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KAHANE-KHINTCHINE INEQUALITIES AND FUNCTIONAL CENTRAL LIMIT THEOREM FOR STATIONARY RANDOM FIELDS

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Abstract

We establish new Kahane-Khintchine inequalities in Orlicz spaces induced by exponential Young functions for stationary real random fields which are bounded or satisfy some finite exponential moment condition. Next, we give sufficient conditions for partial sum processes indexed by classes of sets satisfying some metric entropy condition to converge in distribution to a set-indexed Brownian motion. Moreover, the class of random fields that we study includes ϕ-mixing and martingale difference random fields.

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1 Introduction

Let $(X_k)_{k\in\mathbb{Z}^d}$ be a stationary field of zero mean real-valued random variables. If $\mathcal{A}$ is a collection of Borel subsets of $[0,1]^d$, define the smoothed partial sum process $\{S_n(A)\,;\,A\in\mathcal{A}\}$ by

$$S_n(A)=\sum_{i\in\{1,\ldots,n\}^d}\lambda(nA\cap R_i)X_i \tag{1}$$

where $R_i=]i_1-1,i_1]\times\ldots\times]i_d-1,i_d]$ is the unit cube with upper corner at i and λ is the Lebesgue measure on $\mathbb{R}^d$.

The main aim of this paper is to study the asymptotic behaviour of the sequence of processes $\{S_n(A)\,;\,A\in\mathcal{A}\}$ in terms of the validity of the functional central limit theorem (FCLT) using new Kahane-Khintchine inequalities (cf. section 3). More precisely, we derive the following property: the sequence $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ converges in distribution to a mixture of Brownian motions in the space $C(\mathcal{A})$ of continuous real functions on $\mathcal{A}$ equipped with the metric of uniform convergence.

To measure the size of the collection $\mathcal{A}$ one usually considers the metric entropy with respect to the Lebesgue measure. Dudley [9] proved the existence of a standard Brownian motion with sample paths in the space $C(\mathcal{A})$ if $\mathcal{A}$ has finite entropy integral (i.e. Condition (5) in section 4 holds).

The first weak convergence results for $\mathcal{Q}_d$-indexed partial sum processes were established in the iid case for the collection $\mathcal{Q}_d$ of lower-left quadrants in $[0,1]^d$, that is to say the collection $\{[0,t_1]\times\ldots\times[0,t_d]\,;\,(t_1,\ldots,t_d)\in[0,1]^d\}$. They were proved by Wichura [26] under a finite variance condition and earlier by Kuelbs [17] under additional moment restrictions. When the dimension d is reduced to one, these results coincide with the original invariance principle of Donsker [7]. In 1983, Pyke [22] derived a weak convergence result for the process $\{S_n(A)\,;\,A\in\mathcal{A}\}$ in the iid case provided that the collection $\mathcal{A}$ satisfies an entropy condition with inclusion (i.e. Condition (6) in section 4 holds). However, this FCLT required moment conditions which become more strict as the size of $\mathcal{A}$ increases. Bass [2] and simultaneously Alexander and Pyke [1] extended Pyke's result to iid random fields with finite variance.

For uniform ϕ-mixing and β-mixing random fields, Goldie and Greenwood [12] adapted Bass's approach which is mainly based on Bernstein's inequality for iid random fields. In 1991, Chen [3] proved a FCLT for $\mathcal{Q}_d$-indexed partial sum of non-uniform ϕ-mixing random fields (the non-uniform ϕ-mixing coefficients was introduced by Dobrushin and Nahapetian [6]). Recently, Dedecker [5] gave an L^∞-projective criterion for the partial sum

process $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ to converge to an $\mathcal{A}$-indexed Brownian motion when the collection $\mathcal{A}$ satisfies only the entropy condition (5) of Dudley. This new criterion is valid for martingale difference bounded random fields and provides a new criterion for non-uniform ϕ-mixing bounded random fields. In the unbounded case, Dedecker gave an L^p-version ($p>1$) of his L^∞-projective criterion for $\mathcal{Q}_d$-indexed partial sum of random fields with moments strictly greater than 2. Next, for non-uniform ϕ-mixing random fields, using the chaining method of Bass [2] and establishing Bernstein type inequalities, Dedecker proved the FCLT for the partial sum process $\{S_n(A)\,;\,A\in\mathcal{A}\}$ provided that the collection $\mathcal{A}$ satisfies the more strict entropy condition with inclusion (6) and under both finite fourth moments and an algebraic decay of the mixing coefficients.
In a previous work (see [10]), it is shown that the FCLT may be not valid for p-integrable ($0\leq p<+\infty$) martingale difference random fields. More precisely, the following result is proved.

