# Orthomartingale-coboundary decomposition for stationary random fields

Mohamed EL MACHKOURI and Davide GIRAUDO

Laboratoire de Mathématiques Raphaël Salem
UMR CNRS 6085, Université de Rouen (France)
*mohamed.elmachkouri@univ-rouen.fr*
*davide.giraudo@etu.univ-rouen.fr*

**Abstract**

We provide a new projective condition for a stationary real random field indexed by the lattice $\mathbb{Z}^d$ to be well approximated by an orthomartingale in the sense of Cairoli (1969). Our main result can be viewed as a multidimensional version of the martingale-coboundary decomposition method which the idea goes back to Gordin (1969). It is a powerfull tool for proving limit theorems or large deviations inequalities for stationary random fields when the corresponding result is valid for orthomartingales.

## 1 Introduction and notations

In probability theory, a powerfull approach for proving limit theorems for stationary sequences of random variables is to find a way to approximate such sequences by martingales. This idea goes back to Gordin [12]. It is a powerfull method for proving the central limit theorem (CLT) and the weak invariance principle (WIP) for stationary sequences of dependent random variables satisfying a projective condition (see (1) in Theorem A below). More precisely, let $(X_k)_{k\in\mathbb{Z}}$ be a sequence of real random variables defined on the probability space $(\Omega, \mathcal{F}, \mu)$. We assume that $(X_k)_{k\in\mathbb{Z}}$ is stationary in the sense that its finite-dimensional laws are invariant by translations and we denote by $\nu$ the law of $(X_k)_{k\in\mathbb{Z}}$. Let $f : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ be defined by $f(\omega) = \omega_0$ and $T : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ by $(T\omega)_k = \omega_{k+1}$ for any $\omega$ in $\mathbb{R}^{\mathbb{Z}}$ and any $k$ in $\mathbb{Z}$. Then the sequence $(f \circ T^k)_{k\in\mathbb{Z}}$ defined on the probability space $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \nu)$ is stationary with law $\nu$. So, without loss of generality, we can assume that $X_k = f \circ T^k$ for any $k$ in $\mathbb{Z}$. For any $p \geqslant 1$ and any $\sigma$-algebra $\mathcal{M} \subset \mathcal{F}$, we denote by $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ the space of $p$-integrable real random variables defined on $(\Omega, \mathcal{M}, \mu)$ and we consider the norm $\|.\|_p$ defined by $\|Z\|_p^p = \int_\Omega |Z(\omega)|^p d\mu(\omega)$ for any $Z$ in $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$. We denote also by $\mathbb{L}^p(\Omega, \mathcal{F}, \mu) \ominus \mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ the space of all $Z$ in $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$ such that $\mathbb{E}(Z \mid \mathcal{M}) = 0$ a.s.

**Theorem A (Gordin, 1969)** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T : \Omega \to \Omega$ be a measurable function such that $\mu = T\mu$. Let also $p \geqslant 1$ and $\mathcal{M} \subset \mathcal{F}$ be a $\sigma$-algebra such that $\mathcal{M} \subset T^{-1}\mathcal{M}$. If $f$ belongs to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{i\in\mathbb{Z}} T^{-i}\mathcal{M}, \mu)$ such that*

$$\sum_{k\geqslant 0} \left\| \mathbb{E}\left(f \mid T^k\mathcal{M}\right)\right\|_p < \infty \tag{1}$$

*then there exist $m$ in $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T\mathcal{M}, \mu)$ and $g$ in $\mathbb{L}^p(\Omega, T\mathcal{M}, \mu)$ such that*

$$f = m + g - g \circ T. \tag{2}$$

The term $g - g \circ T$ in (2) is called a coboundary and equation (2) is called the martingale-coboundary decomposition of $f$. Moreover, the stationary sequence $(m \circ T^i)_{i\in\mathbb{Z}}$ is a martingale-difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}}$ (see Definition 1 below) and for any positive integer $n$,

$$S_n(f) = S_n(m) + g - g \circ T^n \tag{3}$$

where $S_n(h) = \sum_{i=0}^{n-1} h \circ T^i$ for any function $h : \Omega \to \mathbb{R}$. Combining (3) with the Billingsley-Ibragimov CLT for martingales (see [3] or [15]), one obtain the CLT for the stationary sequence $(f \circ T^k)_{k\in\mathbb{Z}}$ when the projective condition (1) holds. Similarly, combining (3) with the WIP for martingales (see [4]), we derive the WIP for the stationary sequence $(f \circ T^k)_{k\in\mathbb{Z}}$. Thus, Gordin's method provides a sufficient condition for proving limit theorems for stationary sequences when such a limit theorem holds for martingale-difference sequences. Our aim in this work is to provide an extension of Theorem A for random fields indexed by the lattice $\mathbb{Z}^d$ where $d$ is a positive integer (see Theorem 1).

# 2 Main results

**Definition 1** *We say that a sequence $(X_k)_{k\in\mathbb{Z}}$ of real random variables defined on a probability space $(\Omega, \mathcal{F}, \mu)$ is a martingale-difference (MD) sequence if there exists a filtration $(\mathcal{G}_k)_{k\in\mathbb{Z}}$ such that $\mathcal{G}_k \subset \mathcal{G}_{k+1} \subset \mathcal{F}$ and $X_k$ belongs to $\mathbb{L}^1(\Omega, \mathcal{G}_k, \mu) \ominus \mathbb{L}^1(\Omega, \mathcal{G}_{k-1}, \mu)$ for any $k$ in $\mathbb{Z}$.*

The concept of MD sequences can be extended to the random field setting. One can refer for example to Basu and Dorea [1] or Nahapetian [20] where MD random fields are defined in two differents ways and limit theorems are obtained. In this paper, we are interested by orthomartingale-difference random fields in the sense of Cairoli [5]. A good introduction to this concept is done in the book by Khoshnevisan [16]. Let $d$ be a positive integer. We denote by $\langle d\rangle$ the set $\{1, ..., d\}$. For any $s = (s_1, ..., s_d)$ and any $t = (t_1, ..., t_d)$ in $\mathbb{Z}^d$, we write $s \preccurlyeq t$ (resp. $s \prec t$, $s \succcurlyeq t$ and $s \succ t$) if and only if $s_k \leqslant t_k$ (resp. $s_k < t_k$, $s_k \geqslant t_k$ and $s_k > t_k$) for any $k$ in $\langle d\rangle$ and we denote also $s \wedge t = (s_1 \wedge t_1, ..., s_d \wedge t_d)$.

**Definition 2** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. A family $(\mathcal{G}_i)_{i\in\mathbb{Z}^d}$ of $\sigma$-algebras is a commuting filtration if $\mathcal{G}_i \subset \mathcal{G}_j \subset \mathcal{F}$ for any $i$ and $j$ in $\mathbb{Z}^d$ such that $i \preccurlyeq j$ and*

$$\mathbb{E}\left(\mathbb{E}\left(Z \mid \mathcal{G}_s\right) \mid \mathcal{G}_t\right) = \mathbb{E}\left(Z \mid \mathcal{G}_{s\wedge t}\right) \quad a.s.$$

*for any $s$ and $t$ in $\mathbb{Z}^d$ and any bounded random variable $Z$.*

Definition 2 is known as the "F4 condition".

**Definition 3** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. A random field $(X_k)_{k\in\mathbb{Z}^d}$ is an orthomartingale-difference (OMD) random field if there exists a commuting filtration $(\mathcal{G}_i)_{i\in\mathbb{Z}^d}$ such that $X_k$ belongs to $\mathbb{L}^1(\Omega, \mathcal{G}_k, \mu) \ominus \mathbb{L}^1(\Omega, \mathcal{G}_l, \mu)$ for any $l \not\succcurlyeq k$ and $k$ in $\mathbb{Z}^d$.*

**Remark 1.** Let $k$ be fixed in $\mathbb{Z}^d$ and $S_k = \sum_{0\prec i\preccurlyeq k} X_i$ where $(X_i)_{i\in\mathbb{Z}^d}$ is an OMD random field with respect to a commuting filtration $(\mathcal{G}_i)_{i\in\mathbb{Z}^d}$. Then $S_k$ belongs to $\mathbb{L}^1(\Omega, \mathcal{G}_k, \mu)$ and $\mathbb{E}\left(S_k \mid \mathcal{G}_l\right) = S_l$ for any $l \preccurlyeq k$. We say that $(S_k)_{k\in\mathbb{Z}^d}$ is an orthomartingale (OM) random field.

Arguing as above, without loss of generality, every stationary real random field $(X_k)_{k\in\mathbb{Z}^d}$ can be written as $(f \circ T^k)_{k\in\mathbb{Z}^d}$ where $f : \Omega \to \mathbb{R}$ is a measurable function and for any $k$ in $\mathbb{Z}^d$, $T^k : \Omega \to \Omega$ is a measure-preserving operator satisfying $T^i \circ T^j = T^{i+j}$ for any $i$ and $j$ in $\mathbb{Z}^d$. For any $s$ in $\langle d\rangle$, we denote $T_s = T^{e_s}$ where $e_s = (e_s^{(1)}, \dots, e_s^{(d)})$ is the unique element of $\mathbb{Z}^d$ such that $e_s^{(s)} = 1$ and $e_s^{(i)} = 0$ for any $i$ in $\langle d\rangle\backslash\{s\}$ and $U_s$ is the operator defined by $U_s h = h \circ T_s$ for any function $h : \Omega \to \mathbb{R}$. We define also $U_J$ as the product operator $\Pi_{s\in J} U_s$ for any $\emptyset \subsetneq J \subset \langle d\rangle$ and we write simply $U$ for $U_{\langle d\rangle} = U_1 \circ U_2 \circ \dots \circ U_d$. For any $\emptyset \subsetneq J \subset \langle d\rangle$, we denote also by $|J|$ the number of elements in $J$ and by $J^c$ the set $\langle d\rangle\backslash J$. Finally, the set of nonegative integers will be denoted by $\mathbb{N}$. The main result of this paper is the following.

**Theorem 1** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T^l : \Omega \to \Omega$ be a measure-preserving operator for any $l$ in $\mathbb{Z}^d$ such that $T^i \circ T^j = T^{i+j}$ for any $i$ and $j$ in $\mathbb{Z}^d$. Let $p \geqslant 1$ and let $\mathcal{M} \subset \mathcal{F}$ be a $\sigma$-algebra such that $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^d}$ is a commuting filtration. If $f$ belongs to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{k\in\mathbb{N}^d} T^k\mathcal{M}, \mu)$ and*

$$\sum_{k\in\mathbb{N}^d} \left\|\mathbb{E}\left(f \mid T^k\mathcal{M}\right)\right\|_p < \infty \tag{4}$$

*then $f$ admits the decomposition*

$$f = m + \sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle} \prod_{s\in J}(I - U_s)m_J + \prod_{s=1}^{d}(I - U_s)g, \tag{5}$$

*where $m$, $g$ and $m_J$ belong to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$, $\mathbb{L}^p(\Omega, \prod_{s=1}^d T_s\mathcal{M}, \mu)$ and $\mathbb{L}^p(\Omega, \prod_{s\in J} T_s\mathcal{M}, \mu)$ respectively and $(U^i m)_{i\in\mathbb{Z}^d}$ and $(U^i_{J^c} m_J)_{i\in\mathbb{Z}^{d-|J|}}$ are OMD random fields for $\emptyset \subsetneq J \subsetneq \langle d\rangle$.*

**Remark 2**. One can notice that condition (4) is exactly Gordin's condition (1) when $d = 1$. It is well known that condition (1) is not necessary for $f$ to admit a martingale-coboundary decomposition. In fact, in dimension $d = 1$, a necessary and sufficient condition for $f$ to admit the martingale-coboundary decomposition (2) is the convergence in $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ for $p \geqslant 1$ of the series $\sum_{k\geqslant 0} \mathbb{E}\left(U^k f \mid \mathcal{M}\right)$ (see [22], Theorem 2, condition (7)). So, let $(\delta_j)_{j\geqslant 0}$ be a decreasing sequence of real numbers such that $\sum_{j\geqslant 0} \delta_j^2 < \infty$ and define $a_{2j} = \delta_j$ and $a_{2j+1} = -\delta_j$ for any $j \geqslant 0$. If $(\varepsilon_i)_{i\in\mathbb{Z}}$ is a sequence of iid real random variables with zero-mean and unit variance and $f \circ T^k = \sum_{i\geqslant 0} a_j \varepsilon_{k-j}$ for any $k$ in $\mathbb{Z}$ (we say that $(f \circ T^k)_{k\geqslant 1}$ is a linear process) then $\sum_{k\geqslant 1} \mathbb{E}(U^k f|\mathcal{M})$ converges in $\mathbb{L}^2(\Omega, \mathcal{F}, \mu)$ while the decay of the sequence $\left(\sum_{j\geqslant k} a_j^2\right)_{k\geqslant 1}$ can be arbitrarily slow such that the series $\sum_{k\geqslant 1}\left\|\mathbb{E}(U^k f|\mathcal{M})\right\|_2$ does not converge. That is, $f=\sum_{i\geqslant 0} a_i\varepsilon_{-i}$ is a function which admits the martingale-coboundary decomposition (2) even if Gordin's condition (1) does not hold. Finally, it will be interesting to investigate a necessary and sufficient condition for the orthomartingale-coboundary decomposition (5) when $d\geqslant 2$. This question is still an open problem and will be considered elsewhere.

