Exact convergence rates in the central limit theorem for a class of martingales

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We give optimal convergence rates in the central limit theorem for a large class of martingale difference sequences with bounded third moments. The rates depend on the behaviour of the conditional variances and for stationary sequences the rate $n^{-1/2}\log n$ is reached.

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1 Introduction and notations

The optimal rate of convergence in the central limit theorem for independent random variables $(X_i)_{i\in\mathbb{Z}}$ is well known to be of order $n^{-1/2}$ if the X_i's are centered and have uniformly bounded third moments (cf. Berry [1] and Esseen [8]). For dependent random variables the rate of convergence was also most fully investigated but in many results the rate is not better than $n^{-1/4}$. For example, Philipp [19] obtains a rate of $n^{-1/4}(\log n)^3$ for uniformly mixing sequences, Landers and Rogge [15] obtain a rate of $n^{-1/4}(\log n)^{1/4}$ for a class of Markov chains (see also Bolthausen [3]) and Sunklodas [23] obtains a rate of $n^{-1/4}\log n$ for strong mixing sequences. However, Rio [22] has shown that the rate $n^{-1/2}$ is reached for uniformly mixing sequences of bounded random variables as soon as the sequence $(\phi_p)_{p>0}$ of uniform mixing coefficients satisfies $\sum_{p>0} p\phi_p < \infty$. Jan [13] also established a $n^{-1/2}$ rate of convergence in the central limit theorem for bounded processes taking values in $\mathbb{R}^d$ under some mixing conditions and recently, using a modification of the proof in Rio [22], Le Borgne and Pène [16] obtained the rate $n^{-1/2}$ for stationary processes satisfying a strong decorrelation hypothesis. For bounded martingale difference sequences, Ibragimov [12] has obtained the rate of $n^{-1/4}$ for some stopping partial sums and Ouchti [18] has extended Ibragimov's result to a class of martingales which is related to the one we are going to consider in this paper. Several results on the rate of convergence for the martingale central limit theorem have been obtained for the whole partial sums, one can refer to Hall and Heyde [10] (section 3.6.), Chow and Teicher [5] (Theorem 9.3.2), Kato [14], Bolthausen [2], Haeusler [11], Rinott and Rotar [20] and [21]. In fact, Kato obtains the rate $n^{-1/2}(\log n)^3$ for uniformly bounded variables under the assumption assumption that the conditional variances are almost surely constant. In this paper, we are most interested in results by Bolthausen [2] who obtained the better (in fact optimal) rate $n^{-1/2}\log n$ under somewhat weakened conditions. In this paper, we shall not aim to improve the rate $n^{-1/2}\log n$ but rather introduce a large class of martingales which leads to it. Finally, note that El Machkouri and Volný [7] have shown that the rate of convergence in the central limit theorem can be arbitrary slow for stationary sequences of bounded (strong mixing) martingale difference random variables. Let $n \geq 1$ be a fixed integer. We consider a finite sequence $X = (X_1, ..., X_n)$ of martingale difference random variables (i.e. X_k is $\mathcal{F}_k$-measurable and $E(X_k|\mathcal{F}_{k-1}) = 0$ a.s. where $(\mathcal{F}_k)_{0\leq k\leq n}$ is an increasing filtration and $\mathcal{F}_0$ is the trivial σ-algebra). In the sequel, we use the following notations

$$\sigma_k^2(X) = E(X_k^2|\mathcal{F}_{k-1}), \quad \tau_k^2(X) = E(X_k^2), \quad 1 \leq k \leq n,$$

$$v_n^2(X) = \sum_{k=1}^n \tau_k^2(X) \quad \text{and} \quad V_n^2(X) = \frac{1}{v_n^2(X)} \sum_{k=1}^n \sigma_k^2(X).$$

We denote also $S_n(X) = X_1 + X_2 + ... + X_n$. The central limit theorem established by Brown [4] and Dvoretzky [6] states that under some Lindeberg type condition

$$\Delta_n(X) = \sup_{t\in\mathbb{R}} \left|\mu\left(S_n(X)/v_n(X) \leq t\right) - \Phi(t)\right| \xrightarrow[n\to+\infty]{} 0.$$

For more about central limit theorems for martingale difference sequences one can refer to Hall and Heyde [10]. The rate of convergence of $\Delta_n(X)$ to zero was most fully investigated. Here, we focus on the following result by Bolthausen [2].

Theorem (Bolthausen, 82) *Let $\gamma > 0$ be fixed. There exists a constant $L(\gamma) > 0$ depending only on γ such that for all finite martingale difference sequence $X = (X_1, ..., X_n)$ satisfying $V_n^2(X) = 1$ a.s. and $\|X_i\|_\infty \leq \gamma$ then*

$$\Delta_n(X) \leq L(\gamma)\left(\frac{n\log n}{v_n^3}\right).$$

We are going to show that the method used by Bolthausen [2] in the proof of the theorem above can be extended to a large class of unbounded martingale difference sequences. Note that Bolthausen has already given extensions to unbounded martingale difference sequences which conditional variances become asymptotically nonrandom (cf. [2], Theorems 3 and 4) but his assumptions cannot be compared directly with ours (cf. condition (1) below). So the results are complementary.

2 Main Results

We introduce the following class of martingale difference sequences: a sequence $X = (X_1, ..., X_n)$ belongs to the class $\mathcal{M}_n(\gamma)$ if X is a martingale difference sequence with respect to some increasing filtration $(\mathcal{F}_k)_{0\leq k\leq n}$ such that for any $1 \leq k \leq n$,

$$E(|X_k|^3|\mathcal{F}_{k-1}) \leq \gamma_k\, E(X_k^2|\mathcal{F}_{k-1}) \quad \text{a.s.} \tag{1}$$

where $\gamma = (\gamma_k)_k$ is a sequence of positive reals numbers.

Our first result is the following.

Theorem 1 *There exists a constant $L > 0$ (not depending on n) such that for all finite martingale difference sequence $X = (X_1, ..., X_n)$ which belongs to the class $\mathcal{M}_n(\gamma)$ then*

$$\Delta_n(X) \le L\left(\frac{u_n \ln n}{\min\{v_n,\, 2^n\}} + \|V_n^2(X) - 1\|_\infty^{1/2} \wedge \|V_n^2(X) - 1\|_1^{1/3}\right)$$

where $u_n = \vee_{k=1}^n \gamma_k$.

Theorem 2 *There exists a constant $L > 0$ (not depending on n) such that for all finite martingale difference sequence $X = (X_1, ..., X_n)$ which belongs to the class $\mathcal{M}_n(\gamma)$ and which satisfies $V_n^2(X) = 1$ a.s. then*

$$\Delta_n(X) \le L\left(\frac{u_n \ln n}{\min\{v_n,\, 2^n\}}\right)$$

For any random variable Z we denote $\delta(Z) = \sup_{t\in\mathbb{R}} |\mu(Z \le t) - \Phi(t)|$. We need also the following extension of Lemma 1 in Bolthausen [2] which is of particular interest.