Theorem (El Machkouri, Volný) *Let $(\Omega,\mathcal{F},\mu,T)$ be an ergodic dynamical system with positive entropy where Ω is a Lebesgue space, μ is a probability measure and T is a $\mathbb{Z}^d$-action. For any nonnegative real p, there exist a real function $f\in L^p(\Omega)$ and a collection $\mathcal{A}$ of regular Borel subsets of $[0,1]^d$ such that*

(**1**) *For any k in $\mathbb{Z}^d$, $E\left(f\circ T^k|\sigma(f\circ T^i\,;\,i\neq k)\right)=0$. We say that the random field $(f\circ T^k)_{k\in\mathbb{Z}^d}$ is a strong martingale difference random field.*

(**2**) *The collection $\mathcal{A}$ satisfies the entropy condition with inclusion* (6)*.*

(**3**) *The partial sum process $\{n^{-d/2}S_n(f,A)\,;\,A\in\mathcal{A}\}$ is not tight in the space $C(\mathcal{A})$*

where

$$S_n(f,A):=\sum_{i\in\{1,\dots,n\}^d}\lambda(nA\cap R_i)f\circ T^i.$$

The above theorem shows that not only Dedecker's FCLT for bounded random fields (see [5]) cannot be extended to p-integrable ($0\leq p<+\infty$) random fields but also it lays emphasis on that Bass, Alexander and Pyke's result (see [1], [2]) for iid random fields cannot hold for martingale difference random fields.
In the present work, under a projective condition similar to Dedecker's one, we establish some so-called Kahane-Khintchine inequalities for stationary

real random fields in Orlicz spaces induced by exponential Young functions (cf. section 3). We require the random field to be bounded or to satisfy some finite exponential moment condition (cf. Assumption (2) in section 3). These inequalities extend previous ones for sequences of iid bounded random variables (see for example [14], [15], [20]). With the help of the above inequalities, we are in position to prove the tightness of the sequence of processes $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ in the space $C(\mathcal{A})$ when the collection $\mathcal{A}$ satisfies an entropy condition related to the moments of the random field (i.e. Condition (8) in section 4 holds). The convergence of the finite-dimensional laws is a simple consequence of a central limit theorem (CLT) for stationary real random fields with finite variance (see [4], Theorem 2.2).
Before presenting our results in more details, let us explain the main difference of our approach in tightness's proof of the sequence of processes $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ with Dedecker's one. In fact, Dedecker's proof is based on an exponential inequality of Hoeffding type derived from a Marcinkiewicz-Zygmund type inequality for p-integrable real random fields (cf. Inequality (11) in section 5) by optimizing in p. That is the reason why the boundedness condition is necessary. Our approach combines this Marcinkiewicz-Zygmund type inequality with a property of the norm in Orlicz spaces induced by exponential Young functions (cf. Lemma 1) which allows us to derive the announced Kahane-Khintchine inequalities under only the assumption of some finite exponential moment.

2 Notations

By a stationary real random field we mean any family $(X_k)_{k\in\mathbb{Z}^d}$ of real-valued random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that for any $(k,n)\in\mathbb{Z}^d\times\mathbb{N}^*$ and any $(i_1,...,i_n)\in\mathbb{Z}^{nd}$, the random vectors $(X_{i_1},...,X_{i_n})$ and $(X_{i_1+k},...,X_{i_n+k})$ have the same law.
Let μ be the law of the stationary real random field $(X_k)_{k\in\mathbb{Z}^d}$ and consider the projection f from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$ defined by $f(\omega)=\omega_0$ and the family of translation operators $(T^k)_{k\in\mathbb{Z}^d}$ from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}^{\mathbb{Z}^d}$ defined by $(T^k(\omega))_i=\omega_{i+k}$ for any $k\in\mathbb{Z}^d$ and any ω in $\mathbb{R}^{\mathbb{Z}^d}$. Denote by $\mathcal{B}$ the Borel σ-algebra of $\mathbb{R}$. The random field $(f\circ T^k)_{k\in\mathbb{Z}^d}$ defined on the probability space $(\mathbb{R}^{\mathbb{Z}^d},\mathcal{B}^{\mathbb{Z}^d},\mu)$ is stationary with the same law as $(X_k)_{k\in\mathbb{Z}^d}$. Consequently, without loss of generality, one can suppose that

$$(\Omega,\mathcal{F},\mathbb{P})=(\mathbb{R}^{\mathbb{Z}^d},\mathcal{B}^{\mathbb{Z}^d},\mu)\quad\text{and}\quad X_k=f\circ T^k,\;\;k\in\mathbb{Z}^d.$$

An element A of $\mathcal{F}$ is said to be invariant if $T^k(A) = A$ for any $k \in \mathbb{Z}^d$. We denote by $\mathcal{I}$ the σ-algebra of all measurable invariant sets.
On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i = (i_1, ..., i_d)$ and $j = (j_1, ..., j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{lex} j$ means that either $i_1 < j_1$ or for some p in $\{2, 3, ..., d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \leq q < p$. Let the sets $\{V_i^k\,;\, i \in \mathbb{Z}^d\,,\, k \in \mathbb{N}^*\}$ be defined as follows:

$$V_i^1 = \{j \in \mathbb{Z}^d\,;\, j <_{lex} i\},$$

and for $k \geq 2$

$$V_i^k = V_i^1 \cap \{j \in \mathbb{Z}^d\,;\, |i-j| \geq k\} \quad \text{where} \quad |i-j| = \max_{1 \leq k \leq d} |i_k - j_k|.$$