**Remark 3**. If $d=2$ then (5) reduces to

$$f=m+(I-U_1)m_1+(I-U_2)m_2+(I-U_1)(I-U_2)g,$$

where $m$, $m_1$, $m_2$ and $g$ belong to $\mathbb{L}^p(\Omega,\mathcal{M},\mu)$ such that $(U^i m)_{i\in\mathbb{Z}}$ is an OMD random field and $(U_2^k m_1)_{k\in\mathbb{Z}}$ and $(U_1^k m_2)_{k\in\mathbb{Z}}$ are MD sequences. If $d=3$ then (5) becomes

$$\begin{aligned} f&=m+(I-U_1)m_1+(I-U_2)m_2+(I-U_3)m_3\\ &\quad+(I-U_1)(I-U_2)m_{\{1,2\}}+(I-U_1)(I-U_3)m_{\{1,3\}}+(I-U_2)(I-U_3)m_{\{2,3\}}\\ &\quad+(I-U_1)(I-U_2)(I-U_3)g \end{aligned}$$

where $m$, $m_1$, $m_2$, $m_3$, $m_{\{1,2\}}$, $m_{\{1,3\}}$, $m_{\{2,3\}}$ and $g$ belong to $\mathbb{L}^p(\Omega,\mathcal{M},\mu)$ such that $(U^i m)_{i\in\mathbb{Z}^3}$, $(U^i_{\{2,3\}}m_1)_{i\in\mathbb{Z}^2}$, $(U^i_{\{1,3\}}m_2)_{i\in\mathbb{Z}^2}$ and $(U^i_{\{1,2\}}m_3)_{i\in\mathbb{Z}^2}$ are OMD random fields and $(U_1^k m_{\{2,3\}})_{k\in\mathbb{Z}}$, $(U_2^k m_{\{1,3\}})_{k\in\mathbb{Z}}$ and $(U_3^k m_{\{1,2\}})_{k\in\mathbb{Z}}$ are MD sequences.

**Remark 4**. A decomposition similar to (5) was obtained by Gordin [13] but with reversed orthomartingales and under an assumption on the so-called Poisson equation.

**Proposition 1** *Let $(X_i)_{i\in\mathbb{Z}^d}$ be an OMD random field. There exists a positive constant $\kappa$ such that for any $p\geqslant 2$ and any $n$ in $\mathbb{N}^d$,*

$$\left\|\sum_{0\preccurlyeq k\preccurlyeq n} X_k\right\|_p\leqslant \kappa p^{d/2}\left(\sum_{0\preccurlyeq k\preccurlyeq n}\|X_k\|_p^2\right)^{1/2} \tag{6}$$

*and the constant $p^{d/2}$ in (6) is optimal in the following sense: there exists a stationary OMD random field $(Z_k)_{k\in\mathbb{Z}^d}$ with $\|Z_0\|_\infty=1$ and a positive constant $\kappa$ such that for any $p\geqslant 2$*

$$\inf\left\{C>0\ ;\ \left\|\sum_{0\preccurlyeq k\preccurlyeq n} Z_k\right\|_p\leqslant C\left(\sum_{0\preccurlyeq k\preccurlyeq n}\|Z_k\|_p^2\right)^{1/2}\ \forall n\in\mathbb{N}^d\right\}\geqslant \kappa p^{d/2}. \tag{7}$$

Combining Proposition 1 and Theorem 1, we obtain the following result.

**Proposition 2** *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary real random field defined on a probability space $(\Omega,\mathcal{F},\mu)$ and $(\mathcal{F}_i)_{i\in\mathbb{Z}^d}$ be a commuting filtration such that $X_i$ is $\mathcal{F}_i$-measurable for each $i$ in $\mathbb{Z}^d$. If there exists $p\geqslant 2$ such that $X_0$ belongs to $\mathbb{L}^p(\Omega,\mathcal{F}_0,\mu)\ominus\mathbb{L}^p(\Omega,\cap_{k\in\mathbb{N}^d}\mathcal{F}_{-k},\mu)$ and*

$$\sum_{k\in\mathbb{N}^d}\left\|\mathbb{E}\left(X_0\mid\mathcal{F}_{-k}\right)\right\|_p<\infty \tag{8}$$

*then for any $n = (n_1, ..., n_d)$ in $\mathbb{N}^d$,*

$$\left\| \sum_{0 \preccurlyeq k \preccurlyeq n} X_k \right\|_p \leqslant C_d\, p^{d/2}\, |n|^{d/2} \sum_{k \in \mathbb{N}^d} \left\| \mathbb{E}\left( X_0 \mid \mathcal{F}_{-k} \right) \right\|_p \tag{9}$$

*where $|n| = \prod_{i=1}^d n_i$ and $C_d$ is a positive constant depending only on $d$.*

**Remark 5**. A Young function $\psi$ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies $\lim_{t\to\infty} \psi(t) = \infty$ and $\psi(0) = 0$. We denote by $\mathbb{L}_\psi(\Omega, \mathcal{F}, \mu)$ the Orlicz space associated to the Young function $\psi$, that is the space of real random variables $Z$ defined on $(\Omega, \mathcal{F}, \mu)$ such that $\mathbb{E}\left(\psi(|Z|/c)\right) < \infty$ for some $c > 0$. The Orlicz space $\mathbb{L}_\psi(\Omega, \mathcal{F}, \mu)$ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable $Z$ by $\|Z\|_\psi = \inf\{\, c > 0\,;\, \mathbb{E}[\psi(|Z|/c)] \leqslant 1\,\}$ is a Banach space. For any $p \geqslant 1$, if $\varphi_p$ is the function defined by $\varphi_p(x) = x^p$ for any nonegative real $x$ then $\varphi_p$ is a Young function and the Orlicz space $\mathbb{L}_{\varphi_p}(\Omega, \mathcal{F}, \mu)$ reduces to $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$ equipped with the norm $\|.\|_p$. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [18]. Combining (9) and Lemma 4 in [11], we obtain Kahane-Khintchine inequalities: for any $0 < q < 2/d$, there exists a positive constant $C$ depending only on $d$ and $q$ such that for any $n$ in $\mathbb{N}^d$,

$$\left\| \sum_{0 \preccurlyeq k \preccurlyeq n} X_k \right\|_{\psi_q} \leqslant C\, |n|^{d/2} \sum_{k \in \mathbb{N}^d} \left\| \mathbb{E}\left( X_0 \mid \mathcal{F}_{-k} \right) \right\|_{\psi_{\beta(q)}} \tag{10}$$

where $\beta(q) = 2q/(2 - dq)$ and $\psi_\alpha$ is the Young function defined for any $x \in \mathbb{R}^+$ by

$$\psi_\alpha(x) = \exp((x + h_\alpha)^\alpha) - \exp(h_\alpha^\alpha) \quad \text{with} \quad h_\alpha = ((1-\alpha)/\alpha)^{1/\alpha}\, \mathbb{1}_{\{0<\alpha<1\}}$$

for any real $\alpha > 0$. Using Markov inequality and the definition of the Luxembourg norm, we derive the following large deviations inequalities: for any $0 < q < 2/d$, there exists a positive constant $C$ depending only on $d$ and $q$ such that for any $n$ in $\mathbb{N}^d$ and any positive real $x$,

$$\mu\left( \left| \sum_{0 \preccurlyeq k \preccurlyeq n} X_k \right| > x \right) \leqslant (1 + e^{h_q^q}) \exp\left( - \left( \frac{x}{C\, |n|^{d/2} \sum_{k \in \mathbb{N}^d} \left\| \mathbb{E}\left( X_0 \mid \mathcal{F}_{-k} \right) \right\|_{\psi_{\beta(q)}}} + h_q \right)^q \right). \tag{11}$$

Finally, one can check that (10) and (11) still hold for $q = 2/d$ if $X_0$ is bounded.

**Proposition 3** *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary real random field defined on a probability space $(\Omega, \mathcal{F}, \mu)$ and $(\mathcal{F}_i)_{i\in\mathbb{Z}^d}$ be a commuting filtration such that $X_i$ is $\mathcal{F}_i$-measurable for each $i$ in $\mathbb{Z}^d$. If $X_0$ belongs to $\mathbb{L}^2(\Omega, \mathcal{F}_0, \mu) \ominus \mathbb{L}^2(\Omega, \cap_{k\in\mathbb{N}^d} \mathcal{F}_{-k}, \mu)$ and $\sum_{k\in\mathbb{N}^d} \left\| \mathbb{E}\left( X_0 \mid \mathcal{F}_{-k} \right) \right\|_2 < \infty$ then $\sum_{k\in\mathbb{Z}^d} \left| \mathbb{E}\left( X_0 X_k \right) \right| < \infty$ and*

$$\lim_{|n|\to+\infty} |n|^{-1} \mathbb{E}\left( \left( \sum_{0 \preccurlyeq k \preccurlyeq n} X_k \right)^2 \right) = \sum_{k\in\mathbb{Z}^d} \mathbb{E}\left( X_0 X_k \right)$$

*where $|n| \to +\infty$ means that $\min_{1\leqslant i\leqslant d} n_i \to +\infty$.*

Now, we are able to investigate the WIP for random fields. Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary real random field defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Let also $\mathcal{A}$ be a collection of Borel subsets of $[0,1]^d$ and consider the process $\{S_n(A)\,;\, A \in \mathcal{A}\}$ defined by

$$S_n(A) = \sum_{i\in\langle n\rangle^d} \lambda(nA \cap R_i) X_i \tag{12}$$

where $R_i = ]i_1 - 1, i_1] \times ... \times ]i_d - 1, i_d]$ is the unit cube with upper corner at $i = (i_1, ..., i_d)$ in $\langle n\rangle^d$ and $\lambda$ is the Lebesgue measure on $\mathbb{R}^d$. The collection $\mathcal{A}$ is equipped with the pseudo-metric $\rho$ defined by $\rho(A, B) = \sqrt{\lambda(A\Delta B)}$ for any $A$ and $B$ in $\mathcal{A}$. Let $\varepsilon > 0$ and let $H(\mathcal{A}, \rho, \varepsilon)$ be the logarithm of the smallest number $N(\mathcal{A}, \rho, \varepsilon)$ of open balls of radius $\varepsilon$ with respect to $\rho$ which form a covering of $\mathcal{A}$. The function $H(\mathcal{A}, \rho, .)$ is called the metric entropy of the class $\mathcal{A}$ and allows us to control the size of the collection $\mathcal{A}$. Let $(\mathcal{C}(\mathcal{A}), \|.\|_{\mathcal{A}})$ be the Banach space of continuous real functions on $\mathcal{A}$ equipped with the uniform norm $\|.\|_{\mathcal{A}}$ defined by $\|f\|_{\mathcal{A}} = \sup_{A\in\mathcal{A}} |f(A)|$. A standard Brownian motion indexed by $\mathcal{A}$ is a mean zero Gaussian process $W$ with sample paths in $\mathcal{C}(\mathcal{A})$ such that $\mathrm{Cov}(W(A), W(B)) = \lambda(A\cap B)$ and we know from Dudley [8] that such a process is well defined if $\int_0^1 \sqrt{H(\mathcal{A}, \rho, \varepsilon)}\, d\varepsilon < \infty$. We say that the WIP holds if the sequence of processes $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ converges in distribution in $\mathcal{C}(\mathcal{A})$ to a mixture of $\mathcal{A}$-indexed Brownian motion. The first weak convergence results for $\mathcal{Q}_d$-indexed partial sum processes were established for i.i.d. real random fields where $\mathcal{Q}_d$ is the collection $\{[0, t]\,;\, t \in [0, 1]^d\}$ of lower-left quadrants in $[0, 1]^d$. They were proved by Wichura [25] under a finite variance condition and earlier by Kuelbs [19] under additional moment restrictions. If $d = 1$, these results coincide with the original invariance principle of Donsker [7]. Many others WIP have been established for dependent random fields indexed by large classes of sets. One can refer for example to [6], [9], [10] or [11]. In the sequel, we are going to apply Theorem 1 in order to establish a WIP (Theorem 2) for $\mathcal{Q}_d$-indexed partial sum dependent random fields. Let $n$ be a positive integer. For simplicity, we denote $S_n(t) = S_n([0, t])$ for any $[0, t]$ in $\mathcal{Q}_d$. More precisely, for any $t$ in $[0, 1]^d$,

$$S_n(t) = \sum_{i\in\langle n\rangle^d} \lambda([0, nt] \cap R_i) X_i \tag{13}$$

Recall that the standard $d$-parameter Brownian sheet on $[0, 1]^d$ denoted by $\mathbb{B} = (\mathbb{B}_t)_{t\in[0,1]^d}$ is a mean-zero Gaussian random field such that $\mathrm{Cov}(\mathbb{B}_s, \mathbb{B}_t) = \prod_{i=1}^d s_i \wedge t_i$ for any $s = (s_1, ..., s_d)$ and $t = (t_1, ..., t_d)$ in $[0, 1]^d$. Since the CLT does not hold for general OMD random fields (see [24], example 1, page 12), we restrict ourselves to the case of a filtration generated by iid random variables which is necessarily a commuting filtration (see Proposition 8.1 in [24]).