Lemma 1 *Let X and Y be two real random variables. If there exist real numbers $l > 0$ and $r \ge 1$ such that Y belongs to $L^{lr}(\mu)$ then*

$$\delta(X+Y) \le 2\delta(X) + 3\left\|E\left(|Y|^l|X\right)\right\|_r^{\frac{1}{l+1}} \wedge \left\|E\left(Y^2|X\right)\right\|_\infty^{1/2} \tag{2}$$

and

$$\delta(X) \le 2\delta(X+Y) + 3\left\|E\left(|Y|^l|X\right)\right\|_r^{\frac{1}{l+1}} \wedge \left\|E\left(Y^2|X\right)\right\|_\infty^{1/2}. \tag{3}$$

The proofs of various central limit theorems for stationary sequences of random variables are based on approximating the partial sums of the process by martingales (see Gordin [9], Volný [24]). More precisely, if $(f \circ T^k)_k$ is a p-integrable stationary process where $T : \Omega \to \Omega$ is a bijective, bimeasurable and measure-preserving transformation (in fact, each stationary process has such representation) then there exists necessary and sufficient conditions (cf. Volný [24]) for f to be equal to $h + g - g \circ T$ where $(h \circ T^k)_k$ is a p-integrable stationary martingale difference sequence and g is a p-integrable function. The term $g - g \circ T$ is called a coboundary.
The following theorem gives the rate of convergence in the central limit theorem for stationary processes obtained from a martingale difference sequence which is perturbed by a coboundary.

Theorem 3 *Let $p > 0$ be fixed and let $F = (f \circ T^k)_k$ be a stationary process. If there exist m and g in $L^p(\mu)$ such that $H = (h \circ T^k)_k$ is a martingale difference sequence and $f = h + g - g \circ T$ then*

$$\Delta_n(F) \le 2\Delta_n(H) + \frac{4\|g\|_p^{p/(p+1)}}{n^{p/2(p+1)}}.$$

If $p = \infty$ then

$$\Delta_n(F) \le 2\Delta_n(H) + \frac{4\|g\|_\infty}{n^{1/2}}.$$

3 Proofs

3.1 Proof of theorem 2

In the sequel, we are going to use the following lemma by Bolthausen [2].

Lemma 2 (Bolthausen, 82) *Let $k \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a function which has k derivatives $f^{(1)}, ..., f^{(k)}$ which together with f belong to $L^1(\mu)$. Assume that $f^{(k)}$ is of bounded variation $\|f^{(k)}\|_V$, if X is a random variable and if $\alpha_1 \neq 0$ and α_2 are two real numbers then*

$$|Ef^{(k)}(\alpha_1 X + \alpha_2)| \leq \|f^{(k)}\|_V \sup_{t\in\mathbb{R}} |\mu(X \leq t) - \Phi(t)| + |\alpha_1|^{-(k+1)} \|f\|_1 \sup_x |\phi^{(k)}(x)|$$

where $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

Consider $u = (u_n)_n$ defined by $u_n = (\vee_{k=1}^n \gamma_k)$. Clearly the class $\mathcal{M}_n(\gamma)$ is contained in the class $\mathcal{M}_n(u)$. For any $(u, v) \in \mathbb{R}_+^{\mathbb{N}^*} \times \mathbb{R}_+^*$, we consider the subclass

$$\mathcal{L}_n(u, v) = \left\{X \in \mathcal{M}_n(u) \mid V_n^2(X) = 1,\ v_n(X) = v \text{ a.s.}\right\}$$

and we denote

$$\beta_n(u, v) = \sup \left\{\Delta_n(X) \mid X \in \mathcal{L}_n(u, v)\right\}.$$

In the sequel, we assume that $X = (X_1, ..., X_n)$ belongs to $\mathcal{L}_n(u, v)$, hence $X' = (X_1, ..., X_{n-2}, X_{n-1} + X_n)$ belongs to $\mathcal{L}_{n-1}(4u, v)$ and consequently,

$$\beta_n(u, v) \leq \beta_{n-1}(4u, v).$$

Let $Z_1, Z_2, ..., Z_n$ be independent identically distributed standard normal variables independent of the σ-algebra $\mathcal{F}_n$ (which contains the σ-algebra generated by $X_1, ..., X_n$) and ξ be an extra centered normal variable with variance $\theta^2 > 1 \vee 2u_n^2$ which is independent of anything else. Noting that $\sum_{i=1}^n \sigma_i(X) Z_i / v$ is a standard normal random variable, indeed

$$\begin{aligned} E\left\{e^{it\frac{\sum_{i=1}^n \sigma_{i-1}(X) Z_i}{v}}\right\} &= E\left\{e^{-t^2 \frac{\sum_{j=1}^n \sigma_{j-1}^2(X)}{2v^2}}\right\} \\ &= \exp(-\frac{t^2}{2}) \quad (\text{ Since } \ V_n^2(X) = 1 \text{ a.s.}). \end{aligned}$$

According to Inequality (3) in Lemma 1,

$$\Delta_n(X) \leq 2 \sup_{t\in\mathbb{R}} \left|\Gamma_n(t)\right| + \frac{6\theta}{v}. \tag{4}$$

where

$$\Gamma_n(t) \triangleq \mu\left(\left(S_n(X) + \xi\right)/v \leq t\right) - \mu\left(\left(\sum_{i=1}^n \sigma_i(X) Z_i + \xi\right)/v \leq t\right).$$

For any integer $1 \leq k \leq n$, we consider the following random variables

$$Y_k \triangleq \frac{1}{v} \sum_{i=1}^{k-1} X_i, \quad W_k \triangleq \frac{1}{v}\left(\sum_{i=k+1}^n \sigma_i(X) Z_i + \xi\right),$$

$$H_k \triangleq \frac{1}{v^2}\left(\sum_{i=k+1}^{n} \sigma_i^2(X) + \theta^2\right) \quad \text{and} \quad T_k(t) \triangleq \frac{t - Y_k}{H_k},\ t \in \mathbb{R}$$

with the usual convention $\sum_{i=n+1}^{n} \sigma_i^2(X) = \sum_{i=n+1}^{n} \sigma_i(X) Z_i = 0$ a.s. Moreover, one can notice that conditionned on $\mathcal{G}_k = \sigma(X_1, ..., X_n, Z_k)$, the random variable W_k is centered normal with variance H_k^2. According to the well known Lindeberg's decomposition (cf. [17]), we have