For any subset Γ of $\mathbb{Z}^d$, define $\mathcal{F}_\Gamma = \sigma(X_i\,;\, i \in \Gamma)$. If X_i belongs to $L^1(\mathbb{P})$, set

$$E_k(X_i) = E(X_i|\mathcal{F}_{V_i^k}).$$

<u>*Mixing coefficients for random fields.*</u> Given two σ-algebras $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{F}$, different measures of their dependence have been considered in the literature. We are interested by one of them. The ϕ-mixing coefficient has been introduced by Ibragimov [13] and can be defined by

$$\phi(\mathcal{U}, \mathcal{V}) = \sup\{\|\mathbb{P}(V|\mathcal{U}) - \mathbb{P}(V)\|_\infty\,,\, V \in \mathcal{V}\}.$$

Now, let $(X_k)_{k \in \mathbb{Z}^d}$ be a real random field and denote by $|\Gamma|$ the cardinality of any subset Γ of $\mathbb{Z}^d$. In the sequel, we shall use the following non-uniform ϕ-mixing coefficients defined for any (k, l, n) in $(\mathbb{N}^* \cup \{\infty\})^2 \times \mathbb{N}$ by

$$\phi_{k,l}(n) = \sup\{\phi(\mathcal{F}_{\Gamma_1}, \mathcal{F}_{\Gamma_2}),\, |\Gamma_1| \leq k,\, |\Gamma_2| \leq l,\, d(\Gamma_1, \Gamma_2) \geq n\},$$

where the distance d is defined by $d(\Gamma_1, \Gamma_2) = \min\{|i-j|,\, i \in \Gamma_1,\, j \in \Gamma_2\}$. We say that the random field $(X_k)_{k \in \mathbb{Z}^d}$ is ϕ-mixing if there exists a pair (k, l) in $(\mathbb{N}^* \cup \{\infty\})^2$ such that $\lim_{n \to +\infty} \phi_{k,l}(n) = 0$.
For more about mixing coefficients one can refer to Doukhan [8] or Rio [23].

<u>*Young functions and Orlicz spaces.*</u> Recall that a Young function ψ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies

$$\lim_{t \to +\infty} \psi(t) = +\infty \quad \text{and} \quad \psi(0) = 0.$$

We define the Orlicz space L_ψ as the space of real random variables Z defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[\psi(|Z|/c)] < +\infty$ for some $c > 0$. The Orlicz space L_ψ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable Z by

$$\|Z\|_\psi = \inf\{\, c > 0\,;\, E[\psi(|Z|/c)] \le 1\,\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [16].

3 Kahane-Khintchine inequalities

A real random field $(X_k)_{k\in\mathbb{Z}^d}$ is said to be a martingale difference random field if it satisfies the following condition: for any m in $\mathbb{Z}^d$

$$E(\, X_m \,|\, \sigma(\, X_k\,;\, k <_{lex} m\,)\,) = 0 \quad \text{a.s.}$$

Let $\beta > 0$. We denote by ψ_β the Young function defined for any $x \in \mathbb{R}^+$ by

$$\psi_\beta(x) = \exp((x + h_\beta)^\beta) - \exp(h_\beta^\beta) \quad \text{where} \quad h_\beta = ((1-\beta)/\beta)^{1/\beta}\, 1\!\!1_{\{0<\beta<1\}}.$$

We are interested in Kahane-Khintchine inequalities for a large class of random fields. In fact, we shall give a projective condition (that is to say a condition expressed in terms of a series of conditional expectations) comparable to that introduced by Dedecker to prove a central limit theorem for stationary square-integrable random fields (see [4]) and a functional central limit theorem for stationary bounded random fields (see [5]). Consider the following assumption :

$$\exists q \in]0,2[\quad \exists \theta > 0 \quad E[\exp(\theta |X_0|^{\beta(q)})] < +\infty \tag{2}$$

where $\beta(q) = 2q/(2-q)$ for any $0 < q < 2$. Our first result is the following.

Theorem 1 *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a zero mean stationary real random field which satisfies the assumption (2) for some $0 < q < 2$. There exists a positive universal constant $M_1(q)$ depending only on q such that for any family $(a_i)_{i\in\mathbb{Z}^d}$ of real numbers and any finite subset Γ of $\mathbb{Z}^d$,*

$$\left\| \sum_{i\in\Gamma} a_i X_i \right\|_{\psi_q} \le M_1(q) \left(\sum_{i\in\Gamma} |a_i|\, b_{i,q}(X) \right)^{1/2} \tag{3}$$

where

$$b_{i,q}(X) := |a_i|\, \|X_0\|^2_{\psi_{\beta(q)}} + \sum_{k\in V_0^1} |a_{k+i}| \left\| \sqrt{|X_k E_{|k|}(X_0)|} \right\|^2_{\psi_{\beta(q)}}.$$

If $(X_i)_{i\in\mathbb{Z}^d}$ is bounded then for any $0 < q \leq 2$, there exists a universal positive constant $M_2(q)$ depending only on q such that for any family $(a_i)_{i\in\mathbb{Z}^d}$ of real numbers and any finite subset Γ of $\mathbb{Z}^d$,

$$\left\|\sum_{i\in\Gamma} a_i X_i\right\|_{\psi_q} \leq M_2(q)\left(\sum_{i\in\Gamma}|a_i|\, b_{i,\infty}(X)\right)^{1/2} \tag{4}$$

where

$$b_{i,\infty}(X) := |a_i|\,\|X_0\|_\infty^2 + \sum_{k\in V_0^1}|a_{k+i}|\,\|X_k E_{|k|}(X_0)\|_\infty.$$

Remark 1 If $(X_i)_{i\in\mathbb{Z}^d}$ is a martingale difference random field then

$$b_{i,q}(X) = |a_i|\,\|X_0\|_{\psi_{\beta(q)}}^2 \quad \text{and} \quad b_{i,\infty}(X) = |a_i|\,\|X_0\|_\infty^2.$$

Thus, the inequalities (3) and (4) extend previous ones established for sequences of bounded i.i.d. random variables (see [14], [15], [20]).

