisfies the metric entropy condition (2). Now, we are going to see that the partial sum process $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ defined by (1) is not tight in the space $C(\mathcal{A})$. It is sufficient (Pollard, 1990) to show that there exists $\theta > 0$ such that

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} \mu\left(\sup_{\substack{A,B \in \mathcal{A} \\ \rho(A,B) < \delta}} n^{-d/2}\big|S_n(A) - S_n(B)\big| \ge \theta \right) > 0.$$

For any integer $r \ge 1$, denote $\Lambda_r = \{1, ..., n_r\}^d$ and define W_r as the set of all ω in Ω such that

$$\sum_{i \in \Lambda_r} \mathbb{1}_{\{X_i(\omega) \ge \beta_r\}} \ge k_r.$$

Lemma 5 *There exists a constant $c > 0$ such that for any integer $r \ge 1$,*

$$\mu(W_r) \ge c. \tag{20}$$

Proof of Lemma 5. Let $r \ge 1$ be fixed. For any i in Λ_r, denote

$$Y_i = \mathbb{1}_{\{X_i \ge \beta_r\}} - \mu(X_0 \ge \beta_r).$$

The family $\{Y_i\,;\, i \in \Lambda_r\}$ is a finite sequence of i.i.d. centered random variables bounded by 2. So, using a lower exponential inequality due to Kolmogorov (Ledoux and Talagrand, 1991, Lemma 8.1), it follows that for any $\gamma > 0$, there exist positive numbers $K(\gamma)$ (large enough) and $\varepsilon(\gamma)$ (small enough) depending on γ only, such that for every t satisfying $t \geq K(\gamma)b$ and $2t \leq \varepsilon(\gamma)b^2$,

$$\mu\left(\sum_{i\in\Lambda_r} Y_i > t\right) \geq \exp\left(-(1+\gamma)t^2/2b^2\right)$$

where $b^2 = \sum_{i\in\Lambda_r} EY_i^2$. In particular, there exists a positive universal constant K such that

$$\mu\left(\sum_{i\in\Lambda_r} Y_i > Kb\right) \geq \exp\left(-K^2\right).$$

Noting $c = \exp(-K^2) > 0$ and keeping in mind the definitions of the constant k_r and the random variable Y_i, we derive

$$\mu\left(\sum_{i\in\Lambda_r} \mathbb{1}_{\{X_i \geq \beta_r\}} > Kb + k_r\right) \geq c.$$

Finally, Inequality (20) follows from the fact that $Kb \geq 0$ and the proof of the lemma is complete. The proof of Lemma 5 is complete.

Let ω be fixed in the set W_r and denote

$$\Gamma_r^*(\omega) = \{i \in \Lambda_r\,;\, X_i(\omega) \geq \beta_r\}.$$

By definition of the set W_r, we know that $|\Gamma_r^*(\omega)| \geq k_r$. Let $\Gamma_r(\omega)$ be a subset of $\Gamma_r^*(\omega)$ such that $|\Gamma_r(\omega)| = k_r$ and define

$$A_r(\omega) = \bigcup_{i\in\Gamma_r(\omega)} \left]\frac{i_1-1}{n_r}, \frac{i_1}{n_r}\right] \times ... \times \left]\frac{i_d-1}{n_r}, \frac{i_d}{n_r}\right] \in \mathcal{A}_r \subset \mathcal{A}.$$

For any ω in W_r and any i in Λ_r, we have

$$\lambda(n_r A_r(\omega) \cap R_i) = \mathbb{1}_{\Gamma_r(\omega)}(i).$$

Consequently, we have

$$\begin{aligned}
n_r^{-d/2} S_{n_r}(A_r(\omega)) &= n_r^{-d/2} \sum_{i\in\Lambda_r} \lambda(n_r A_r(\omega)\cap R_i) X_i(\omega) \\
&= n_r^{-d/2} \sum_{i\in\Gamma_r(\omega)} X_i(\omega) \\
&\geq n_r^{-d/2} |\Gamma_r(\omega)| \beta_r \\
&= n_r^{-d/2} k_r \beta_r \\
&= n_r^{d/2} \mu(X_0 \geq \beta_r)\beta_r \\
&= \frac{1}{2} n_r^{d/2} \beta_r^{-p} \\
&= \frac{1}{2}.
\end{aligned}$$

Thus, for any integer $r \geq 1$ and any ω in W_r, we have

$$\left| n_r^{-d/2} S_{n_r}(A_r(\omega)) \right| \geq 1/2. \tag{21}$$

Let $\delta > 0$ be fixed. There exists an integer R such that for any $r \geq R$ and any ω in W_r, $\lambda(A_r(\omega)) = k_r/n_r^d \leq \delta^2$. Then, using the lower bounds (20) and (21), it follows that for any $r \geq R$,

$$\begin{aligned}
&\mu\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}} \left| n_r^{-d/2} S_{n_r}(A) - n_r^{-d/2} S_{n_r}(B) \right| \geq 1/2 \right) \\
&\geq \mu\left(\sup_{\substack{A\in\mathcal{A}\\ \lambda(A)<\delta^2}} \left| n_r^{-d/2} S_{n_r}(A) \right| \geq 1/2 \right) \\
&\geq \mu\left(\left\{ \omega \in W_r \;\middle|\; \left| n_r^{-d/2} S_{n_r}(A_r(\omega)) \right| \geq 1/2 \right\} \right) \\
&= \mu(W_r) \geq c > 0.
\end{aligned}$$

Finally, we have shown that for any $\delta > 0$,

$$\limsup_{n\to+\infty} \mu\left(\sup_{\substack{A,B\in\mathcal{A}\\ \rho(A,B)<\delta}} \left| n^{-d/2} S_n(A) - n^{-d/2} S_n(B) \right| \geq 1/2 \right) \geq c > 0.$$

The proof of Theorem 3 is complete.

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