$$\begin{aligned}
\Gamma_n(t) &= \sum_{k=1}^{n} \mu\left(Y_k + W_k + \frac{X_k}{v} \le t\right) - \mu\left(Y_k + W_k + \frac{\sigma_k(X) Z_k}{v} \le t\right) \\
&= \sum_{k=1}^{n} \mu\left(\frac{W_k}{H_k} \le T_k(t) - \frac{X_k}{vH_k}\right) - \mu\left(\frac{W_k}{H_k} \le T_k(t) - \frac{\sigma_k(X) Z_k}{vH_k}\right) \\
&= \sum_{k=1}^{n} E\left(E\left(\mathbb{1}_{\frac{W_k}{H_k} \le T_k(t) - \frac{X_k}{vH_k}} |\mathcal{G}_k\right)\right) - E\left(E\left(\mathbb{1}_{\frac{W_k}{H_k} \le T_k(t) - \frac{\sigma_k(X) Z_k}{vH_k}} |\mathcal{G}_k\right)\right) \\
&= \sum_{k=1}^{n} E\left(\Phi\left(T_k(t) - \frac{X_k}{vH_k}\right)\right) - E\left(\Phi\left(T_k(t) - \frac{\sigma_k(X) Z_k}{vH_k}\right)\right)
\end{aligned}$$

Now, for any integer $1 \le k \le n$ and any random variable ζ_k, there exists a random variable $|\varepsilon_k| < 1$ a.s. such that

$$\Phi\left(T_k(t) - \zeta_k\right) = \Phi\left(T_k(t)\right) - \zeta_k \Phi^{'}\left(T_k(t)\right) + \frac{\zeta_k^2}{2} \Phi^{''}\left(T_k(t)\right) - \frac{\zeta_k^3}{6} \Phi^{'''}\left(T_k(t) - \varepsilon_k \zeta_k\right) \text{ a.s.}$$

So, we derive

$$\begin{aligned}
\Gamma_n(t) = \sum_{k=1}^{n} E\Big\{ &\left(-\frac{X_k}{vH_k} + \frac{\sigma_k(X) Z_k}{vH_k}\right) \Phi^{'}(T_k(t)) + \left(\frac{X_k^2}{2v^2 H_k^2} - \frac{\sigma_k^2(X) Z_k^2}{2v^2 H_k^2}\right) \Phi^{''}(T_k(t)) \\
&- \left(\frac{X_k^3}{6v^3 H_k^3}\right) \Phi^{'''}\left(T_k(t) - \frac{\varepsilon_k X_k}{vH_k}\right) + \left(\frac{\sigma_k^3(X) Z_k^3}{6v^3 H_k^3}\right) \Phi^{'''}\left(T_k(t) - \frac{\varepsilon_k^{'} \sigma_k(X) Z_k}{vH_k}\right) \Big\}.
\end{aligned}$$

Since $V_n^2(X) = 1$ a.s. we derive that H_k and $T_k(t)$ are $\mathcal{F}_{k-1}$-measurable, hence

$$\Gamma_n(t) = \sum_{k=1}^{n} \frac{1}{6v^3} E\Big\{ -\frac{X_k^3}{H_k^3} \Phi^{'''}\left(T_k(t) - \frac{\varepsilon_k X_k}{vH_k}\right) + \frac{\sigma_k^3(X) Z_k^3}{H_k^3} \Phi^{'''}\left(T_k(t) - \frac{\varepsilon_k^{'} \sigma_k(X) Z_k}{vH_k}\right) \Big\}$$

and consequently

$$\left|\Gamma_n(t)\right| \le \frac{1}{6v^3}\left(S_1 + S_2\right) \tag{5}$$

where

$$S_1 = \sum_{k=1}^{n} E\Big\{ \frac{|X_k|^3}{H_k^3} \Big| \Phi^{'''}\left(T_k(t) - \frac{\varepsilon_k X_k}{vH_k}\right) \Big| \Big\}$$

and

$$S_2 = \sum_{k=1}^{n} E\Big\{ \frac{\sigma_k^3(X) |Z_k|^3}{H_k^3} \Big| \Phi^{'''}\left(T_k(t) - \frac{\varepsilon_k^{'} \sigma_k(X) Z_k}{vH_k}\right) \Big| \Big\}.$$

Consider the stopping times $\nu(j)_{j=0,..,n}$ defined by $\nu(0) = 0$, $\nu(n) = n$ and for any $1 \le j < n$

$$\nu(j) = \inf\left\{k \ge 1 \,|\, \sum_{i=1}^{k} \sigma_i^2(X) \ge \frac{jv^2}{n} \quad \text{a.s.}\right\}.$$

Noting that $\{1, ..., n\} = \cup_{j=1}^{n} \{\nu(j-1)+1, ..., \nu(j)\}$ a.s. we derive

$$S_1 = \sum_{j=1}^{n} E\Bigg\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} \frac{|X_k|^3}{H_k^3} \left| \Phi''' \left(T_k(t) - \frac{\varepsilon_k X_k}{vH_k} \right) \right| \Bigg\},$$

moreover, for any $\nu(j-1) < k \le \nu(j)$ we have

$$\begin{aligned}
H_k^2 &\ge \frac{1}{v^2}\left(\sum_{i=\nu(j)+1}^{n} \sigma_i^2(X) + \theta^2 \right) \\
&= \frac{1}{v^2}\left(\sum_{i=1}^{n} \sigma_i^2(X) - \sum_{i=1}^{\nu(j)-1} \sigma_i^2(X) - \sigma_{\nu(j)}^2(X) + \theta^2 \right) \\
&\ge \frac{1}{v^2}\left(v^2 - \frac{jv^2}{n} - u_n^2 + \theta^2 \right) \\
&\triangleq m_j^2 \quad \text{a.s.}
\end{aligned}$$

Similarly,

$$\begin{aligned}
H_k^2 &\le \frac{1}{v^2}\left(\sum_{i=\nu(j-1)+1}^{n} \sigma_i^2(X) + \theta^2 \right) \\
&= \frac{1}{v^2}\left(\sum_{i=1}^{n} \sigma_i^2(X) - \sum_{i=1}^{\nu(j-1)} \sigma_i^2(X) + \theta^2 \right) \\
&\le \frac{1}{v^2}\left(v^2 - \frac{(j-1)v^2}{n} + \theta^2 \right) \\
&\triangleq M_j^2 \quad \text{a.s.}
\end{aligned}$$

Now, for any $\nu(j-1) < k \le \nu(j)$ put

$$R_k \triangleq \frac{1}{v} \sum_{i=\nu(j-1)+1}^{k-1} X_i, \quad A_k \triangleq \left\{ \frac{|R_k|}{m_j} \le \frac{|t - Y_{\nu(j-1)+1}|}{2M_j} \right\}$$

and for any positive integer q consider the real function ψ_q defined for any real x by $\psi_q(x) \triangleq \sup\{|\Phi'''(y)| \,;\, y \ge \frac{|x|}{2} - q\}$. On the other hand, on the set