Using Serfling's inequality (see [19] or [24]), we deduce from Theorem 1 the following result for stationary ϕ-mixing real random fields.

Corollary 1 *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a zero mean stationary real random field which satisfies the assumption (2) for some $0 < q < 2$. For any family $(a_i)_{i\in\mathbb{Z}^d}$ of real numbers and any finite subset Γ of $\mathbb{Z}^d$,*

$$\left\|\sum_{i\in\Gamma} a_i X_i\right\|_{\psi_q} \leq M_1(q)\,\|X_0\|_{\psi_{\beta(q)}}\left(\sum_{i\in\Gamma}|a_i|\,\widetilde{b}_{i,q}(X)\right)^{1/2}$$

where

$$\widetilde{b}_{i,q}(X) := |a_i| + C(q)\sum_{k\in V_0^1}|a_{k+i}|\sqrt{\phi_{\infty,1}(|k|)},$$

$M_1(q)$ is the positive constant introduced in Theorem 1 and $C(q)$ is a positive universal constant depending only on q.
If $(X_i)_{i\in\mathbb{Z}^d}$ is bounded then for any $0 < q \leq 2$, any family $(a_i)_{i\in\mathbb{Z}^d}$ of real numbers and any finite subset Γ of $\mathbb{Z}^d$,

$$\left\|\sum_{i\in\Gamma} a_i X_i\right\|_{\psi_q} \leq M_2(q)\,\|X_0\|_\infty\left(\sum_{i\in\Gamma}|a_i|\,\widetilde{b}_{i,\infty}(X)\right)^{1/2}$$

where

$$\widetilde{b}_{i,\infty}(X) := |a_i| + 2\sum_{k\in V_0^1}|a_{k+i}|\,\phi_{\infty,1}(|k|),$$

One can notice that in the unbounded case we were able to give Kahane-Khintchine inequalities only in Orlicz spaces L_{ψ_q} when $0 < q < 2$ but for bounded random field we established these inequalities even in the space L_{ψ_2}. That is the reason why we cannot give a proof of the FCLT for random fields with finite exponential moments (Theorem 2) under Dudley's entropy condition (5) unlike as in the case of bounded random fields (see [5]).

4 Functional central limit theorem

Let $\mathcal{A}$ be a collection of Borel subsets of $[0,1]^d$. We focus on the sequence of processes $\{S_n(A)\,;\, A \in \mathcal{A}\}$ defined by (1). As a function of A, this process is continuous with respect to the pseudo-metric $\rho(A,B) = \sqrt{\lambda(A\Delta B)}$.
To measure the size of $\mathcal{A}$ one considers the metric entropy: denote by $H(\mathcal{A},\rho,\epsilon)$ the logarithm of the smallest number of open balls of radius ϵ with respect to ρ which form a covering of $\mathcal{A}$. The function $H(\mathcal{A},\rho,.)$ is the entropy of the class $\mathcal{A}$. A more strict tool is the metric entropy with inclusion: assume that $\mathcal{A}$ is totally bounded with inclusion i.e. for each positive ϵ there exists a finite collection $\mathcal{A}(\epsilon)$ of Borel subsets of $[0,1]^d$ such that for any $A \in \mathcal{A}$, there exist A^- and A^+ in $\mathcal{A}(\epsilon)$ with $A^- \subseteq A \subseteq A^+$ and $\rho(A^-,A^+) \le \epsilon$. Denote by $\mathbb{H}(\mathcal{A},\rho,\epsilon)$ the logarithm of the cardinality of the smallest collection $\mathcal{A}(\epsilon)$. The function $\mathbb{H}(\mathcal{A},\rho,.)$ is the entropy with inclusion (or bracketing entropy) of the class $\mathcal{A}$. Let $C(\mathcal{A})$ be the space of continuous real functions on $\mathcal{A}$, equipped with the norm $\|.\|_{\mathcal{A}}$ defined by

$$\|f\|_{\mathcal{A}} = \sup_{A\in\mathcal{A}} |f(A)|.$$

A standard Brownian motion indexed by $\mathcal{A}$ is a mean zero Gaussian process W with sample paths in $C(\mathcal{A})$ and Cov(W(A),W(B))= $\lambda(A\cap B)$. From Dudley [9] we know that such a process exists if

$$\int_0^1 \sqrt{H(\mathcal{A},\rho,\epsilon)}\, d\epsilon < +\infty. \tag{5}$$

Since $H(\mathcal{A},\rho,.) \le \mathbb{H}(\mathcal{A},\rho,.)$, the standard Brownian motion W is well defined if

$$\int_0^1 \sqrt{\mathbb{H}(\mathcal{A},\rho,\epsilon)}\, d\epsilon < +\infty. \tag{6}$$

We say that the sequence $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ satisfies the functional central limit theorem (FCLT) if it converges in distribution to a mixture of

$\mathcal{A}$-indexed Brownian motions in the space $C(\mathcal{A})$ (which means that the limiting process is of the form ηW, where W is a standard Brownian motion indexed by $\mathcal{A}$ and η is a nonnegative random variable independent of W).