**Theorem 2** *Let $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ be an iid real random field defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Denote by $(\mathcal{F}_i)_{i\in\mathbb{Z}^d}$ the commuting filtration where $\mathcal{F}_i$ is the $\sigma$-algebra generated by $\varepsilon_j$ for $j \preccurlyeq i$ and $i$ in $\mathbb{Z}^d$. Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary real random field such that $X_0$ belongs to $\mathbb{L}^2(\Omega, \mathcal{F}_0, \mu) \ominus \mathbb{L}^2(\Omega, \cap_{k\in\mathbb{N}^d}\mathcal{F}_{-k}, \mu)$ and (8) holds for $p = 2$. Then the sequence of processes $\{n^{-d/2}S_n(t)\,;\, t \in [0, 1]^d\}$ converges in distribution in $\mathcal{C}(\mathcal{Q}_d)$ to $\sqrt{\eta}\mathbb{B}$ where $\mathbb{B}$ is a standard $d$-Brownian sheet and $\eta = \sum_{k\in\mathbb{Z}^d} \mathbb{E}\,(X_0X_k)$.*

**Remark 6**. El Machkouri et al. [11] and Wang and Woodroofe [24] obtained also a WIP for random fields $(X_k)_{k\in\mathbb{Z}^d}$ which can be expressed as a functional of iid real random variables but under the more restrictive condition that $X_0$ belongs to $\mathbb{L}^p(\Omega,\mathcal{F},\mu)$ with $p>2$. In a recent work, Wang and Volný [23] obtained the WIP for $p=2$ under a multidimensional version of the so-called Hannan's condition for time series. Their condition is less restrictive than (8) but condition (8) gives also an orthomartingale approximation for the considered random field which is of independent interest (see Theorem 1). In particular, (8) provides not only a WIP but also large deviations inequalities (see Proposition 2 and Remark 5).

**Proposition 4** *Let $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ be an iid real random field defined on a probability space $(\Omega,\mathcal{F},\mu)$ such that $\varepsilon_0$ has zero mean and belongs to $\mathbb{L}^p(\Omega,\mathcal{F},\mu)$ for some $p\geqslant 2$. Consider the linear random field $(X_k)_{k\in\mathbb{Z}^d}$ defined for any $k$ in $\mathbb{Z}^d$ by $X_k=\sum_{j\in\mathbb{N}^d}a_j\varepsilon_{k-j}$ where $(a_j)_{j\in\mathbb{N}^d}$ is a family of real numbers satisfying $\sum_{j\in\mathbb{N}^d}a_j^2<\infty$. Then the condition (8) holds if and only if*

$$\sum_{k\in\mathbb{N}^d}\sqrt{\sum_{j\succcurlyeq k}a_j^2}<\infty. \tag{14}$$

**Remark 7**. Proposition 4 ensures that the conclusion of Theorem 2 still hold for linear random fields with iid innovations under assumption (14). Let $(X_k)_{k\in\mathbb{Z}^d}$ be a linear random field defined as in Proposition 4. In [24], Wang and Woodroofe obtained a WIP for $(X_k)_{k\in\mathbb{Z}^d}$ under a weaker condition than (14) but again with the additional assumption that $\varepsilon_0$ belongs to $\mathbb{L}^p(\Omega,\mathcal{F},\mu)$ with $p>2$. In [2], Biermé and Durieu obtained also a WIP under the so-called stability condition $\sum_{k\in\mathbb{N}^d}|a_k|<\infty$ which is less restrictive than (14). In fact, let $a_k:=k_1^{-\alpha}\ldots k_d^{-\alpha}$ for $1<\alpha<3/2$ and $k=(k_1,...,k_d)$ in $\mathbb{N}^d$ then the linear process $(X_k)_{k\in\mathbb{Z}^d}$ satisfies the stability condition but (14) does not hold. Indeed, (14) is equivalent to the convergence of the series $\sum_{j\geqslant 0}\sqrt{\sum_{l\geqslant j}l^{-2\alpha}}$ and this last one is not convergent since there exists a positive constant $C_\alpha$ such that $\sum_{l\geqslant j}l^{-2\alpha}\geqslant C_\alpha\,j^{1-2\alpha}$ for each $j\geqslant 0$. Nevertheless, (14) provides the orthomartingale-coboundary decomposition (5) while it is not the case for the stability condition even when $d=1$. In fact, let $(b_k)_{k\geqslant 0}$ be a decreasing sequence of real numbers such that $b_k$ goes to zero as $k$ goes to infinity and $\sum_{k\geqslant 0}b_k^2=+\infty$ and let $a_k=b_k-b_{k+1}$ for any $k\geqslant 0$. So, we have $\sum_{k\geqslant 0}|a_k|<\infty$ but if $\mathcal{F}_0$ is the $\sigma$-algebra generated by all $\varepsilon_j$ for $j\preccurlyeq 0$ then $\left\|\sum_{i=1}^N\mathbb{E}(X_i|\mathcal{F}_0)\right\|_2^2=\sum_{l\geqslant 0}(b_{l+1}-b_{l+N})^2$ for any positive integer $N$. Since, for any positive integer $L$, we have

$$\sup_N\sum_{l=0}^{L}(b_{l+1}-b_{l+N})^2\geqslant\sum_{l=0}^{L}b_{l+1}^2,$$

we obtain $\sup_N\left\|\sum_{i=1}^N\mathbb{E}(X_i|\mathcal{M})\right\|_2^2=+\infty$. Consequently, the martingale-coboundary decomposition (2) does not hold.

We now provide an application of Theorem 1 to the WIP in Hölder spaces. We consider for $0<\gamma\leqslant 1$ the space $\mathbb{H}_\gamma([0,1]^d)$ as the space of all continuous functions $g$ for which there exists a constant $K$ such that $|g(s) - g(t)| \leqslant K\|s-t\|^\gamma$ for each $s$ and $t$ in $[0,1]^d$ where $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^d$. We endow this function space with the norm $\|g\| := |g(0)| + \sup_{s,t\in[0,1]^d, s\neq t} |g(t)-g(s)|/\|t-s\|^\gamma$ and we consider the partial sum process $(S_n(t))_{t\in[0,1]^d}$ defined by (13) as an element of $\mathbb{H}_\gamma([0,1]^d)$.

**Theorem 3** *If the assumptions of Theorem 2 hold with $p > 4\times(\log_2(4d/(4d-3)))^{-1}$ then the sequence of processes $\{n^{-d/2}S_n(t)\,;\, t\in[0,1]^d\}_{n\geqslant 1}$ converges in distribution in $\mathbb{H}_\gamma([0,1]^d)$ to $\sqrt{\eta}\mathbb{B}$ for each $\gamma < 1/2 - d/p$ where $\mathbb{B}$ is a standard d-Brownian sheet and $\eta = \sum_{k\in\mathbb{Z}^d}\mathbb{E}\left(X_0X_k\right)$.*

**Remark 8**. In [21], a necessary and sufficient condition was obtained for iid random fields to satisfy the WIP in Hölder spaces. Our result provides a sufficient condition for stationary real random fields which can be expressed as a functional of iid real random variables.

# 3 Proofs

In this section, the letter $\kappa$ will denote a universal positive constant which the value may change from line to line. The proof of Theorem 1 will be done by induction on $d$. We shall need the following lemma.

**Lemma 1** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $d$ be a positive integer and $T^l : \Omega \to \Omega$ be a measure-preserving operator for any $l$ in $\mathbb{Z}^{d+1}$ such that $T^i \circ T^j = T^{i+j}$ for any $i$ and $j$ in $\mathbb{Z}^{d+1}$. Let $p \geqslant 1$ and $\mathcal{M} \subset \mathcal{F}$ be a $\sigma$-algebra such that $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^{d+1}}$ is a commuting filtration. Assume that $f$ belongs to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{k\in\mathbb{N}^{d+1}} T^k\mathcal{M}, \mu)$ and*

$$\sum_{k\in\mathbb{N}^{d+1}} \left\|\mathbb{E}\left(f \mid T^k\mathcal{M}\right)\right\|_p < \infty. \tag{15}$$

*Then there exist $M \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_{d+1}\mathcal{M}, \mu)$ and $G \in \mathbb{L}^p(\Omega, T_{d+1}\mathcal{M}, \mu)$ such that*

$$f = M + G - G\circ T_{d+1} \tag{16}$$

*and*

$$\sum_{k\in\mathbb{N}^d} \left\|\mathbb{E}\left(M \mid T^{(k,0)}\mathcal{M}\right)\right\|_p + \left\|\mathbb{E}\left(G \mid T^{(k,0)}\mathcal{M}\right)\right\|_p < \infty. \tag{17}$$

*Proof of Lemma 1.* First, the decomposition (16) is a direct consequence of Theorem $A$ (see section 1). Moreover, a carefull reading of the proof of Theorem $A$ (see Volný [22]) ensures the following expressions of $M$ and $G$:

$$M = \sum_{j\geqslant 0} \mathbb{E}[U_{d+1}^j f \mid \mathcal{M}] - \mathbb{E}[U_{d+1}^j f \mid T_{d+1}\mathcal{M}] \quad \text{and} \quad G = \sum_{j\geqslant 0} \mathbb{E}[U_{d+1}^j f \mid T_{d+1}\mathcal{M}]. \tag{18}$$

Let $k = (k_1, ..., k_d)$ be fixed in $\mathbb{N}^d$. Since

$$\mathbb{E}\left(M \mid T^{(k,0)}\mathcal{M}\right) = \sum_{j\geqslant 0} \mathbb{E}[U_{d+1}^j f \mid T^{(k,0)}\mathcal{M}] - \sum_{j\geqslant 0} \mathbb{E}[U_{d+1}^j f \mid T^{(k,1)}\mathcal{M}],$$

we derive

$$\left\|\mathbb{E}\left(M \mid T^{(k,0)}\mathcal{M}\right)\right\|_p \leqslant 2\sum_{j\geqslant 0}\left\|\mathbb{E}[U_{d+1}^j f \mid T^{(k,0)}\mathcal{M}]\right\|_p = 2\sum_{j\geqslant 0}\left\|\mathbb{E}[f \mid T^{(k,j)}\mathcal{M}]\right\|_p .$$

Finally, using (15), we obtain

$$\sum_{k\in\mathbb{N}^d}\left\|\mathbb{E}\left(M \mid T^{(k,0)}\mathcal{M}\right)\right\|_p \leqslant 2\sum_{k\in\mathbb{N}^d}\sum_{j\geqslant 0}\left\|\mathbb{E}[f \mid T^{(k,j)}\mathcal{M}]\right\|_p < \infty. \tag{19}$$

Similarly, we have also $\sum_{k\in\mathbb{N}^d}\left\|\mathbb{E}\left(G \mid T^{(k,0)}\mathcal{M}\right)\right\|_p < \infty$. The proof of Lemma 1 is complete.