$A_k \cap \{|X_k| \le q\}$ we have

$$\begin{aligned}
\left|T_k(t) - \frac{\varepsilon_k X_k}{vH_k}\right| &= \left|\frac{t - Y_{\nu(j-1)+1}}{H_k} - \frac{R_k}{H_k} - \frac{\varepsilon_k X_k}{vH_k}\right| \\
&\ge \frac{|t - Y_{\nu(j-1)+1}|}{H_k} - \frac{|R_k|}{H_k} - \frac{|X_k|}{vH_k} \\
&\ge \frac{|t - Y_{\nu(j-1)+1}|}{M_j} - \frac{|R_k|}{m_j} - \frac{q}{\theta} \\
&\ge \frac{|t - Y_{\nu(j-1)+1}|}{2M_j} - q \quad \text{a.s.} \quad (\text{since } \theta \ge 1).
\end{aligned}$$

Thus

$$\left|\Phi''' \left(T_k(t) - \frac{\varepsilon_k X_k}{vH_k}\right)\right| \mathbb{1}_{A_k \cap |X_k| \le q} \le \psi_q \left(\frac{t - Y_{\nu(j-1)+1}}{M_j}\right) \mathbb{1}_{A_k \cap |X_k| \le q}.$$

So, for any $1 \le j \le n$ we have

$$\begin{aligned}
&E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} \frac{|X_k|^3}{H_k^3} \Big| \Phi''' \left(T_k(t) - \frac{\varepsilon_k X_k}{vH_k}\right) \Big| \mathbb{1}_{A_k \cap \{|X_k| \le q\}} \Big\} \\
&\le E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} \frac{|X_k|^3}{H_k^3} \Big| \psi_q \left(\frac{t - Y_{\nu(j-1)+1}}{M_j}\right) \Big| \Big\} \\
&= E\Big\{ E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} \frac{|X_k|^3}{H_k^3} |\mathcal{F}_{\nu(j-1)} \Big\} \Big| \psi_q \left(\frac{t - Y_{\nu(j-1)+1}}{M_j}\right) \Big| \Big\}. \qquad (6)
\end{aligned}$$

On the other hand, for any $1 \le j \le n$ we have

$$\begin{aligned}
&E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} \frac{|X_k|^3}{H_k^3} |\mathcal{F}_{\nu(j-1)} \Big\} \\
&= E\Big\{ \sum_{k=\nu(j-1)+1}^{n} \frac{|X_k|^3}{H_k^3} |\mathcal{F}_{\nu(j-1)} \Big\} - E\Big\{ \sum_{k=\nu(j)+1}^{n} \frac{|X_k|^3}{H_k^3} |\mathcal{F}_{\nu(j-1)} \Big\} \\
&= \sum_{l=1}^{n} \sum_{k=l+1}^{n} \left(E\Big\{ \frac{|X_k|^3}{H_k^3} \mathbb{1}_{\nu(j-1)=l} |\mathcal{F}_{\nu(j-1)} \Big\} - E\Big\{ \frac{|X_k|^3}{H_k^3} \mathbb{1}_{\nu(j)=l} |\mathcal{F}_{\nu(j-1)} \Big\} \right) \\
&= \sum_{l=1}^{n} \sum_{k=l+1}^{n} \left(E\Big\{ E(\frac{|X_k|^3}{H_k^3} |\mathcal{F}_{k-1}) \mathbb{1}_{\nu(j-1)=l} |\mathcal{F}_{\nu(j-1)} \Big\} - E\Big\{ E(\frac{|X_k|^3}{H_k^3} |\mathcal{F}_{k-1}) \mathbb{1}_{\nu(j)=l} |\mathcal{F}_{\nu(j-1)} \Big\} \right) \\
&= E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} E(\frac{|X_k|^3}{H_k^3} |\mathcal{F}_{k-1}) |\mathcal{F}_{\nu(j-1)} \Big\}. \qquad (7)
\end{aligned}$$

By using the inequality (6), (7) and the fact that $X \in \mathcal{L}_n(u,\, v)$, we have

$$\begin{aligned}
&E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} \frac{|X_k|^3}{H_k^3} \Big| \Phi''' \left(T_k(t) - \frac{\varepsilon_k X_k}{v H_k} \right) \Big| \mathbb{1}_{A_k \cap \{|X_k| \le q\}} \Big\} \\
&= E\Big\{ E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} E\left(\frac{|X_k|^3}{H_k^3} |\mathcal{F}_{k-1} \right) |\mathcal{F}_{\nu(j-1)} \Big\} \Big| \psi_q \left(\frac{t - Y_{\nu(j-1)+1}}{M_j} \right) \Big| \Big\} \\
&\le \frac{u_n}{m_j^3} E\Big\{ E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} \sigma_k^2(X) |\mathcal{F}_{\nu(j-1)} \Big\} \Big| \psi_q \left(\frac{t - Y_{\nu(j-1)+1}}{M_j} \right) \Big| \Big\}.
\end{aligned}$$

Moreover, note that

$$\begin{aligned}
\sum_{k=\nu(j-1)+1}^{\nu(j)} \sigma_k^2(X) &= \sum_{k=1}^{\nu(j)} \sigma_k^2(X) - \sum_{k=1}^{\nu(j-1)} \sigma_k^2(X) \\
&\le \frac{(j+1)v^2}{n} - \frac{(j-1)v^2}{n} = \frac{2v^2}{n} \quad \text{a.s.} \qquad (8)
\end{aligned}$$

Thus, for each $1 \le j \le n$,

$$\begin{aligned}
&E\Big\{ \sum_{k=\nu(j-1)+1}^{\nu(j)} \frac{|X_k|^3}{H_k^3} \Big| \Phi''' \left(T_k(t) - \frac{\varepsilon_k X_k}{v H_k} \right) \Big| \mathbb{1}_{A_k \cap \{|X_k| \le q\}} \Big\} \\
&\le \frac{2u_n\, v^2}{n m_j^3} E\Big\{ \Big| \psi_q \left(\frac{t - Y_{\nu(j-1)+1}}{M_j} \right) \Big| \Big\}
\end{aligned}$$

Using Lemma 2, noting that $\|\psi_q\|_\infty \le 1$ and keeping in mind the notation $\delta(Z) \triangleq \sup_{t \in \mathbb{R}} |\mu(Z \le t) - \Phi(t)|$ there exists a positive constant c_3 such that

$$E\left\{ \psi_q \left(\frac{t - Y_{\nu(j-1)+1}}{M_j} \right) \right\} \le \delta(Y_{\nu(j-1)+1}) + c_3 M_j.$$