In the sequel, we shall give a projective criterion which implies the tightness of the sequence $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ in $C(\mathcal{A})$ under the assumption (2) of finite exponential moments and provided that the class $\mathcal{A}$ satisfies an entropy condition related to the moments of the random field (i.e. Condition (8) holds). The case of bounded stationary real random fields was studied by Dedecker in [5] where he proved that the FCLT holds under the L^∞-projective criterion

$$\sum_{k\in V_0^1}\|X_kE_{|k|}(X_0)\|_\infty<+\infty$$

and for any collection $\mathcal{A}$ satisfying only Dudley's entropy condition (5). For any Borel set A in $[0,1]^d$, let ∂A be the boundary of A. We say that A is regular if $\lambda(\partial A)=0$.

Theorem 2 *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a zero mean stationary real random field which satisfies the assumption (2) for some $0<q<2$ and assume that*

$$\sum_{k\in V_0^1}\left\|\sqrt{|X_kE_{|k|}(X_0)|}\right\|_{\psi_{\beta(q)}}^2<+\infty. \tag{7}$$

Let $\mathcal{A}$ be a collection of regular Borel subsets of $[0,1]^d$ satisfying the following entropy condition

$$\int_0^1(H(\mathcal{A},\rho,\epsilon))^{1/q}\,d\epsilon<+\infty. \tag{8}$$

Then

(1) *For the σ-algebra $\mathcal{I}$ of invariant sets defined in section 2, we have*

$$\sum_{k\in\mathbb{Z}^d}\left\|\sqrt{|E(X_0X_k|\mathcal{I})|}\right\|_{\psi_{\beta(q)}}^2<+\infty. \tag{9}$$

Denote by η the nonnegative and $\mathcal{I}$-measurable random variable

$$\eta=\sum_{k\in\mathbb{Z}^d}E(X_0X_k|\mathcal{I}).$$

(**2**) *The sequence of processes $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ converges in distribution in $C(\mathcal{A})$ to $\sqrt{\eta}W$ where W is a standard Brownian motion indexed by $\mathcal{A}$ and independent of $\mathcal{I}$.*

In Theorem 2, one can see that we control the size of the class $\mathcal{A}$ via the classical metric entropy (without inclusion). In fact, all the earlier results we know (in particular [1], [2], [5]) about the FCLT for unbounded processes indexed by large classes of sets deal with the more strict bracketing entropy.

Using Serfling's inequality (see [19] or [24]), we derive from Theorem 2 the following result for stationary ϕ-mixing real random fields.

Corollary 2 *Theorem* 2 *still holds if we replace the condition* (7) *by*

$$\sum_{k\in\mathbb{Z}^d}\sqrt{\phi_{\infty,1}(|k|)}<+\infty. \tag{10}$$

5 Proofs

We need the following lemma which can be obtained using the expansion of the exponential function (see [25]).

Lemma 1 *Let β be a positive real number and Z be a real random variable. There exist positive universal constants A_β and B_β depending only on β such that*

$$A_\beta\sup_{p>2}\frac{\|Z\|_p}{p^{1/\beta}}\leq\|Z\|_{\psi_\beta}\leq B_\beta\sup_{p>2}\frac{\|Z\|_p}{p^{1/\beta}}.$$

Recall that in [5], Dedecker established the following Marcinkiewicz-Zygmund type inequality for nonstationary real random fields.

Proposition (Dedecker, 2001) *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a zero mean real random field and Γ be a finite subset of $\mathbb{Z}^d$. For any $p>2$,*

$$\left\|\sum_{i\in\Gamma}X_i\right\|_p\leq\left(2\,p\sum_{i\in\Gamma}c_i(X)\right)^{1/2} \tag{11}$$

where

$$c_i(X):=\|X_i^2\|_{\frac{p}{2}}+\sum_{k\in V_i^1}\|X_kE_{|k-i|}(X_i)\|_{\frac{p}{2}}.$$

Now, recall that $\beta(q)=2q/(2-q)$ for any $0<q<2$ and define $1/\beta(2)=0$. Combining Lemma 1 and Inequality (11), we derive the following estimation.

Lemma 2 *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a zero mean real random field. For any $0<q\le 2$ there exists a positive universal constant B_q depending only on q such that for any family $(a_i)_{i\in\mathbb{Z}^d}$ of real numbers and any finite subset Γ of $\mathbb{Z}^d$,*

$$\left\|\sum_{i\in\Gamma} a_i X_i\right\|_{\psi_q} \le \sqrt{2}\,B_q \sup_{p>2}\frac{1}{p^{1/\beta(q)}}\left(\sum_{i\in\Gamma} c_i(aX)\right)^{1/2} \tag{12}$$

where

$$c_i(aX) := a_i^2\|X_i\|_p^2 + |a_i|\sum_{k\in V_i^1}|a_k|\,\|X_k E_{|k-i|}(X_i)\|_{\frac{p}{2}}.$$