*Proof of Theorem 1.* We are going to prove Theorem 1 by induction on the dimension $d$. First, for $d = 1$, the result reduces to Gordin's martingale-difference coboundary decomposition (see Theorem A above). Let $d$ be a positive integer. We assume that our result is true for $d$ and we have to show that it is true for $d + 1$. We thus consider a measure-preserving operator $T^l : \Omega \to \Omega$ for any $l$ in $\mathbb{Z}^{d+1}$ such that $T^i \circ T^j = T^{i+j}$ for any $i$ and $j$ in $\mathbb{Z}^{d+1}$. Let $p \geqslant 1$ and let $\mathcal{M} \subset \mathcal{F}$ be a $\sigma$-algebra such that $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^{d+1}}$ is a commuting filtration. Assume that $f$ belongs to $\mathbb{L}^p(\Omega,\mathcal{M},\mu)$ and satisfies (15). By Lemma 1, there exist $M \in \mathbb{L}^p(\Omega,\mathcal{M},\mu)\ominus\mathbb{L}^p(\Omega,T_{d+1}\mathcal{M},\mu)$ and $G \in \mathbb{L}^p(\Omega,T_{d+1}\mathcal{M},\mu)$ such that (16) and (17) hold. So, by the induction hypothesis, we have

$$M = m' + \sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle}\prod_{s\in J}(I-U_s)m'_J + \prod_{s=1}^{d}(I-U_s)g', \tag{20}$$

$$G = m'' + \sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle}\prod_{s\in J}(I-U_s)m''_J + \prod_{s=1}^{d}(I-U_s)g'' \tag{21}$$

where

- $m'$ and $m''$ belong to $\mathbb{L}^p(\Omega,\mathcal{M},\mu)\ominus\mathbb{L}^p(\Omega,T_i\mathcal{M},\mu)$ for each $i$ in $\langle d\rangle$.
- $m'_J$ and $m''_J$ belong to $\mathbb{L}^p(\Omega,\prod_{s\in J}T_s\mathcal{M},\mu)\ominus\mathbb{L}^p(\Omega,T_i\prod_{s\in J}T_s\mathcal{M},\mu)$ for each $i$ in $\langle d\rangle\setminus J$.
- $g'$ and $g''$ belong to $\mathbb{L}^p(\Omega,\prod_{s=1}^{d}T_s\mathcal{M},\mu)$;

Since $\mathbb{E}[M \mid T_{d+1}\mathcal{M}] = 0$ and using (20), we derive

$$\mathbb{E}[m' \mid T_{d+1}\mathcal{M}] = -\sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle}\mathbb{E}\left[\prod_{s\in J}(I-U_s)m'_J \mid T_{d+1}\mathcal{M}\right] - \mathbb{E}\left[\prod_{s=1}^{d}(I-U_s)g' \mid T_{d+1}\mathcal{M}\right]. \tag{22}$$

Let $\emptyset \subsetneq J \subsetneq \langle d \rangle$ be fixed and recall that $|J|$ is the number of elements of $J$. So, if $J = \{j_1, \ldots, j_{|J|}\}$ then

$$\begin{aligned}\mathbb{E}\left[\prod_{s\in J}(I-U_s)m'_J \mid T_{d+1}\mathcal{M}\right] &= \mathbb{E}\left[\sum_{i=0}^{|J|}(-1)^i\prod_{s=1}^{i}U_{j_s}m'_J \mid T_{d+1}\mathcal{M}\right]\\ &= \sum_{i=0}^{|J|}(-1)^i\mathbb{E}\left[\prod_{s=1}^{i}U_{j_s}m'_J \mid T_{d+1}\mathcal{M}\right]\\ &= \sum_{i=0}^{|J|}(-1)^i\prod_{s=1}^{i}U_s\mathbb{E}\left[m'_J \mid \prod_{s=1}^{i}T_{j_s}T_{d+1}\mathcal{M}\right]\end{aligned}$$

where we used the convention $\prod_{s=1}^{0}U_s = I$ and the property $\mathbb{E}[U_s h \mid \mathcal{G}] = U_s\mathbb{E}[h \mid T_s\mathcal{G}]$ for any $s$ in $\langle d \rangle$, any $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ and any integrable function $h$. Let $0 \leqslant i \leqslant |J|$ be fixed. Since $(T^{-k}\mathcal{M})_{k\in\mathbb{Z}^{d+1}}$ is a commuting filtration, we have

$$\mathbb{E}\left[m'_J \mid \prod_{s=1}^{i}T_{j_s}T_{d+1}\mathcal{M}\right] = \mathbb{E}\left[\mathbb{E}\left[m'_J \mid \prod_{s=1}^{i}T_{j_s}\mathcal{M}\right] \mid T_{d+1}\mathcal{M}\right].$$

Using the measurability of $m'_J$ with respect to $\prod_{s=1}^{i}T_{j_s}\mathcal{M}$, we obtain

$$\mathbb{E}\left[m'_J \mid \prod_{s=1}^{i}T_{j_s}T_{d+1}\mathcal{M}\right] = \mathbb{E}[m'_J \mid T_{d+1}\mathcal{M}].$$

Consequently,

$$\mathbb{E}\left[\prod_{s\in J}(I-U_s)m'_J \mid T_{d+1}\mathcal{M}\right] = \sum_{i=0}^{|J|}(-1)^i\prod_{s=1}^{i}U_s\mathbb{E}\left[m'_J \mid T_{d+1}\mathcal{M}\right] = \prod_{s\in J}(I-U_s)\mathbb{E}\left[m'_J \mid T_{d+1}\mathcal{M}\right].$$

Similarly, since $g'$ is $\prod_{s=1}^{d}T_s\mathcal{M}$-measurable, we have also

$$\mathbb{E}\left[\prod_{s=1}^{d}(I-U_s)g' \mid T_{d+1}\mathcal{M}\right] = \prod_{s=1}^{d}(I-U_s)\mathbb{E}\left[g' \mid T_{d+1}\mathcal{M}\right].$$

Using (22), we derive

$$\mathbb{E}[m' \mid T_{d+1}\mathcal{M}] = -\sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle}\prod_{s\in J}(I-U_s)\mathbb{E}\left[m'_J \mid T_{d+1}\mathcal{M}\right] - \prod_{s=1}^{d}(I-U_s)\mathbb{E}\left[g' \mid T_{d+1}\mathcal{M}\right]. \quad (23)$$

So, denoting $m := m' - \mathbb{E}[m' \mid T_{d+1}\mathcal{M}]$ and combining (20) and (23) we obtain

$$M = m + \sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle}\prod_{s\in J}(I-U_s)\left(m'_J - \mathbb{E}\left[m'_J \mid T_{d+1}\mathcal{M}\right]\right) + \prod_{s=1}^{d}(I-U_s)\left(g' - \mathbb{E}\left[g' \mid T_{d+1}\mathcal{M}\right]\right). \quad (24)$$

Moreover, $m$ is $\mathcal{M}$-measurable and $\mathbb{E}[m \mid T_s\mathcal{M}] = 0$ for each $s$ in $\langle d+1\rangle$. Combining (21) and (24), one can write

$$f = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d\rangle} \prod_{s\in J}(I - U_s)\left(m'_J - \mathbb{E}\left[m'_J \mid T_{d+1}\mathcal{M}\right]\right) + \prod_{s=1}^{d}(I - U_s)\left(g' - \mathbb{E}\left[g' \mid T_{d+1}\mathcal{M}\right]\right)$$
$$+ (I - U_{d+1})\left(m'' + \sum_{\emptyset \subsetneq J \subsetneq \langle d\rangle} \prod_{s\in J}(I - U_s)m''_J + \prod_{s=1}^{d}(I - U_s)g''\right),$$

which is of the form (5) for $d+1$ instead of $d$. Indeed, let $\emptyset \subsetneq J \subsetneq \langle d+1\rangle$ be fixed. If $d+1 \in J$, we denote

$$m_J = \begin{cases} m'' & \text{if } J = \{d+1\} \\ m''_{J\setminus\{d+1\}} & \text{if } J \setminus \{d+1\} \neq \emptyset \end{cases} \tag{25}$$

and if $d+1 \notin J$, we denote

$$m_J = \begin{cases} m'_J - \mathbb{E}\left[m'_J \mid T_{d+1}\mathcal{M}\right] & \text{if } J \neq \langle d\rangle \\ g' - \mathbb{E}\left[g' \mid T_{d+1}\mathcal{M}\right] & \text{if } J = \langle d\rangle. \end{cases} \tag{26}$$

Finally, denoting $g = g''$, we obtain

$$f = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d+1\rangle} \prod_{s\in J}(I - U_s)m_J + \prod_{s=1}^{d+1}(I - U_s)g,$$

The proof of Theorem 1 is complete.

*Proof of Proposition 1.* For simplicity, we consider only the case $d = 2$. Let $(X_{i,j})_{(i,j)\in\mathbb{Z}^2}$ be an OMD random field with respect to a commuting filtration $(\mathcal{F}_{i,j})_{(i,j)\in\mathbb{Z}^2}$. Let $(n_1, n_2)$ be fixed in $\mathbb{N}^2$ and consider $(Y_i)_{i\in\mathbb{Z}}$ defined for any $i$ in $\mathbb{Z}$ by $Y_i = \sum_{j=0}^{n_2} X_{i,j}$. One can notice that $(Y_i)_{i\in\mathbb{Z}}$ is a MD sequence with respect to the filtration $(\vee_{j\in\mathbb{Z}}\mathcal{F}_{i,j})_{i\in\mathbb{Z}}$. Consequently, by Burkholder's inequality, we have

$$\left\|\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} X_{i,j}\right\|_p \leqslant \kappa\sqrt{p}\left(\sum_{i=0}^{n_1}\|Y_i\|_p^2\right)^{1/2}.$$

Moreover, since for any $i$ in $\mathbb{Z}$, $(X_{i,j})_{j\in\mathbb{Z}}$ is a MD sequence with respect to the filtration $(\vee_{i\in\mathbb{Z}}\mathcal{F}_{i,j})_{j\in\mathbb{Z}}$, we have also

$$\|Y_i\|_p = \left\|\sum_{j=0}^{n_2} X_{i,j}\right\|_p \leqslant \kappa\sqrt{p}\left(\sum_{j=0}^{n_2}\|X_{i,j}\|_p^2\right)^{1/2}.$$

Consequently, we obtain

$$\left\|\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} X_{i,j}\right\|_p \leqslant \kappa p\left(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\|X_{i,j}\|_p^2\right)^{1/2}. \tag{27}$$

In order to prove the optimality of the constant $p$ in (27), arguing as in Wang and Woodroofe [24] (Example 1, page 12), we consider a sequence $(\eta_i)_{i\in\mathbb{Z}}$ of iid real random variables satisfying $\mu(\eta_0 = 1) = \mu(\eta_0 = -1) = 1/2$. Let also $(\eta_i')_{i\in\mathbb{Z}}$ be an independent copy of $(\eta_i)_{i\in\mathbb{Z}}$ and consider the filtrations $(\mathcal{G}_k)_{k\in\mathbb{Z}}$ and $(\mathcal{H}_k)_{k\in\mathbb{Z}}$ defined for any $k$ in $\mathbb{Z}$ by $\mathcal{G}_k = \sigma(\eta_s; s \leqslant k)$ and $\mathcal{H}_k = \sigma(\eta_s'; s \leqslant k)$. For any $(i, j)$ in $\mathbb{Z}^2$, we denote $Z_{i,j} = \eta_i\eta_j'$. Then $(Z_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is an OMD random field with respect to the commuting filtration $(\mathcal{F}_{i,j})_{(i,j)\in\mathbb{Z}^2}$ defined by $\mathcal{F}_{i,j} = \mathcal{G}_i \vee \mathcal{H}_j$ for any $(i, j)$ in $\mathbb{Z}^2$. Let $C$ be a positive constant such that for any $(n_1, n_2)$ in $\mathbb{N}^2$,

$$\left\|\sum_{i=0}^{n_1}\eta_i\right\|_p \times \left\|\sum_{j=0}^{n_2}\eta_j'\right\|_p = \left\|\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} Z_{i,j}\right\|_p \leqslant C\left(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\|Z_{i,j}\|_p^2\right)^{1/2} \leqslant C\sqrt{n_1 n_2}.$$

Applying the CLT for iid real random variables, we derive $C \geqslant \|N\|_p^2$ where $N$ is a standard normal random variable. Since there exists $\kappa > 0$ such that $\|N\|_p^2 \geqslant \kappa p$, we derive (7). The proof of Proposition 1 is complete.