Now, using Lemma 1 and the inequality

$$E\left\{ \left(\sum_{k=\nu(j-1)+1}^{n} X_k \right)^2 \Big| \mathcal{F}_{\nu(j-1)} \right\} \le v^2 \left(1 - \frac{j-1}{n} \right) \quad \text{a.s.}$$

we obtain

$$\begin{aligned}
\delta(Y_{\nu(j-1)+1}) &\le 2\, \delta(S_n(X)/v) + 3 \Big\| E\Big\{ \frac{1}{v^2} \left(\sum_{k=\nu(j-1)+1}^{n} X_k \right)^2 \Big| Y_{\nu(j-1)+1} \Big\} \Big\|_\infty^{1/2} \\
&= 2\, \Delta_n(X) + 3 \Big\| E\Big\{ \frac{1}{v^2} \left(\sum_{k=\nu(j-1)+1}^{n} X_k \right)^2 \Big| Y_{\nu(j-1)+1} \Big\} \Big\|_\infty^{1/2} \\
&\le 2\, \beta_{n-1}(4u, v) + 3 \left(1 - \frac{j-1}{n} \right)^{1/2} \qquad (9)
\end{aligned}$$

and so

$$E\left\{\psi_q\left(\frac{t-Y_{\nu(j-1)+1}}{M_j}\right)\right\} \le 2\,\beta_{n-1}(4u,v)+3\left(1-\frac{j-1}{n}\right)^{1/2}+c_1M_j.$$

Using this estimate and the dominated convergence theorem, we derive for any integer $1\le j\le n$,

$$\begin{aligned}(\star) &= E\Bigg\{\sum_{k=\nu(j-1)+1}^{\nu(j)}\frac{|X_k|^3}{H_k^3}\left|\Phi^{'''}\left(T_k(t)-\frac{\varepsilon_kX_k}{vH_k}\right)\right|\mathbb{1}_{A_k}\Bigg\}\\ &\le \frac{c_4u_n}{m_j^3}\times\frac{v^2}{n}\times\left(\beta_{n-1}(4u,v)+\left(1-\frac{j-1}{n}\right)^{1/2}+M_j\right). \qquad (10)\end{aligned}$$

On the other hand, for any integer $\nu(j-1)<k\le\nu(j)$

$$A_k^c\subset B_j \triangleq \left\{\max_{\nu(j-1)<i\le\nu(j)}\frac{|R_i|}{m_j}>\frac{|t-Y_{\nu(j-1)+1}|}{2M_j}\right\}.$$

Since the set A_k is $\mathcal{F}_k$, we have

$$\begin{aligned}(\star\star) &= E\Bigg\{\sum_{k=\nu(j-1)+1}^{\nu(j)}\frac{|X_k|^3}{H_k^3}\left|\Phi^{'''}\left(T_k(t)-\frac{\varepsilon_kX_k}{vH_k}\right)\right|\mathbb{1}_{A_k^c}\Bigg\}\\ &\le \|\Phi^{'''}\|_\infty E\left\{\sum_{k=\nu(j-1)+1}^{\nu(j)}\frac{|X_k|^3}{H_k^3}\mathbb{1}_{A_k^c}\right\}\\ &\le u_nE\left\{\sum_{k=\nu(j-1)+1}^{\nu(j)}\frac{\sigma_k^2(X)}{H_k^3}\mathbb{1}_{A_k^c}\right\}\\ &\le u_nE\left\{\sum_{k=\nu(j-1)+1}^{\nu(j)}\frac{\sigma_k^2(X)}{H_k^3}\mathbb{1}_{B_j}\right\}.\end{aligned}$$

By using inequality 8 and the fact that $H_k\ge m_j$ for any $k\in\{\nu(j-1)+1,\cdots,\nu(j)\}$, we have

$$\begin{aligned}(\star\star) &\le \frac{2u_n}{m_j^3}\times\frac{v^2}{n}\times\mu(B_j)\\ &\le \frac{2u_n}{m_j^3}\times\frac{v^2}{n}\times\mu\left(\max_{\nu(j-1)<i\le\nu(j)}|R_i|>\frac{m_j|t-Y_{\nu(j-1)+1}|}{2M_j}\right)\\ &\le \frac{2u_n}{m_j^3}\times\frac{v^2}{n}\times E\left(\min\left\{1,\frac{4M_j^2}{m_j^2|t-Y_{\nu(j-1)+1}|^2}E\left(\max_{\nu(j-1)<i\le\nu(j)}|R_i|^2|\mathcal{F}_{\nu(j-1)}\right)\right\}\right). \qquad (11)\end{aligned}$$

Noticing that the sequence of random variables

$$\overline{R}_i=\begin{cases}R_i, & \text{si } \nu(j-1)+1\le i\le\nu(j);\\ R_{\nu(j)}, & \text{si } \nu(j)+1\le i\le n.\end{cases}$$

is a martingale adapted to the filtration $(\mathcal{F}_{i-1})_{i\leq n}$, thus

$$\begin{aligned} E\left(\max_{\nu(j-1)<i\leq\nu(j)}|R_i|^2|\mathcal{F}_{\nu(j-1)}\right) &= E\left(\max_{\nu(j-1)<i\leq n}|\overline{R}_i|^2|\mathcal{F}_{\nu(j-1)}\right) \\ &\leq 4E(|\overline{R}_n|^2|\mathcal{F}_{\nu(j-1)}) \\ &= 4E(|R_{\nu(j)}|^2|\mathcal{F}_{\nu(j-1)}). \end{aligned} \tag{12}$$

By the inequality (8), (11) and (12), we have

$$\begin{aligned} (\star\star) &\leq \frac{2u_n}{m_j^3}\times\frac{v^2}{n}\times E\left(\min\left\{1,\frac{16M_j^2}{m_j^2|t-Y_{\nu(j-1)+1}|^2}E\left(|R_{\nu(j)}|^2|\mathcal{F}_{\nu(j-1)}\right)\right\}\right) \\ &\leq \frac{2u_n}{m_j^3}\times\frac{v^2}{n}\times E\left(\min\left\{1,\frac{32M_j^2}{nm_j^2|t-Y_{\nu(j-1)+1}|^2}\right\}\right). \end{aligned}$$

By applying lemma 2 with $f(x)=\min(1;\,x^{-2})$, we have

$$\begin{aligned} E\left(\min\left\{1,\frac{32M_j^2}{nm_j^2|t-Y_{\nu(j-1)+1}|^2}\right\}\right) &\leq \delta(Y_{\nu(j-1)+1})+\frac{\sqrt{32}}{\sqrt{2n\pi\,m_j}}M_j \\ &\leq \delta(Y_{\nu(j-1)+1})+c_3M_j, \end{aligned}$$

where c_3 is a strictly positive constant.
By the inequality (9), we have

$$E\left(\min\left\{1,\frac{32M_j^2}{nm_j^2|t-Y_{\nu(j-1)+1}|^2}\right\}\right)\leq 2\beta_{n-1}(4u,\,v)+3\left(1-\frac{j-1}{n}\right)^{1/2}+c_3M_j.$$