5.1 Proof of Theorem 1

Assume that $(X_i)_{i\in\mathbb{Z}^d}$ is a zero mean stationary real random field satisfying the condition (2) for some $0<q<2$ and $(a_i)_{i\in\mathbb{Z}^d}$ is a family of real numbers. Let i in Γ be fixed. We have

$$\begin{aligned}
c_i(aX) &:= a_i^2\|X_i\|_p^2 + |a_i|\sum_{k\in V_i^1}|a_k|\,\|X_k E_{|k-i|}(X_i)\|_{\frac{p}{2}}\\
&= a_i^2\|X_i\|_p^2 + |a_i|\sum_{k\in V_i^1}|a_k|\left\|\sqrt{|X_k E_{|k-i|}(X_i)|}\right\|_p^2\\
&= a_i^2\|X_0\|_p^2 + |a_i|\sum_{k\in V_0^1}|a_{k+i}|\left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|_p^2.
\end{aligned}$$

Moreover, by Lemma 1, there exists a positive universal constant $A_{\beta(q)}$ depending only on q such that

$$\sup_{p>2}\frac{\|X_0\|_p}{p^{1/\beta(q)}} \le A_{\beta(q)}^{-1}\,\|X_0\|_{\psi_{\beta(q)}} \tag{13}$$

and for any k in V_0^1,

$$\sup_{p>2}\frac{1}{p^{1/\beta(q)}}\left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|_p \le A_{\beta(q)}^{-1}\left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|_{\psi_{\beta(q)}}. \tag{14}$$

Combining (12), (13) and (14), we derive the following estimation

$$\left\|\sum_{i\in\Gamma} a_i X_i\right\|_{\psi_q} \le M_1(q)\left(\sum_{i\in\Gamma}|a_i|\,b_{i,q}(X)\right)^{1/2}$$

where

$$b_{i,q}(X) := |a_i| \, \|X_0\|^2_{\psi_{\beta(q)}} + \sum_{k \in V_0^1} |a_{k+i}| \left\| \sqrt{|X_k E_{|k|}(X_0)|} \right\|^2_{\psi_{\beta(q)}}$$

and $M_1(q)$ denotes the constant $\sqrt{2} B_q A^{-1}_{\beta(q)}$. The first part of Theorem 1 is proved.
Now, assume that the random field $(X_i)_{i\in\mathbb{Z}^d}$ is bounded, let $0 < q \le 2$ be fixed and recall that $1/\beta(2) = 0$. For any i in Γ,

$$c_i(aX) \le a_i^2 \|X_0\|^2_\infty + |a_i| \sum_{k\in V_0^1} |a_{k+i}| \, \|X_k E_{|k|}(X_0)\|_\infty.$$

So, using Inequality (12), we infer that

$$\left\| \sum_{i\in\Gamma} a_i X_i \right\|_{\psi_q} \le M_2(q) \left(\sum_{i\in\Gamma} |a_i| \, b_{i,\infty}(X) \right)^{1/2}$$

where

$$b_{i,\infty}(X) := |a_i| \, \|X_0\|^2_\infty + \sum_{k\in V_0^1} |a_{k+i}| \, \|X_k E_{|k|}(X_0)\|_\infty$$

and $M_2(q)$ denotes the constant $\sqrt{2} B_q 2^{-1/\beta(q)}$. The proof of Theorem 1 is complete. $\square$

5.2 Proof of Corollary 1

Let i in Γ be fixed. Consider

$$\widetilde{b}_{i,q}(X) := |a_i| + C(q) \sum_{k\in V_0^1} |a_{k+i}| \sqrt{\phi_{\infty,1}(|k|)}$$

and

$$\widetilde{b}_{i,\infty}(X) := |a_i| + 2 \sum_{k\in V_0^1} |a_{k+i}| \, \phi_{\infty,1}(|k|)$$

where $C(q)$ is a positive universal constant depending only on q that we will define later. It is sufficient to prove that

$$b_{i,q}(X) \le \|X_0\|^2_{\psi_{\beta(q)}} \, \widetilde{b}_{i,q}(X) \tag{15}$$

and

$$b_{i,\infty}(X) \le \|X_0\|^2_\infty \, \widetilde{b}_{i,\infty}(X). \tag{16}$$

Let k in V_0^1 be fixed. By Lemma 1, there exists a positive universal constant $B_{\beta(q)}$ depending only on q such that

$$\begin{aligned}\left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|_{\psi_{\beta(q)}}^2 &\leq B_{\beta(q)}^2 \sup_{p>2} \frac{1}{p^{2/\beta(q)}} \left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|_p^2 \\ &= B_{\beta(q)}^2 \sup_{p>2} \frac{1}{p^{2/\beta(q)}} \|X_k E_{|k|}(X_0)\|_{\frac{p}{2}} \\ &\leq B_{\beta(q)}^2 \sup_{p>2} \frac{1}{p^{2/\beta(q)}} \|X_0\|_p \, \|E_{|k|}(X_0)\|_p.\end{aligned}$$

Using Serfling's inequality (see [19] or [24]), we derive for any $p > 2$,

$$\begin{aligned}\|E_{|k|}(X_0)\|_p &\leq 2\, \|X_0\|_p \, \phi_{\infty,1}(|k|)^{\frac{p-1}{p}} \\ &\leq 2\, \|X_0\|_p \sqrt{\phi_{\infty,1}(|k|)}.\end{aligned}$$