*Proof of Proposition 2.* We start with the following lemma.

**Lemma 2** *If $f$ is a function satisfying the assumptions of Theorem 1 for some $p \geqslant 2$ then there exists a constant $C_d$ depending only on $d$ such that*

$$\max\left\{\|m\|_p, \|m_J\|_p, \|g\|_p\right\} \leqslant C_d\,\Delta_d(f, p), \tag{28}$$

*where $m$, $g$ and $m_J$ are defined by* (5) *and* $\Delta_d(f,p) := \sum_{k\in\mathbb{N}^d}\left\|\mathbb{E}[f \mid T^k\mathcal{M}]\right\|_p$.

*Proof of Lemma 2.* We prove this lemma by induction on $d$. The case $d = 1$ is a direct consequence of (18). Let $d$ be a positive integer and let $p \geqslant 2$. We assume that Lemma 2 is true for $d$ and we are going to prove that it is true for $d + 1$. Assume that $\Delta_{d+1}(f, p)$ is finite. Using Lemma 1 and arguing as in the proof of Theorem 1, we have

$$f = m + \sum_{\emptyset\subsetneq J\subsetneq\langle d+1\rangle}\prod_{s\in J}(I - U_s)m_J + \prod_{s=1}^{d+1}(I - U_s)g$$

where $m_J$ is given by (25) and (26), $g = g''$ and $m := m' - \mathbb{E}[m' \mid T_{d+1}\mathcal{M}]$ (see the last part of the proof of Theorem 1). Keeping in mind (16) and arguing as in the proof of Lemma 1 (see (19)), we derive

$$\max\left\{\Delta_d(M, p), \Delta_d(G, p)\right\} \leqslant 2\Delta_{d+1}(f, p). \tag{29}$$

The induction hypothesis yields $\|m'\|_p \leqslant C_d\Delta_d(M,p)$. Since $\|m\|_p \leqslant 2\|m'\|_p$, using (29), we obtain

$$\|m\|_p \leqslant 4C_d\Delta_{d+1}(f,p).$$

Similarly, we have

$$\|g\| = \|g''\|_p \leqslant C_d\Delta_d(G,p) \leqslant 2C_d\Delta_{d+1}(f,p).$$

Let $J$ be a nonempty subset of $\langle d+1\rangle$.

- If $d+1 \in J$ then using (25) and the induction hypothesis, we have $\|m_J\|_p \leqslant C_d\Delta_d(G,p)$. Hence by (29),

$$\|m_J\|_p \leqslant 2C_d\Delta_{d+1}(f,p).$$

- Similarly, using (26), if $d+1 \notin J$ and $J \neq \langle d\rangle$ then

$$\|m_J\|_p \leqslant 2\|m'_J\|_p \leqslant 2C_d\Delta_d(M,p) \leqslant 4C_d\Delta_{d+1}(f,p)$$

and $\left\|m_{\langle d\rangle}\right\|_p \leqslant 2\|g'\|_p \leqslant 4C_d\Delta_{d+1}(f,p)$.

Finally, it suffices to define $C_{d+1} := 4C_d$. The proof of Lemma 2 is complete.

Without loss of generality, one can write $X_i = f\circ T^i$ for any $i$ in $\mathbb{Z}^d$ where $f = X_0$ and $(T^i)_{i\in\mathbb{Z}^d}$ is a family of measure-preserving operators on $\Omega$ such that $T^k\circ T^l = T^{k+l}$ for any $k$ and $l$ in $\mathbb{Z}^d$. Let $n$ be fixed in $\mathbb{N}^d$. In the sequel, we denote $\Lambda_n = \{i\in\mathbb{N}^d\,;\, 0\preccurlyeq i\preccurlyeq n\}$ and $S_n(h) = \sum_{i\in\Lambda_n} h\circ T^i$ for any function $h$ defined on $\Omega$. Applying Theorem 1, we have

$$S_n(f) = S_n(m) + \sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle} S_n\left(\prod_{s\in J}(I-U_s)m_J\right) + S_n\left(\prod_{s=1}^{d}(I-U_s)g\right). \tag{30}$$

Let $\emptyset\subsetneq J\subsetneq\langle d\rangle$ and $k=(k_1,...,k_d)\in\mathbb{N}^d$ be fixed and define $k^{(J)} = (k_i)_{i\in\langle d\rangle\backslash J}$. We have

$$S_n\left(\prod_{s\in J}(I-U_s)m_J\right) = \prod_{s\in J}(I-U_s^{n_s+1})\sum_{0\preccurlyeq k^{(J)}\preccurlyeq n^{(J)}}\prod_{i\in\langle d\rangle\backslash J}U_i^{k_i}m_J.$$

Since the operator $\prod_{s\in J}(I-U_s^{n_s+1})$ may be written as a sum of $2^{|J|}$ isometries, the inequality

$$\left\|S_n\left(\prod_{s\in J}(I-U_s)m_J\right)\right\|_p \leqslant 2^{|J|}\left\|\sum_{0\preccurlyeq k^{(J)}\preccurlyeq n^{(J)}}\prod_{i\in\langle d\rangle\backslash J}U_i^{k_i}m_J\right\|_p \tag{31}$$

takes place and an application of Proposition 1 yields

$$\left\|\sum_{0\preccurlyeq k^{(J)}\preccurlyeq n^{(J)}}\prod_{i\in\langle d\rangle\backslash J}U_i^{k_i}m_J\right\|_p \leqslant \kappa p^{d/2}\left(\prod_{s\in J}n_s\right)^{1/2}\|m_J\|_p. \tag{32}$$

Combining (31) and (32), we get

$$\left\| S_n\left(\prod_{s\in J}(I-U_s)m_J\right)\right\|_p \leqslant 2^{|J|}\kappa p^{d/2}|n|^{1/2}\left\|m_J\right\|_p. \tag{33}$$

Moreover, since

$$S_n\left(\prod_{s=1}^{d}(I-U_s)g\right)=\prod_{s=1}^{d}(I-U_s^{n+1})g$$

and $\prod_{s=1}^{d}(I-U_s^{n+1})$ is a sum of $2^d$ isometries, if follows that

$$\left\| S_n\left(\prod_{s=1}^{d}(I-U_s)g\right)\right\|_p \leqslant 2^{d}\kappa p^{d/2}|n|^{1/2}\left\|g\right\|_p. \tag{34}$$

By Proposition 1, we have also

$$\left\|S_n\left(m\right)\right\|_p \leqslant \kappa p^{d/2}|n|^{1/2}\left\|m\right\|_p. \tag{35}$$

Combining (30), (33), (34) and (35), we obtain

$$\left\|S_n(f)\right\|_p \leqslant \kappa p^{d/2}|n|^{1/2}\left(\left\|m\right\|_p+\sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle}2^{|J|}\left\|m_J\right\|_p+2^d\left\|g\right\|_p\right). \tag{36}$$

Finally, applying Lemma 2 yields

$$\left\|S_n(f)\right\|_p \leqslant \kappa p^{d/2}|n|^{1/2}C_d\Delta_d(f,p)\left(1+\sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle}2^{|J|}+2^d\right).$$

The proof of Proposition 2 is complete.

*Proof of Proposition 3.* First, since $(X_k)_{k\in\mathbb{Z}^d}$ is stationary, we have

$$\sum_{k\in\mathbb{Z}^d}|\mathbb{E}(X_0X_k)|\leqslant 2^d\sum_{k\in\mathbb{N}^d}|\mathbb{E}(X_0X_k)|\leqslant\|X_0\|_2\sum_{k\in\mathbb{N}^d}\|\mathbb{E}(X_0|\mathcal{F}_{-k})\|_2<\infty.$$

Let $n=(n_1,...,n_d)$ be fixed in $\mathbb{N}^d$. Then,

$$|n|^{-1}\mathbb{E}\left(\left(\sum_{k\in\Lambda_n}X_k\right)^2\right)=\sum_{k\in\mathbb{Z}^d}|n|^{-1}|\Lambda_n\cap(\Lambda_n-k)|\mathbb{E}(X_0X_k)$$

where $\Lambda_n=\{i\in\mathbb{N}^d\,;\,0\preccurlyeq i\preccurlyeq n\}$ and $\Lambda_n-k=\{i-k\,;\,i\in\Lambda_n\}$ for any $k$ in $\mathbb{Z}^d$. Moreover,

$$|n|^{-1}|\Lambda_n\cap(\Lambda_n-k)||\mathbb{E}(X_0X_k)|\leqslant|\mathbb{E}(X_0X_k)|$$

and $\lim_{|n|\to+\infty}|n|^{-1}|\Lambda_n\cap(\Lambda_n-k)|=1$ for any $k$ in $\mathbb{Z}^d$. Finally, it suffices to apply the Lebesgue convergence theorem. The proof of Proposition 3 is complete.

*Proof of Theorem 2.* Let $(\Omega,\mathcal{F},\mu,\{T^k\}_{k\in\mathbb{Z}^d})$ be a dynamical system (that is, $(\Omega,\mathcal{F},\mu)$ is a probability space and $T^k:\Omega\to\Omega$ is a measure-preserving transformation for any $k$ in $\mathbb{Z}^d$ satisfying $T^i\circ T^j=T^{i+j}$ for any $i$ and any $j$ in $\mathbb{Z}^d$) and let $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ be a field of iid real random variables defined on $(\Omega,\mathcal{F},\mu)$. Let $\mathcal{M}\subset\mathcal{F}$ be the $\sigma$-algebra generated by the random variables $\varepsilon_i$ for $i\preccurlyeq 0$ and let $f:\Omega\to\mathbb{R}$ be $\mathcal{M}$-measurable. We consider the stationary real random field $(f\circ T^i)_{i\in\mathbb{Z}^d}$ and the partial sum process $\left\{S_n(f,t)\,;\,t\in[0,1]^d\right\}_{n\geqslant 1}$ defined for any integer $n\geqslant 1$ and any $t$ in $[0,1]^d$ by

$$S_n(f,t)=\sum_{i\in\langle n\rangle^d}\lambda([0,nt]\cap R_i)f\circ T^i \tag{37}$$

where $\lambda$ is the Lebesgue measure on $\mathbb{R}^d$ and $R_i=]i_1-1,i_1]\times\ldots\times]i_d-1,i_d]$ is the unit cube with upper corner $i=(i_1,...,i_d)$ in $\langle n\rangle^d$. As usual, we have to prove the convergence of the finite-dimensional laws and the tightness of the sequence of processes $\left\{n^{-d/2}S_n(f,t)\,;\,t\in[0,1]^d\right\}_{n\geqslant 1}$. We start with the tightness property: it suffices to establish for any $\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^d\\ |s-t|<\delta}}n^{-d/2}|S_n(f,s)-S_n(f,t)|>\varepsilon\right)=0$$

where $|x|=\max_{k\in\langle d\rangle}|x_k|$ for any $x=(x_1,...,x_d)$ in $[0,1]^d$. For simplicity, we are going to consider only the case $d=2$. By Theorem 1, we have

$$f=m+(I-U_1)m_1+(I-U_2)m_2+(I-U_1)(I-U_2)g, \tag{38}$$

where $m$, $m_1$, $m_2$ and $g$ are square-integrable functions defined on $\Omega$ such that $(U^i m)_{i\in\mathbb{Z}^2}$ is an OMD random field and $(U_2^k m_1)_{k\in\mathbb{Z}}$ and $(U_1^k m_2)_{k\in\mathbb{Z}}$ are MD sequences. In the sequel, for any real $x$, we denote by $[x]$ the integer part of $x$. Let $n\geqslant 1$ and $t=(t_1,t_2)$ in $[0,1]^2$. For any $1\leqslant i\leqslant[nt_1]+1$ and any $1\leqslant j\leqslant[nt_2]+1$, we denote $\lambda_{i,j}(t)=\lambda\left([0,nt]\cap R_{(i,j)}\right)$. We have

$$S_n((I-U_1)m_1,t)=\sum_{i=1}^{[nt_1]+1}\sum_{j=1}^{[nt_2]+1}\lambda_{i,j}(t)U^{(i,j)}(I-U_1)m_1=\sum_{j=1}^{[nt_2]+1}U_2^j\sum_{i=1}^{[nt_1]+1}\lambda_{i,j}(t)\left(U_1^i m_1-U_1^{i+1}m_1\right).$$