Thus there exists a positive constant c_4 such that

$$(\star\star)\leq\frac{c_4u_n}{m_j^3}\times\frac{v^2}{n}\times\left(\beta_{n-1}(4u,\,v)+\left(1-\frac{j-1}{n}\right)^{1/2}+M_j\right) \tag{13}$$

From (10) and (13), there exists a positive constant c_5 such that

$$(\star)+(\star\star)\leq\frac{c_5u_n}{m_j^3}\times\frac{v^2}{n}\times\left(\beta_{n-1}(4u,v)+\left(1-\frac{j-1}{n}\right)^{1/2}+M_j\right).$$

Finally, we obtain the following estimate

$$\begin{aligned} S_1 &\triangleq \sum_{k=1}^n E\left\{\frac{|X_k|^3}{H_k^3}\left|\Phi'''\left(T_k(t)-\frac{\varepsilon_k\theta_k}{vH_k}\right)\right|\right\} \\ &\leq c_5u_n\times\frac{v^2}{n}\times\left(\beta_{n-1}(4u,v)\sum_{j=1}^n\frac{1}{m_j^3}+\sum_{j=1}^n\frac{1}{m_j^3}\left(1-\frac{j-1}{n}\right)^{1/2}+\sum_{j=1}^n\frac{M_j}{m_j^3}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{m_j^3} &= \sum_{j=1}^{n} \frac{1}{(1-\frac{j}{n}+\frac{u_n^2}{v^2}+\frac{\theta^2-2u_n^2}{v^2})^{3/2}} \\
&\leq \sum_{j=1}^{n} \frac{1}{1-\frac{j}{n}+\frac{u_n^2}{v^2}} \times \frac{v}{\sqrt{\theta^2-2u_n^2}} \\
&\leq c_5 \frac{v\, n\, \ln n}{\sqrt{\theta^2-2u_n^2}} \qquad (\text{ since } v^2 \leq n\, u_n^2),
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{m_j^3}(1-\frac{j-1}{n})^{1/2} &\leq \sum_{j=1}^{n} \frac{1}{(1-\frac{j}{n}+\frac{u_n^2}{v^2})^{3/2}}(1-\frac{j-1}{n})^{1/2} \quad (\text{ since } \theta^2 > 2u_n^2) \\
&\leq \sum_{j=1}^{n} \frac{1}{1-\frac{j-1}{n}} \quad (\text{ since } v^2 \leq n\, u_n^2) \\
&\leq c_5\, n\, \ln n,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{n} \frac{M_j}{m_j^3} &= \sum_{j=1}^{n} \frac{1}{m_j^3}(1-\frac{j-1}{n}+\frac{\theta^2}{v^2})^{1/2} \\
&\leq \sum_{j=1}^{n} \frac{1}{m_j^3}(1-\frac{j-1}{n})^{1/2} + \frac{\theta}{v}\sum_{j=1}^{n}\frac{1}{m_j^3} \\
&\leq c_5\, n\, \ln n(1+\frac{\theta}{\sqrt{\theta^2-2u_n^2}}).
\end{aligned}$$

Hence

$$S_1 \leq c_5\, u_n \times \frac{v^2}{n} \times \left(\beta_{n-1}(4u,v)\frac{v\, n\, \ln n}{\sqrt{\theta^2-2u_n^2}} + n\, \ln n\,(2+\frac{\theta}{\sqrt{\theta^2-2u_n^2}})\right). \quad (14)$$

Note that to obtain the above estimates of S_1, we have only use the fact that the martingale difference sequence X belongs to the class $\mathcal{L}_n(u,v)$. Since the sequence $\sigma Z \triangleq (\sigma_1(X)Z_1, ..., \sigma_n(X)Z_n)$ belongs to $\mathcal{L}_n(4u/\sqrt{2\pi}, v)$, we are able to reach a similar estimate for S_2:

$$S_2 \leq c_6 u_n \times \frac{v^2}{n} \times \left(\beta_{n-1}(16u/\sqrt{2\pi},v)\frac{v\, n\, \ln n}{\sqrt{\theta^2-2u_n^2}} + n\, \ln n\,(2+\frac{\theta}{\sqrt{\theta^2-2u_n^2}})\right). \quad (15)$$

where c_6 is a positive constant.
Using (4), (5), (14) and (15) there exist a positive constant c such that

$$\begin{aligned}
&\beta_n(u,v) && (16) \\
&\leq c\, u_n\left(\beta_{n-1}(16u/\sqrt{2\pi},v)\frac{\ln n}{\sqrt{\theta^2-2u_n^2}} + n\,\frac{\ln n}{v}\,(2+\frac{\theta}{\sqrt{\theta^2-2u_n^2}})\right) + \frac{6\theta}{v}. && (17)
\end{aligned}$$

Putting

$$D_n(v) \triangleq \sup\left\{\frac{\beta_n(u,v)}{u_n \log n}\ ;\ u \in \mathbb{R}_+^{\mathbb{N}^*}\right\}.$$

and $\theta^2 \triangleq \left(2+4c^2\ln^2 n\right)u_n^2$, by the inequality 16, we have

$$D_n(v) \le \frac{D_{n-1}(v)}{2} + \frac{C}{v} \tag{18}$$

where C is a positive constant which does not depend on n. Finally, we conclude that

$$D_n(v) \le 2\frac{C}{v} + \frac{1}{2^n} \le \frac{4C}{\min(v;\,2^n)}.$$

Thus

$$\beta_n(u,v) \le 4C\frac{u_n\ \ln n}{\min(v;\,2^n)}.$$

The proof of Theorem 2 is complete.