Consequently,

$$\left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|_{\psi_{\beta(q)}}^2 \leq 2\, B_{\beta(q)}^2 \left(\sup_{p>2} \frac{1}{p^{1/\beta(q)}} \|X_0\|_p\right)^2 \sqrt{\phi_{\infty,1}(|k|)}.$$

Using Inequality (13) and putting $C(q) = 2\, B_{\beta(q)}^2 \, A_{\beta(q)}^{-2}$, we obtain

$$\left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|_{\psi_{\beta(q)}}^2 \leq C(q)\, \|X_0\|_{\psi_{\beta(q)}}^2 \sqrt{\phi_{\infty,1}(|k|)}. \tag{17}$$

Finally, Inequality (15) is a simple consequence of (17). The first part of Corollary 1 is proved.
Now, assume that the random field $(X_i)_{i\in\mathbb{Z}^d}$ is bounded. Serfling's inequality (see [19] or [24]) implies

$$\|E_{|k|}(X_0)\|_\infty \leq 2\, \|X_0\|_\infty \, \phi_{\infty,1}(|k|).$$

Consequently, we obtain for any k in V_0^1,

$$\|X_k E_{|k|}(X_0)\|_\infty \leq 2\, \|X_0\|_\infty^2 \, \phi_{\infty,1}(|k|)$$

which implies Inequality (16). The proof of Corollary 1 is complete. $\square$

5.3 Proof of Theorem 2

Let k in V_0^1 be fixed. Consider the tail σ-algebra $\mathcal{F}_{-\infty} = \cap_{i\in\mathbb{N}^*}\mathcal{F}_{V_0^i}$. Using the same argument as in Georgii ([11], Proposition 14.9), we derive that the σ-algebra $\mathcal{I}$ of invariant sets is included in the $\mathbb{P}$-completion of $\mathcal{F}_{-\infty}$. So, for any nonnegative real p, we have

$$\|E(X_0X_k|\mathcal{I})\|_p \leq \|E(X_0X_k|\mathcal{F}_{-\infty})\|_p \leq \|X_kE_{|k|}(X_0)\|_p. \tag{18}$$

By Lemma 1, there exists a positive universal constant $A_{\beta(q)}$ depending only on q such that

$$\left\|\sqrt{|X_kE_{|k|}(X_0)|}\right\|^2_{\psi_{\beta(q)}} \geq A^2_{\beta(q)} \sup_{p>2}\frac{1}{p^{2/\beta(q)}}\left\|\sqrt{|X_kE_{|k|}(X_0)|}\right\|^2_p \tag{19}$$

Since

$$\left\|\sqrt{|X_kE_{|k|}(X_0)|}\right\|^2_p = \|X_kE_{|k|}(X_0)\|_{\frac{p}{2}}, \tag{20}$$

the inequality (19) implies

$$\left\|\sqrt{|X_kE_{|k|}(X_0)|}\right\|^2_{\psi_{\beta(q)}} \geq A^2_{\beta(q)} \sup_{p>2}\frac{1}{p^{2/\beta(q)}}\|X_kE_{|k|}(X_0)\|_{\frac{p}{2}}$$

and the inequality (18) gives

$$\begin{aligned}\left\|\sqrt{|X_kE_{|k|}(X_0)|}\right\|^2_{\psi_{\beta(q)}} &\geq A^2_{\beta(q)} \sup_{p>2}\frac{1}{p^{2/\beta(q)}}\|E(X_0X_k|\mathcal{I})\|_{\frac{p}{2}}\\ &= A^2_{\beta(q)} \sup_{p>2}\frac{1}{p^{2/\beta(q)}}\left\|\sqrt{|E(X_0X_k|\mathcal{I})|}\right\|^2_p\\ &\geq A^2_{\beta(q)}\, B^{-2}_{\beta(q)}\left\|\sqrt{|E(X_0X_k|\mathcal{I})|}\right\|^2_{\psi_{\beta(q)}} \qquad \text{(by Lemma 1)}\end{aligned}$$

where $B_{\beta(q)}$ is the positive universal constant in Lemma 1.
So, using the stationarity of the random field and the assumption (7), we derive the assertion (9).
Now, if ϵ is a positive real number, (19) and (20) imply that

$$\left\|\sqrt{|X_kE_{|k|}(X_0)|}\right\|^2_{\psi_{\beta(q)}} \geq (2+\epsilon)^{-2/\beta(q)}\, A^2_{\beta(q)}\,\|X_kE_{|k|}(X_0)\|_1.$$

Consequently, the condition (7) is more strict than the projective criterion

$$\sum_{k\in V_0^1}\|X_kE_{|k|}(X_0)\|_1 < +\infty$$

initially introduced by Dedecker [4] as a sufficient condition for the central limit theorem (CLT) for stationary real random fields with finite variance. Therefore, the random variable η is nonnegative (see [4], Proposition 3).