Using Abel's transformation and noting that $\lambda_{i+1,j}(t) = \lambda_{i,j}(t)$ for any $1 \leqslant i \leqslant [nt_1] - 1$ and any $1 \leqslant j \leqslant [nt_2] + 1$, we obtain that $S_n((I - U_1)m_1, t)$ equals

$$\sum_{j=1}^{[nt_2]+1} U_2^j \left\{ \lambda_{[nt_1]+1,j}(t) \left( U_1 m_1 - U_1^{[nt_1]+2} m_1 \right) - \sum_{i=1}^{[nt_1]} \left( U_1 m_1 - U_1^{i+1} m_1 \right) \left( \lambda_{i+1,j}(t) - \lambda_{i,j}(t) \right) \right\}$$

$$= \sum_{j=1}^{[nt_2]+1} U_2^j \left\{ \lambda_{[nt_1]+1,j}(t) \left( U_1 m_1 - U_1^{[nt_1]+2} m_1 \right) - \left( U_1 m_1 - U_1^{[nt_1]+1} m_1 \right) \left( \lambda_{[nt_1]+1,j}(t) - \lambda_{[nt_1],j}(t) \right) \right\}$$

$$= U_1(I - U_1^{[nt_1]+1}) \sum_{j=1}^{[nt_2]+1} \lambda_{[nt_1]+1,j}(t)\, U_2^j m_1 - U_1(I - U_1^{[nt_1]}) \sum_{j=1}^{[nt_2]+1} \left( \lambda_{[nt_1]+1,j}(t) - \lambda_{[nt_1],j}(t) \right) U_2^j m_1.$$

Moreover, since $\lambda_{i,j}(t) = \lambda_{i,1}(t)$ for any $1 \leqslant i \leqslant [nt_1] + 1$ and any $1 \leqslant j \leqslant [nt_2]$, we derive

$$\begin{aligned} S_n((I - U_1)m_1, t) = {} & U_1(I - U_1^{[nt_1]+1}) \lambda_{[nt_1]+1,1}(t) \sum_{j=1}^{[nt_2]} U_2^j m_1 \\ & + U_1(I - U_1^{[nt_1]+1}) \lambda_{[nt_1]+1,[nt_2]+1}(t)\, U_2^{[nt_2]+1} m_1 \\ & - U_1(I - U_1^{[nt_1]}) \left( \lambda_{[nt_1]+1,1}(t) - \lambda_{[nt_1],1}(t) \right) \sum_{j=1}^{[nt_2]} U_2^j m_1 \\ & - U_1(I - U_1^{[nt_1]}) \left( \lambda_{[nt_1]+1,[nt_2]+1}(t) - \lambda_{[nt_1],[nt_2]+1}(t) \right) U_2^{[nt_2]+1} m_1. \end{aligned}$$

So, we obtain

$$\sup_{t \in [0,1]^2} |S_n((I - U_1)m_1, t)| \leqslant 4 \max_{1 \leqslant l,k \leqslant n+2} U_1^l U_2^k |m_1| + 4 \max_{1 \leqslant l,k \leqslant n+2} U_1^l \left| \sum_{j=1}^{k} U_2^j m_1 \right|. \tag{39}$$

Let $x > 0$ be fixed. Since $m_1 \in \mathbb{L}^2(\Omega, \mathcal{F}, \mu)$, we have

$$\mu \left( \max_{1 \leqslant l,k \leqslant n+2} U_1^l U_2^k |m_1| > nx \right) \leqslant \kappa n^2 \mu \left( m_1^2 > n^2 x^2 \right) \xrightarrow[n \to \infty]{} 0. \tag{40}$$

In the other part,

$$\mu \left( \max_{1 \leqslant l,k \leqslant n+2} U_1^l \left| \sum_{j=1}^{k} U_2^j m_1 \right| > xn \right) = \mu \left( \max_{1 \leqslant l \leqslant n+2} U_1^l \left( \frac{1}{\sqrt{n}} \max_{1 \leqslant k \leqslant n+2} \left| \sum_{j=1}^{k} U_2^j m_1 \right| \right)^2 > nx^2 \right). \tag{41}$$

**Lemma 3** *Let $(Z_n)_{n \geqslant 1}$ be a uniformly integrable sequence of real random variables. For any $s$ in $\langle d \rangle$,*

$$\limsup_{n \to \infty} \mu \left( \max_{1 \leqslant i_1, .., i_s \leqslant n} U_1^{i_1} \dots U_s^{i_s} |Z_n| > n^s \right) = 0.$$

*Proof of Lemma 3.* Let $n$ be a positive integer. For any $s$ in $\langle d\rangle$, we denote

$$p_n(s) := \mu\left(\max_{1\leqslant i_1,\ldots,i_s\leqslant n} U_1^{i_1}\ldots U_s^{i_s}\left|Z_n\right| > n^s\right).$$

Let $R$ be a positive real number. We have

$$p_n(s) \leqslant \frac{2R}{n^s} + n^s\,\mu\left(\left|Z_n\right|\ \mathbb{1}_{\{|Z_n|>R\}} > \frac{n^s}{2}\right) \leqslant \frac{2R}{n^s} + 2\sup_{k\geqslant 1}\mathbb{E}\left(\left|Z_k\right|\ \mathbb{1}_{\{|Z_k|>R\}}\right).$$

Consequently, $\limsup_{n\to\infty} p_n(s) \leqslant 2\sup_{k\geqslant 1}\mathbb{E}\left(\left|Z_k\right|\ \mathbb{1}_{\{|Z_k|>R\}}\right) \xrightarrow[R\to\infty]{} 0$. The proof of Lemma 3 is complete.

**Lemma 4** *The sequence* $\left\{\left(\frac{1}{\sqrt{n}}\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1\right|\right)^2 ;\ n\geqslant 1\right\}$ *is uniformly integrable.*

*Proof of Lemma 4.* Since $(U_2^k m_1)_{k\in\mathbb{Z}}$ is a MD sequence, using Doob's inequality, we derive

$$\left\|\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1\right|\right\|_2 \leqslant 2\left\|\sum_{j=1}^{n+2} U_2^j m_1\right\|_2 \leqslant \kappa\sqrt{n}\left\|m_1\right\|_2.$$

So, $\left\{\left(\frac{1}{\sqrt{n}}\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1\right|\right)^2 ;\ n\geqslant 1\right\}$ is bounded in $\mathbb{L}^1(\Omega,\mathcal{F},\mu)$. Let $M$ be a fixed positive constant. We have $m_1 = m_1' + m_1''$ where

$$\begin{aligned}
m_1' &= m_1\,\mathbb{1}_{|m_1|\leqslant M} - \mathbb{E}\left(m_1\,\mathbb{1}_{|m_1|\leqslant M}\mid T_2\mathcal{M}\right)\\
m_1'' &= m_1\,\mathbb{1}_{|m_1|> M} - \mathbb{E}\left(m_1\,\mathbb{1}_{|m_1|> M}\mid T_2\mathcal{M}\right).
\end{aligned}$$

Moreover, if $A$ belongs to $\mathcal{F}$ then

$$\begin{aligned}
\int_A\left(\frac{1}{\sqrt{n}}\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1\right|\right)^2 d\mu &\leqslant 2\int_A\left(\frac{1}{\sqrt{n}}\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1'\right|\right)^2 d\mu\\
&\quad + 2\int_A\left(\frac{1}{\sqrt{n}}\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1''\right|\right)^2 d\mu.
\end{aligned}$$

Since $(U_2^k m_1')_{k\in\mathbb{Z}}$ and $(U_2^k m_1'')_{k\in\mathbb{Z}}$ are MD sequences, using Schwarz's inequality, we obtain

$$\begin{aligned}
\int_A\left(\frac{1}{\sqrt{n}}\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1\right|\right)^2 d\mu &\leqslant 2\left\|\frac{1}{\sqrt{n}}\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1'\right|\right\|_4^2\sqrt{\mu(A)}\\
&\quad + 2\left\|\frac{1}{\sqrt{n}}\max_{1\leqslant k\leqslant n+2}\left|\sum_{j=1}^k U_2^j m_1''\right|\right\|_2^2.
\end{aligned}$$

Keeping in mind that $m_1'$ is bounded by $M$ and using again Doob's inequality, there exists a positive constant $\kappa_0$ such that

$$\int_A \left( \frac{1}{\sqrt{n}} \max_{1 \leqslant k \leqslant n+2} \left| \sum_{j=1}^{k} U_2^j m_1 \right| \right)^2 d\mu \leqslant \kappa_0 \left( M^2 \sqrt{\mathbb{P}(A)} + \mathbb{E} \left( m_1^2 \mathbb{1}_{|m_1| > M} \right) \right).$$

Let $\varepsilon > 0$ be fixed and let $M > 0$ such that $\kappa_0 \mathbb{E} \left( m_1^2 \mathbb{1}_{|m_1| > M} \right) \leqslant \frac{\varepsilon}{2}$. One can choose the measurable set $A$ in $\mathcal{F}$ such that $\kappa_0 M^2 \sqrt{\mu(A)} \leqslant \frac{\varepsilon}{2}$ and consequently

$$\sup_{n \geqslant 1} \int_A \left( \frac{1}{\sqrt{n}} \max_{1 \leqslant k \leqslant n+2} \left| \sum_{j=1}^{k} U_2^j m_1 \right| \right)^2 d\mu \leqslant \varepsilon.$$

The proof of Lemma 4 is complete.

Combining (39), (40), (41), Lemma 3 and Lemma 4, we obtain

$$\limsup_{n \to \infty} \mu \left( \sup_{t \in [0,1]^2} |S_n((I - U_1)m_1, t)| > xn \right) = 0. \tag{42}$$

In a similar way, we derive also

$$\limsup_{n \to \infty} \mu \left( \sup_{t \in [0,1]^2} |S_n((I - U_2)m_2, t)| > xn \right) = 0. \tag{43}$$

Now, noting that $\lambda_{i,j}(t) = \lambda_{i,1}(t)$ for any $1 \leqslant i \leqslant [nt_1] + 1$ and any $1 \leqslant j \leqslant [nt_2]$, we have $S_n((I - U_1)(I - U_2)g, t)$ equals

$$\begin{aligned}
&\sum_{i=1}^{[nt_1]+1} \sum_{j=1}^{[nt_2]+1} \lambda_{i,j}(t)\, U^{(i,j)}(I - U_1)(I - U_2)g \\
&= \sum_{i=1}^{[nt_1]+1} U_1^i (I - U_1) \left( \lambda_{i,1}(t) \sum_{j=1}^{[nt_2]} (U_2^j - U_2^{j+1})g + \lambda_{i,[nt_2]+1}(t) U_2^{[nt_2]+1} (I - U_2) g \right) \\
&= U_2(I - U_2^{[nt_2]}) \sum_{i=1}^{[nt_1]+1} \lambda_{i,1}(t)(U_1^i - U_1^{i+1})g + U_2^{[nt_2]+1}(I - U_2) \sum_{i=1}^{[nt_1]+1} \lambda_{i,[nt_2]+1}(t)(U_1^i - U_1^{i+1})g.
\end{aligned}$$

Since $\lambda_{i,j}(t) = \lambda_{1,j}(t)$ for any $1 \leqslant i \leqslant [nt_1]$ and any $1 \leqslant j \leqslant [nt_2] + 1$, we derive

$$\begin{aligned}
S_n((I - U_1)(I - U_2)g, t) &= \lambda_{1,1}(t) U_2 (I - U_2^{[nt_2]}) U_1 (I - U_1^{[nt_1]}) g \\
&\quad + \lambda_{[nt_1]+1,1}(t) U_2 (I - U_2^{[nt_2]}) U_1^{[nt_1]+1} (I - U_1)\, g \\
&\quad + \lambda_{1,[nt_2]+1}(t) U_2^{[nt_2]+1} (I - U_2) U_1 (I - U_1^{[nt_1]+1}) g \\
&\quad + \lambda_{[nt_1]+1,[nt_2]+1}(t) U_2^{[nt_2]+1} (I - U_2) U_1^{[nt_1]+1} (I - U_1) g.
\end{aligned}$$