3.2 Proof of Theorem 1

Let $X = (X_1, ..., X_n)$ in $\mathcal{M}_n(u)$. Following an idea by Bolthausen [2], we are going to define a new martingale difference sequence $\hat{X}$ which satisfies $V_n^2(\hat{X}) = 1$ a.s. Denote for each $d \in \mathbb{R}_+^*$,

$$\hat{n}(d) = n + [2d/u_n^2],\ \ \hat{k}(d) = (v_n^2 + d - v_n^2 V_n^2)/u_n^2,\ \ k(d) = [\hat{k}(d)],$$

$$d_1 = \|v_n^2 V_n^2(X) - v_n^2\|_1,\ \ d_\infty = \|v_n^2 V_n^2(X) - v_n^2\|_\infty$$

and

$$\hat{u}_i = \begin{cases} u_i, & \text{for } i \le n; \\ u_n, & \text{for } n+1 \le i \le \hat{n}(d). \end{cases}$$

where [.] denotes the integer part function. Consider the random variables $\hat{X}_1, ..., \hat{X}_{\hat{n}+1}$ defined as follows:

$$\begin{cases} \hat{X}_i = X_i \quad \text{a.s.} & 1 \le i \le n \\ \mu\big(\hat{X}_i = \pm\, u_n | \mathcal{F}_n\big) = \frac{1}{2} \quad \text{a.s.} & n+1 \le i \le n + k(d) \\ \mu\big(\hat{X}_{n+k(d)+1} = \pm\, [\hat{k}(d) - k(d)]^{\frac{1}{2}} u_n | \mathcal{F}_n\big) = \frac{1}{2} \quad \text{a.s.} & \\ \hat{X}_i = 0 & \text{else.} \end{cases}$$

We put

$$V_{\hat{n}(d)}^2(\hat{X}) = \frac{1}{v_{\hat{n}(d)}^2}\sum_{i=1}^{\hat{n}(d)} E(\hat{X}_i^2 | \hat{\mathcal{F}}_{i-1}),$$

$$\hat{v}_{\hat{n}(d)}^2 = \sum_{i=1}^{\hat{n}(d)} E(\hat{X}_i^2) \text{ and } \hat{\mathcal{F}}_l = \sigma(\hat{X}_1, \cdots, \hat{X}_l).$$

Lemma 3 *For each $i \le \hat{n}(d)$, we have*

$$\hat{v}_{\hat{n}(d)}^2 - v_n^2 = d,\ \ V_{\hat{n}(d)}^2(\hat{X}) = 1 \ \ \textit{and}\ \ E(|\hat{X}_i|^3 | \hat{\mathcal{F}}_{i-1}) \le \hat{u}_i E(X_i^2 | \hat{\mathcal{F}}_{i-1}) \ \textit{a.s.}$$

Proof of Lemma 3: By definition of $\hat{X}$, we have

$$\begin{aligned}
\hat{v}^2_{\hat{n}(d)} &= v_n^2 + \sum_{i=n+1}^{\hat{n}(d)} E\big[E\big(\hat{X}_i^{\,2}|\mathcal{F}_n\big)\big] \\
&= v_n^2 + \sum_{i=n+1}^{\hat{n}(d)} E\big[u_n^2 \, \mathbb{1}_{i\leq n+k(d)} + u_n^2[\hat{k}(d)-k(d)] \, \mathbb{1}_{i=n+k(d)+1}|\big] \\
&= v_n^2 + u_n^2 E[\hat{k}(d)] \\
&= v_n^2 + d
\end{aligned}$$

and

$$\begin{aligned}
V^2_{\hat{n}(d)}(\hat{X}) &= \frac{1}{\hat{v}^2_{\hat{n}(d)}} \sum_{i=1}^{\hat{n}(d)} E(\hat{X}_i^2|\hat{\mathcal{F}}_{i-1}) = \frac{1}{\hat{n}(d)}\Big(v_n^2 V_n^2(X) + \sum_{i=n+1}^{\hat{n}(d)} E(\hat{X}_i^2|\hat{\mathcal{F}}_{i-1})\Big) \\
&= \frac{1}{\hat{v}^2_{\hat{n}(d)}}\Big(v_n^2 V_n^2(X) + \sum_{i=n+1}^{\hat{n}(d)} u_n^2 \, \mathbb{1}_{i\leq n+k(d)} + u_n^2[\hat{k}(d)-k(d)] \, \mathbb{1}_{i=n+k(d)+1}\Big) \\
&= \frac{1}{\hat{v}^2_{\hat{n}(d)}}\Big(v_n^2 V_n^2(X) + u_n^2 k(d) + u_n^2[\hat{k}(d)-k(d)]\Big) \\
&= \frac{1}{\hat{v}^2_{\hat{n}(d)}}\Big(v_n^2 V_n^2(X) + v_n^2 + d - v_n^2 V_n^2(X)\Big) \\
&= \frac{v_n^2 + d}{\hat{v}^2_{\hat{n}(d)}} = 1.
\end{aligned}$$

On the other hand, for each $n+1 \leq i \leq \hat{n}(d)$, we obtain

$$E(|\hat{X}_i|^3|\hat{\mathcal{F}}_{i-1}) = \begin{cases} u_n^3, & \text{if } i \leq n+k(d); \\ u_n^3[\hat{k}(d)-k(d)]^{3/2}, & \text{if } i = n+k(d)+1; \\ 0, & \text{else.} \end{cases}$$

and

$$E(|\hat{X}_i|^2|\hat{\mathcal{F}}_{i-1}) = \begin{cases} u_n^2, & \text{if } i \leq n+k(d); \\ u_n^2[\hat{k}(d)-k(d)]^{3/2}, & \text{if } i = n+k(d)+1; \\ 0, & \text{else.} \end{cases}$$

Thus, for each $0 \leq i \leq \hat{n}(d)$, we obtain

$$E(|\hat{X}_i|^3|\hat{\mathcal{F}}_{i-1}) \leq \hat{u}_i E(\hat{X}_i^2|\hat{\mathcal{F}}_{i-1}) \;\text{ a.s.}$$

The proof of lemma 3 is complete.

One can easily check that

$$\Delta_n(X) \leq \sup_{t\in\mathbb{R}} |\mu(S_n(X)/\hat{v}_{\hat{n}(d)} \leq t) - \Phi(t)| + \sup_{t\in\mathbb{R}} |\Phi\left(\frac{v_n t}{\hat{v}_{\hat{n}(d)}}\right) - \Phi(t)|.$$

Noting that $\hat{v}^2 - v_n^2 = d$ and using Lemma 1 with $l = 2$ and $r = 1$, if $d \triangleq d_1$ there exist a positive constant c such that

$$\begin{aligned}\Delta_n(X) &\leq 2\Delta_{\hat{n}(d_1)}(\hat{X}) + 2\Big\| E\Big([\frac{1}{\hat{v}_{\hat{n}(d_1)}} \sum_{i=n+1}^{\hat{n}(d_1)} \hat{X}_i]^2 | S_n(X)\Big)\Big\|_1^{1/3} + \frac{1}{\sqrt{2\pi}} (\frac{\hat{v}_n(d_1) - v_n}{v_n}) \\ &\leq 2\Delta_{\hat{n}(d_1)}(\hat{X}) + 2\frac{d_1^{1/3}}{v_n^{2/3}} + \frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}}\frac{d_1^{1/2}}{v_n} \\ &\leq 2\Delta_{\hat{n}(d_1)}(\hat{X}) + c\frac{d_1^{1/3}}{v_n^{2/3}} \ \text{(one can suppose that } d_1 \leq v_n^2),\end{aligned}$$