As usual, we have to prove the convergence of the finite-dimensional laws and the tightness of the partial sum process $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ in $C(\mathcal{A})$. The convergence of the finite-dimensional laws is a simple consequence of both the CLT for random fields ([4], Theorem 2.2) and the following lemma (see [5]).
For any subset Γ of $\mathbb{Z}^d$ we consider

$$\partial\Gamma = \big\{ i \in \Gamma\,;\, \exists j \notin \Gamma \text{ such that } |i-j| = 1 \big\}.$$

For any Borel set A of $[0,1]^d$, we denote by $\Gamma_n(A)$ the finite subset of $\mathbb{Z}^d$ defined by $\Gamma_n(A) = nA \cap \mathbb{Z}^d$.

Lemma 3 (Dedecker, 2001) *Let A be a regular Borel set of $[0,1]^d$ with $\lambda(A) > 0$. We have*

$$(i)\ \lim_{n\to+\infty} \frac{|\Gamma_n(A)|}{n^d} = \lambda(A) \qquad (ii)\ \lim_{n\to+\infty} \frac{|\partial\Gamma_n(A)|}{|\Gamma_n(A)|} = 0.$$

Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary random field with mean zero and finite variance. Assume that $\sum_{k\in\mathbb{Z}^d} |E(X_0X_k)| < +\infty$. Then

$$\lim_{n\to+\infty} \quad n^{-d/2}\big|\big|S_n(A) - \sum_{k\in\Gamma_n(A)} X_k\big|\big|_2 = 0.$$

Remark 2 The series $\sum_{k\in\mathbb{Z}^d} |E(X_0X_k)|$ converges under the assumption (7). In fact, one can check that

$$\sum_{k\in\mathbb{Z}^d} |E(X_0X_k)| \le E(X_0^2) + 2\sum_{k\in V_0^1} \|X_kE_{|k|}(X_0)\|_1.$$

Now, using the Kahane-Khintchine inequalities established in section 3, we shall see that the partial sum process $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ is tight in the space $C(\mathcal{A})$. It is sufficient (see [21]) to check the following property:

$$\lim_{\delta\to 0}\limsup_{n\to+\infty} E\left(\sup_{\rho(A,B)<\delta} \big|n^{-d/2}S_n(A) - n^{-d/2}S_n(B)\big|\right) = 0. \qquad (21)$$

Recall that the random field $(X_k)_{k\in\mathbb{Z}^d}$ satisfies the assumption (2) for some $0 < q < 2$. Let A and B be two elements of the class $\mathcal{A}$ and let n be a positive integer. For any k in the set $\{1, ..., n\}^d$, we consider the element $a_k = \lambda(nA \cap R_k) - \lambda(nB \cap R_k)$ of $[-1, 1]$. The Kahane-Khintchine inequality (3) stated in Theorem 1 provides the following

$$\begin{aligned}
\|S_n(A) - S_n(B)\|_{\psi_q} &= \left\| \sum_{k \in \{1,\ldots,n\}^d} a_k \, X_k \right\|_{\psi_q} \\
&\leq K_q(X) \left(\sum_{k \in \{1,\ldots,n\}^d} |a_k| \right)^{1/2} \\
&\leq K_q(X) \left(\sum_{k \in \{1,\ldots,n\}^d} \lambda(n(A \Delta B) \cap R_k) \right)^{1/2} \\
&= K_q(X) \sqrt{\lambda(n(A \Delta B))} \\
&= K_q(X) \, n^{d/2} \, \rho(A, B)
\end{aligned}$$

where

$$K_q(X) = M_1(q) \left(\|X_0\|^2_{\psi_{\beta(q)}} + \sum_{k \in V_0^1} \left\| \sqrt{|X_k E_{|k|}(X_0)|} \right\|^2_{\psi_{\beta(q)}} \right)^{1/2}.$$

That is to say, for any positive integer n and any elements A and B of $\mathcal{A}$,

$$\|n^{-d/2} S_n(A) - n^{-d/2} S_n(B)\|_{\psi_q} \leq K_q(X) \, \rho(A, B). \tag{22}$$

The inequality (22) means that the partial sum process $\{n^{-d/2} S_n(A) \,;\, A \in \mathcal{A}\}$ is lipschitzian uniformly in n. Now, suppose that the metric entropy condition (8) holds. Applying Theorem 11.6 in Ledoux and Talagrand [18], we infer that the sequence $\{n^{-d/2} S_n(A) \,;\, A \in \mathcal{A}\}$ satisfies the following property: for each positive ϵ there exists a positive real δ, depending only on ϵ and on the value of the entropy integral (8), such that

$$E \left(\sup_{\rho(A,B) < \delta} |n^{-d/2} S_n(A) - n^{-d/2} S_n(B)| \right) < \epsilon.$$

The condition (21) is then satisfied and the process $\{n^{-d/2} S_n(A) \,;\, A \in \mathcal{A}\}$ is tight in $C(\mathcal{A})$. The proof of Theorem 2 is complete. $\square$

5.4 Proof of Corollary 2

From Inequality (17) in the proof of Corollary 1, we have

$$\sum_{k\in V_0^1}\left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|_{\psi_{\beta(q)}}^2 \leq C(q)\,\|X_0\|_{\psi_{\beta(q)}}^2 \sum_{k\in V_0^1}\sqrt{\phi_{\infty,1}(|k|)}.$$

Consequently, the condition (10) is more strict than the condition (7) and the proof of Corollary 2 is complete. □

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