Thus

$$\sup_{t\in[0,1]^2} |S_n((I-U_1)(I-U_2)g,t)| \leqslant \kappa \max_{1\leqslant k,l\leqslant n+2} U_1^k U_2^l |g|$$

and for any positive $x$,

$$\mu\left(\sup_{t\in[0,1]^2} |S_n((I-U_1)(I-U_2)g,t)| > xn\right) \leqslant \kappa n^2 \mu\left(g^2 > n^2x^2\right) \xrightarrow[n\to\infty]{} 0. \tag{44}$$

Now, it suffices to prove the tightness of the process $\{\frac{1}{n}S_n(m,t); t\in[0,1]^2\}_{n\geqslant 1}$. That is, for any positive $x$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mu\left(\sup_{\substack{s,t\in[0,1]^2\\ |s-t|<\delta}} |S_n(m,s)-S_n(m,t)| > xn\right) = 0. \tag{45}$$

Let $n$ be a positive integer and let $s=(s_1,s_2)$ and $t=(t_1,t_2)$ be fixed in $[0,1]^2$. We denote $\Delta_n(s,t) = S_n(m,s)-S_n(m,t)$ and for any $i$ and $j$ in $\langle n\rangle$,

$$\beta_{i,j} = \lambda_{i,j}(s) - \lambda_{i,j}(t) = \lambda\left([0,ns]\cap R_{(i,j)}\right) - \lambda\left([0,nt]\cap R_{(i,j)}\right).$$

Noting that $\beta_{i,j}=0$ for any $1\leqslant i\leqslant [n(s_1\wedge t_1)]$ and any $1\leqslant j\leqslant [n(s_2\wedge t_2)]$, we have $\Delta_n(s,t) = \Delta_n'(s,t) + \Delta_n''(s,t)$ where

$$\Delta_n'(s,t) = \sum_{i=[n(s_1\wedge t_1)]+1}^{[n(s_1\vee t_1)]+1} \sum_{j=1}^{[n(s_2\wedge t_2)]+1} \beta_{i,j}\, U^{(i,j)}m \quad\text{and}\quad \Delta_n''(s,t) = \sum_{i=1}^{[n(s_1\wedge t_1)]+1} \sum_{j=[n(s_2\wedge t_2)]+1}^{[n(s_2\vee t_2)]+1} \beta_{i,j}\, U^{(i,j)}m.$$

Moreover, $\Delta_n'(s,t) = \Delta_{1,n}'(s,t) + \Delta_{2,n}'(s,t) + \Delta_{3,n}'(s,t) + \Delta_{4,n}'(s,t)$ where

$$\Delta_{1,n}'(s,t) = \sum_{i=[n(s_1\wedge t_1)]+2}^{[n(s_1\vee t_1)]} \sum_{j=1}^{[n(s_2\wedge t_2)]} \beta_{i,j}\, U^{(i,j)}m$$

$$\Delta_{2,n}'(s,t) = \sum_{j=1}^{[n(s_2\wedge t_2)]} \beta_{[n(s_1\vee t_1)]+1,j}\, U^{([n(s_1\vee t_1)]+1,j)}m$$

$$\Delta_{3,n}'(s,t) = \sum_{j=1}^{[n(s_2\wedge t_2)]} \beta_{[n(s_1\wedge t_1)]+1,j}\, U^{([n(s_1\wedge t_1)]+1,j)}m$$

$$\Delta_{4,n}'(s,t) = \sum_{i=[n(s_1\wedge t_1)]+1}^{[n(s_1\vee t_1)]+1} \beta_{i,[n(s_2\wedge t_2)]+1}\, U^{(i,[n(s_2\wedge t_2)]+1)}m.$$

Let $\alpha$ in $\{-1,+1\}$ such that $\beta_{i,j}=\alpha$ if $[n(s_1\wedge t_1)]+2\leqslant i\leqslant [n(s_1\vee t_1)]$ and $1\leqslant j\leqslant [n(s_2\wedge t_2)]$ So,

$$\Delta_{1,n}'(s,t) = \alpha \sum_{i=[n(s_1\wedge t_1)]+2}^{[n(s_1\vee t_1)]} \sum_{j=1}^{[n(s_2\wedge t_2)]} U^{(i,j)}m$$

and for any positive $x$,

$$\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta'_{1,n}(s,t)|>nx\right)\leqslant\sum_{k=0}^{[\frac{1}{\delta}]}\mu\left(\max_{\substack{1\leqslant p\leqslant n\\r\in[0,\delta]}}\left|\sum_{i=[nk\delta]+2}^{[n(k\delta+r)]}\sum_{j=1}^{p}U^{(i,j)}m\right|>nx\right)$$
$$=\sum_{k=0}^{[\frac{1}{\delta}]}\mu\left(\max_{\substack{1\leqslant p\leqslant n\\r\in[0,\delta]}}\left|\sum_{i=1}^{[n(k\delta+r)]-[nk\delta]-1}\sum_{j=1}^{p}U^{(i,j)}m\right|>nx\right).$$

Since $[n(k\delta+r)]-[nk\delta]-1$ is an integer smaller than $[nr]$, we obtain

$$\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta'_{1,n}(s,t)|>nx\right)\leqslant\left(1+\frac{1}{\delta}\right)\mu\left(\max_{\substack{1\leqslant p\leqslant n\\1\leqslant q\leqslant[n\delta]}}\left|\sum_{i=1}^{q}\sum_{j=1}^{p}U^{(i,j)}m\right|>nx\right)$$
$$=\left(1+\frac{1}{\delta}\right)\mu\left(\max_{\substack{1\leqslant p\leqslant n\\1\leqslant q\leqslant[n\delta]}}\left(\frac{1}{n\sqrt{\delta}}\sum_{i=1}^{q}\sum_{j=1}^{p}U^{(i,j)}m\right)^2>\frac{x^2}{\delta}\right)$$
$$\leqslant\left(\frac{1+\delta}{x^2}\right)\mathbb{E}_{\frac{x^2}{\delta}}\left(\max_{\substack{1\leqslant p\leqslant n\\1\leqslant q\leqslant[n\delta]}}\left(\frac{1}{n\sqrt{\delta}}\sum_{i=1}^{q}\sum_{j=1}^{p}U^{(i,j)}m\right)^2\right)$$

where we used the notation $\mathbb{E}_A(Z)=\mathbb{E}\left(Z\,\mathbb{1}_{|Z|>A}\right)$ for any $A>0$ and any $Z$ in $\mathbb{L}^1(\Omega,\mathcal{F},\mu)$.

**Lemma 5** *The family* $\left\{\max_{\substack{1\leqslant p\leqslant n\\1\leqslant q\leqslant[n\delta]}}\left(\frac{1}{n\sqrt{\delta}}\sum_{i=1}^{q}\sum_{j=1}^{p}U^{(i,j)}m\right)^2\ ;\ n\geqslant 1,\delta>0\right\}$ *is uniformly integrable.*

*Proof of Lemma 5.* The proof follows the same lines as the proof of Lemma 4 using Cairoli's maximal inequality for orthomartingales (see [16], Theorem 2.3.1) instead of Doob's inequality for martingales. It is left to the reader.

So, we obtain

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta'_{1,n}(s,t)|>nx\right)=0. \tag{46}$$

In the other part, since $\beta_{[n(s_1\vee t_1)]+1,j}=\beta_{[n(s_1\vee t_1)]+1,1}$ for any $1\leqslant j\leqslant[n(s_2\wedge t_2)]$, we have

$$\Delta'_{2,n}(s,t)=\beta_{[n(s_1\vee t_1)]+1,1}U_1^{[n(s_1\vee t_1)]+1}\sum_{j=1}^{[n(s_2\wedge t_2)]}U_2^j m$$

and consequently

$$\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta'_{2,n}(s,t)|\leqslant\max_{\substack{1\leqslant k\leqslant n+1\\1\leqslant l\leqslant n}}U_1^k\left|\sum_{j=1}^{l}U_2^j m\right|.$$

So,

$$\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta'_{2,n}(s,t)|>nx\right)\leqslant\mu\left(\max_{1\leqslant k\leqslant n+1}U_1^k\left(\max_{1\leqslant l\leqslant n}\frac{1}{\sqrt{n}}\left|\sum_{j=1}^{l}U_2^jm\right|\right)^2>nx^2\right). \tag{47}$$

Since $(U_2^km)_{k\in\mathbb{Z}}$ is a MD sequence, arguing as in Lemma 4, the sequence $\left\{\left(\max_{1\leqslant l\leqslant n}\frac{1}{\sqrt{n}}\left|\sum_{j=1}^{l}U_2^jm\right|\right)^2\right\}_{n\geqslant 1}$ is uniformly integrable. Combining (47) and Lemma 3, we derive that for any $\delta>0$,

$$\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta'_{2,n}(s,t)|>nx\right)=0. \tag{48}$$

Similarly, we have also

$$\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta'_{3,n}(s,t)|>nx\right)=0 \tag{49}$$

for any $\delta>0$. Moreover, for any $[n(s_1\wedge t_1)]+1\leqslant i\leqslant[n(s_1\vee t_1)]$, we have $\beta_{i,[n(s_2\wedge t_2)]+1}=\beta_{[n(s_1\wedge t_1)]+1,[n(s_2\wedge t_2)]+1}$ and consequently

$$\begin{aligned}\Delta'_{4,n}(s,t)&=\beta_{[n(s_1\wedge t_1)]+1,[n(s_2\wedge t_2)]+1}\,U_2^{[n(s_2\wedge t_2)]+1}\sum_{i=[n(s_1\wedge t_1)]+1}^{[n(s_1\vee t_1)]}U_1^im\\&\quad+\beta_{[n(s_1\vee t_1)]+1,[n(s_2\wedge t_2)]+1}\,U^{([n(s_1\vee t_1)]+1,[n(s_2\wedge t_2)]+1)}m\end{aligned}$$

and

$$\begin{aligned}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta'_{4,n}(s,t)|>nx\right)&\leqslant\mu\left(\max_{1\leqslant k\leqslant n+1}U_2^k\left(\max_{1\leqslant l\leqslant[n\delta]}\frac{1}{\sqrt{n\delta}}\left|\sum_{j=1}^{l}U_1^jm\right|\right)^2>\frac{nx^2}{2\delta}\right)\\&\quad+2n^2\mu\left(m^2>\frac{n^2x^2}{4}\right)\end{aligned}$$

Arguing as in Lemma 3, the family $\left\{\left(\max_{1\leqslant l\leqslant[n\delta]}\frac{1}{\sqrt{n\delta}}\left|\sum_{j=1}^{l}U_1^jm\right|\right)^2;\ n\geqslant 1,\delta>0\right\}$ is uniformly integrable since $(U_1^km)_{k\in\mathbb{Z}}$ is a MD sequence. By Lemma 4, we obtain for any $\delta>0$,

$$\limsup_{n\to\infty}\mu\left(\max_{1\leqslant k\leqslant n+1}U_2^k\left(\max_{1\leqslant l\leqslant[n\delta]}\frac{1}{\sqrt{n\delta}}\left|\sum_{j=1}^{l}U_1^jm\right|\right)^2>\frac{nx^2}{2\delta}\right)=0.$$

Moreover, $n^2\mu\left(m^2 > \frac{n^2x^2}{4}\right)$ goes to zero as $n$ goes to infinity since $m$ belongs to $\mathbb{L}^2(\Omega, \mathcal{F}, \mu)$. Consequently, for any $\delta > 0$,

$$\limsup_{n\to\infty} \mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}} |\Delta'_{4,n}(s,t)| > nx\right) = 0. \tag{50}$$

Combining (46), (48), (49) and (50), we obtain

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}} |\Delta'_{n}(s,t)| > nx\right) = 0. \tag{51}$$

Similarly, one can check that

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}} |\Delta''_{n}(s,t)| > nx\right) = 0. \tag{52}$$

Finally, keeping in mind $\Delta_n(s,t) = \Delta'_n(s,t) + \Delta''_n(s,t)$ and combining (51) and (52), we obtain (45). Now, we are going to prove the convergence of the finite-dimensional laws. In fact, combining (5), (42), (43) and (44), we have

$$\limsup_{n\to\infty} \mu\left(\sup_{t\in[0,1]^2} |S_n(f-m,t)| > xn\right) = 0. \tag{53}$$

Let $(t,n)$ be fixed in $[0,1]^2 \times \mathbb{N}^2$ and denote $\Lambda_n(t) = [0,nt] \cap \mathbb{N}^2$. We have

$$S_n(m,t) - \sum_{i\in\Lambda_n(t)} m\circ T^i = \sum_{i\in W_n(t)} a_i\, m\circ T^i \tag{54}$$

where $a_i = \lambda([0,nt]\cap R_i) - \mathbb{1}_{i\in\Lambda_n(t)}$ and $W_n(t)$ is the set of all $i$ in $\langle n\rangle^d$ such that $R_i\cap[0,nt] \neq \emptyset$ and $R_i \cap (\mathbb{R}^2 \setminus [0,nt]) \neq \emptyset$. Noting that $|a_i| \leqslant 1$ and combining (54) and Proposition 1, we obtain

$$\left\|S_n(m,t) - \sum_{i\in\Lambda_n(t)} m\circ T^i\right\|_2 \leqslant C\,\|m\|_2 \left(\sum_{i\in W_n(t)} a_i^2\right)^{1/2} \leqslant C\,\|m\|_2 \sqrt{|W_n(t)|} \tag{55}$$

where $C$ is a positive constant and $|W_n(t)|$ denotes the number of elements in $W_n(t)$. Since $|W_n(t)| = O(n)$, we derive

$$n^{-1}\left\|S_n(m,t) - \sum_{i\in\Lambda_n(t)} m\circ T^i\right\|_2 = O\left(\frac{1}{\sqrt{n}}\right). \tag{56}$$

Finally, combining (53), (56) and Proposition 3 and arguing as in the proof of Theorem 4.1 by Wang and Woodroofe [24], we derive the convergence of the finite dimensional laws of $\{n^{-1}S_n(f,t)\,;\,t\in[0,1]^2\}$. The proof of Theorem 2 is complete.