where c is a positive constant. Using Lemma 3 and applying Theorem 2, we derive

$$\begin{aligned}\Delta_n(X) &\leq 2L\Big(\frac{\hat{u}_{\hat{n}(d_1)} \ln \hat{n}(d_1)}{\min\big(\hat{v}_{\hat{n}(d_1)};\, 2^{\hat{n}(d_1)}\big)} + \frac{d_1^{1/3}}{v_n^{2/3}}\Big) \\ &\leq 2L\Big(\frac{u_n \ln[n(1 + \frac{1}{v_n^2})]}{\min\big(v_n;\, 2^n\big)} + \frac{d_1^{1/3}}{v_n^{2/3}}\Big) \\ &\leq 4L\Big(\frac{u_n \ln n}{\min\big(v_n;\, 2^n\big)} + \frac{d_1^{1/3}}{v_n^{2/3}}\Big) \quad \text{because } d_1 \leq v_n^2.\end{aligned}$$

where L is a strictly positive constant.
Similarly if $d \triangleq d_\infty$ then

$$\begin{aligned}\Delta_n(X) &\leq 2L\Big(\frac{\hat{u}_{\hat{n}(d_\infty)} \ln \hat{n}(d_\infty)}{\min\big(\hat{v}_{\hat{n}(d_\infty)};\, 2^{\hat{n}(d_\infty)}\big)} + \frac{d_\infty^{1/3}}{v_n^{2/3}}\Big) \\ &\leq 4L\Big(\frac{u_n \ln n}{\min\big(v_n;\, 2^n\big)} + \frac{d_\infty^{1/3}}{v_n^{2/3}}\Big).\end{aligned}$$

Finally, we have

$$\Delta_n(X) \leq 4L\left(\frac{u_n \ln n}{\min(v_n;\, 2^n)} + \min\left\{ \frac{d_1^{1/3}}{v_n^{2/3}}, \frac{d_\infty^{1/2}}{v_n} \right\}\right).$$

The proof of Theorem 1 is complete. $\Diamond$

3.3 Proof of Theorem 3 and Lemma 1

Applying the inequality (3.3) in Lemma 1 for $Y = n^{-1/2}(g - g \circ T^n)$, $l = p$ and $r = 1$

$$\begin{aligned}\Delta_n(F) &\leq 2\Delta_n(H) + 2\Big\| E\Big(\Big|\frac{g - g\circ T^n}{n^{1/2}}\Big|^p \Big| \sum_{i=1}^{n} h \circ T^i\Big)\Big\|_1^{1/(p+1)} \\ &\leq 2\Delta_n(H) + 2\frac{\|g - g\circ T^n\|_p^{p/(p+1)}}{n^{p/2(p+1)}} \\ &\leq 2\Delta_n(H) + 4\frac{\|g\|_p^{p/(p+1)}}{n^{p/2(p+1)}}.\end{aligned}$$

If $p = +\infty$, we obtain

$$\begin{aligned}
2\Delta_n(F) &\le 2\Delta_n(H) + 2\left\| E\left(\left(\frac{g - g\circ T^n}{n^{1/2}} \right)^2 \left| \sum_{i=1}^n h\circ T^i \right) \right\|_\infty^{1/2} \\
&\le 2\Delta_n(H) + 4\frac{\|g\|_\infty}{n^{1/2}}.
\end{aligned}$$

The proof of the theorem (3) is complete. $\Diamond$

Let X and Y be two real random variables. We put for each $k > 0$ and $r \ge 1$, denote $\beta = \|E\left(|Y|^k|X\right)\|_r$ and consider $q \in \mathbb{R}\cup\{\infty\}$ such that $1/r + 1/q = 1$. Let $\lambda > 0$ and t be two real numbers we have

$$\begin{aligned}
\mu\,(X + Y \le t) &\ge \mu(X \le t-\lambda,\, Y \le t - X) \\
&= \mu(X \le t-\lambda) - \mu(X \le t-\lambda,\, Y > |t-X|) \\
&\ge \mu(X \le t-\lambda) - E\left\{ \mathbb{1}_{X\le t-\lambda}\, \mu(|Y| > |t-X|\,|X) \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
E\left\{ \mathbb{1}_{X\le t-\lambda}\mu(|Y| > |t-X|\,|X) \right\} &\le E\left\{ |t-X|^{-k} E(|Y|^k|X)\, \mathbb{1}_{X\le t-\lambda} \right\} \\
&\le \beta\|E\{ \mathbb{1}_{X\le t-\lambda}|t-X|^{-k}\}\|_q \\
&\le \beta\lambda^{-k},
\end{aligned}$$

we obtain

$$\mu(X+Y\le t) \ge \mu(X \le t-\lambda) - \beta\lambda^{-k}.$$

Consequently

$$\mu(X+Y\le t) - \Phi(t) \ge \mu(X \le t-\lambda) - \Phi(t-\lambda) - \frac{\lambda}{\sqrt{2\pi}} - \beta\lambda^{-k}$$

and taking $\lambda = \left(\beta\sqrt{2\pi}\right)^{1/(k+1)}$, there exists a positive constant c such that

$$\delta(X+Y) \ge \delta(X) - c\beta^{1/(k+1)}. \tag{19}$$

On the other hand

$$\begin{aligned}
\mu(X+Y\le t) &\le \mu(X \le t+\lambda) + \mu(X \ge t+\lambda, |Y| \ge |t-X|) \\
&= \mu(X \le t+\lambda) + E\left\{ \mathbb{1}_{X>t+\lambda}\, \mu(|Y| \ge |t-X|\,|X) \right\}
\end{aligned}$$

and

$$\begin{aligned}
E\left\{ \mathbb{1}_{X>t+\lambda}\, \mu(|Y| \le |t-X|\,|X) \right\} &\le E\left\{ \mathbb{1}_{X>t+\lambda}\, E(|Y|^k\,|X)\,|t-X|^{-k} \right\} \\
&\le \beta\|E(\mathbb{1}_{X>t+\lambda}\,|t-X|^{-k})\|_q \\
&\le \beta\lambda^{-k}.
\end{aligned}$$

Consequently

$$\mu(X+Y\le t) \le \mu(X\le t+\lambda) + \beta\lambda^{-k}$$

and

$$\mu(X+Y\le t) - \Phi(t) \le \mu(X \le t+\lambda) - \Phi(t+\lambda) + \frac{\lambda}{\sqrt{2\pi}} + \beta\lambda^{-k}.$$

Taking $\lambda = (\beta\sqrt{2\pi})^{1/(k+1)}$, there exists a positive constant $c^{'}$ such that

$$\delta(X+Y) \leq \delta(X) + c^{'}\beta^{1/(k+1)}. \tag{20}$$

Combining (19) and (20) with Lemma 1 in Bolthausen [2] completes the proof of Lemma 1.

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