*Proof of Proposition 4.* Since $X_0=\sum_{j\in\mathbb{N}^d}a_j\varepsilon_{-j}$ where $(a_j)_{j\in\mathbb{N}^d}$ is a family of real numbers satisfying $\sum_{j\in\mathbb{N}^d}a_j^2<\infty$ and $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ is an iid zero-mean real random field, we have

$$\mathbb{E}(X_0\mid\mathcal{F}_{-k})=\sum_{j\succcurlyeq k}a_j\varepsilon_{-j}\tag{57}$$

where $k\in\mathbb{N}^d$ and $\mathcal{F}_{-k}$ is the $\sigma$-algebra generated by $\varepsilon_i$ for any $i\preccurlyeq -k$. We recall the Rosenthal's inequality ([14], Theorem 2.12): for any $p\geqslant 2$, there exists a positive constant $C$ depending only on $p$ such that if $(Y_j)_{j\geqslant1}$ is a sequence of independent zero-mean random variables and $n$ is a positive integer then

$$\frac{1}{C}\left(\sum_{j=1}^{n}\mathbb{E}[Y_j^2]\right)^{p/2}+\frac{1}{C}\sum_{j=1}^{n}\mathbb{E}|Y_j|^p\leqslant\mathbb{E}\left|\sum_{j=1}^{n}Y_j\right|^p\leqslant C\left(\sum_{j=1}^{n}\mathbb{E}[Y_j^2]\right)^{p/2}+C\sum_{j=1}^{n}\mathbb{E}|Y_j|^p.\tag{58}$$

Combining (57) and (58), we obtain that (8) holds if and only if (14) holds. The proof of Proposition 4 is complete.

*Proof of Theorem 3.* We shall use Theorem 1 in [17] which states that if a sequence of random processes $\{Y_n(t)\,;\,t\in[0,1]^d\}_{n\geqslant1}$ whose finite dimensional distributions are weakly convergent and for some constants $\alpha$, $\beta$ and $K$ such that

$$\beta\in(0,1]\quad\text{and}\quad\alpha\beta>\frac{2}{\log_2\left(\frac{4d}{4d-3}\right)}$$

and

$$\mu\left\{|Y_n(t)-Y_n(s)|\geqslant\varepsilon\right\}\leqslant\frac{K}{\varepsilon^\alpha}\|s-t\|^{\alpha\beta}\tag{59}$$

for any $s$ and $t$ in $[0,1]^d$, any $\varepsilon>0$ and any positive integer $n$ then $(Y_n(\cdot))_{n\geqslant1}$ converges weakly to some process in $\mathbb{H}_\gamma([0,1]^d)$ where $0<\gamma<\beta-d/\alpha$. Since the finite-dimensional laws of the process $\{n^{-d/2}S_n(t)\,;\,t\in[0,1]^d\}_{n\geqslant1}$ are weakly convergent (cf. Theorem 2), it suffices to convert the moment inequality given by Proposition 2 into an inequality involving $\mu\left\{|S_n(t)-S_n(s)|\geqslant n^{d/2}\varepsilon\right\}$ in order to check that condition (59) is satisfied with $\alpha=p$, $\beta=1/2$ and $Y_n(t)=n^{-d/2}S_n(t)$. We shall do the proof for $d=2$. Let $s=(s_1,s_2)$ and $t=(t_1,t_2)$ be fixed in $[0,1]^2$ and $n$ be a positive integer. Without loss of generality, we assume that $s_1>t_1$ and $s_2<t_2$ (similar arguments can be used to treat the general case). Let $s_1'=k_1/n$ and $t_1'=(l_1+1)/n$ where $(k_1,l_1)$ is the unique element of $\langle n\rangle^2$ such that $k_1/n\leqslant s_1<(k_1+1)/n$ and $l_1/n\leqslant t_1<(l_1+1)/n$. In other words, keeping in mind that [.] denotes the integer part function, we have $s_1'=[ns_1]/n$ and $t_1'=([nt_1]+1)/n$ and similarly, we define $s_2' = ([ns_2]+1)/n$ and $t_2' = [nt_2]/n$. With these notations, we have

$$\begin{aligned}|S_n(t) - S_n(s)| = |S_n(t_1,t_2) - S_n(s_1,s_2)| \\ \leqslant |S_n(t_1,t_2) - S_n(t_1,t_2')| + |S_n(t_1',t_2') - S_n(t_1,t_2')| \\ + |S_n(t_1',t_2') - S_n(s_1',s_2')| + |S_n(s_1',s_2') - S_n(s_1',s_2)| \\ + |S_n(s_1',s_2) - S_n(s_1,s_2)|.\end{aligned}$$

Since

$$|S_n(t_1,t_2) - S_n(t_1,t_2')| = (t_2 - t_2')\left|\sum_{i=1}^{[nt_1]} X_{i,[nt_2]} + (t_1' - t_1)X_{[nt_1]+1,[nt_2]}\right|$$

and $t_2 - t_2' \leqslant 1/n$, by Proposition 2, there exists a positive constant $C$ such that

$$\mathbb{E}\,|S_n(t_1,t_2) - S_n(t_1,t_2')|^p \leqslant C\,p^p(t_2-t_2')^p n^{p/2} \leqslant C\,p^p(t_2-t_2')^{p/2}. \tag{60}$$

Similarly,

$$\mathbb{E}\,|S_n(t_1',t_2') - S_n(t_1,t_2')|^p \leqslant C\,p^p(t_1'-t_1)^{p/2} \tag{61}$$
$$\mathbb{E}\,|S_n(s_1',s_2') - S_n(s_1',s_2)|^p \leqslant C\,p^p(s_2'-s_2)^{p/2} \tag{62}$$
$$\mathbb{E}\,|S_n(s_1',s_2) - S_n(s_1,s_2)|^p \leqslant C\,p^p(s_1-s_1')^{p/2}. \tag{63}$$

Moreover, from Proposition 2, for any positive integer $n$ and any $i$ and $j$ in $\langle n\rangle^2$, we have

$$\mathbb{E}\left|\frac{1}{n}S_n\left(\frac{i}{n}\right) - \frac{1}{n}S_n\left(\frac{j}{n}\right)\right|^p \leqslant C\,p^p\left\|\frac{i}{n} - \frac{j}{n}\right\|^{p/2}. \tag{64}$$

Combining (60), (61), (62), (63) and (64) and using the elementary convexity inequality $(a_1+a_2+a_3+a_4+a_5)^p \leqslant 5^{p-1}(a_1^p+a_2^p+a_3^p+a_4^p+a_5^p)$ for any non-negative $a_1, a_2, a_3, a_4$ and $a_5$, we derive

$$\mathbb{E}|S_n(t) - S_n(s)|^p \leqslant \kappa\|s-t\|^{p/2}.$$

Finally, using Markov's inequality, we obtain (59). The proof of Theorem 3 is complete.

**Acknowledgements.** We are grateful for an anonymous referee for useful comments and for Dalibor Volný and Yizao Wang for many stimulating discussions.

# References

[1] A. K. Basu and C. C. Y. Dorea. On functional central limit theorem for stationary martingale random fields. *Acta. Math. Hung.*, 33:307–316, 1979.

[2] H. Biermé and O. Durieu. Invariance principles for self-similar set-indexed random fields. *Trans. Amer. Math. Soc.*, 366(11):5963–5989, 2014.

[3] P. Billingsley. The Lindeberg-Lévy theorem for martingales. *Proc. Amer. Math. Soc.*, 12:788–792, 1961.

[4] P. Billingsley. *Convergence of probability measures.* Wiley, 1968.

[5] R. Cairoli. Un théorème de convergence pour martingales à indices multiples. *C. R. Acad. Sci. Paris Sér. A-B*, 269:A587–A589, 1969.

[6] J. Dedecker. Exponential inequalities and functional central limit theorems for random fields. *ESAIM: Probability and Statistics*, 5:77–104, 2001.

[7] M. D. Donsker. An invariance principle for certain probability limit theorems. *Mem. Amer. Math. Soc.*, 6:1–12, 1951.

[8] R. M. Dudley. Sample functions of the Gaussian process. *Ann. Probab.*, 1:66–103, 1973.

[9] M. El Machkouri. Kahane-Khintchine inequalities and functional central limit theorem for stationary random fields. *Stoch. Proc. and Their Appl.*, 120:285–299, 2002.

[10] M. El Machkouri and L. Ouchti. Invariance principles for standard-normalized and self-normalized random fields. *ALEA Lat. Am. J. Probab. Math. Stat.*, 2:177–194, 2006.

[11] M. El Machkouri, D. Volný, and W. B. Wu. A central limit theorem for stationary random fields. *Stochastic Process. Appl.*, 123(1):1–14, 2013.

[12] M. I. Gordin. The central limit theorem for stationary processes. *Soviet Math.Dokl.*, pages 1174–1176, 1969.

[13] M. I. Gordin. Martingale-co-boundary representation for a class of stationary random fields. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 364(Veroyatnost i Statistika. 14.2):88–108, 236, 2009.

[14] P. Hall and C. C. Heyde. *Martingale limit theory and its application.* Academic Press, New York, 1980.

[15] I. A. Ibragimov. A central limit theorem for a class of dependent random variables. *Theory Probab. Appl.*, 8:83–89, 1963.

[16] D. Khoshnevisan. *Multiparameter processes.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2002.

[17] Jana Klicnarová. Central limit theorem for Hölder processes on $\mathbb{R}^m$-unit cube. *Comment. Math. Univ. Carolin.*, 48(1):83–91, 2007.

[18] M. A. Krasnosel'skii and Y. B. Rutickii. *Convex Functions and Orlicz Spaces.* P. Noordhoff LTD-Groningen-The Netherlands, 1961.

[19] J. Kuelbs. The invariance principle for a lattice of random variables. *Ann. Math. Statist.*, 39:382–389, 1968.

[20] B. Nahapetian and A. N. Petrosian. Martingale-difference Gibbs random fields and central limit theorem. *Ann. Acad. Sci. Fenn., Series A-I Math.*, 17:105–110, 1992.

[21] A. Račkauskas, C. Suquet, and V. Zemlys. A Hölderian functional central limit theorem for a multi-indexed summation process. *Stochastic Process. Appl.*, 117(8):1137–1164, 2007.

[22] D. Volný. Approximating martingales and the central limit theorem for strictly stationary processes. *Stochastic Processes and Their Applications*, 44:41–74, 1993.

[23] D. Volný and Y. Wang. An invariance principle for stationary random fields under Hannan's condition. *Stochastic Process. Appl.*, 124(12):4012–4029, 2014.

[24] Y. Wang and M. Woodroofe. A new condition for the invariance principle for stationary random fields. *Statist. Sinica*, 23(4):1673–1696, 2013.

[25] M. J. Wichura. Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters. *Ann. Math. Statist.*, 40:681–687, 1969.