Parameter estimation for the non-ergodic Ornstein-Uhlenbeck processes driven by Gaussian process

Mohamed El Machkouri[1], Khalifa Es-Sebaiy[2] and Youssef Ouknine[3]

Abstract: The statistical analysis for equations driven by fractional Gaussian process (fGp) is obviously more recent. The development of stochastic calculus with respect to the fGp allowed to study such models. In the present paper we consider the drift parameter estimation problem for the non-ergodic Ornstein-Uhlenbeck process defined as $dX_t = \theta X_t dt + dG_t,\ t \geq 0$, with an unknown parameter $\theta > 0$, where G is a Gaussian process. We provide sufficient conditions, based the properties of G, ensuring the strong consistency and the asymptotic distributions of our estimator $\widetilde{\theta}_t$ of θ based on the observation $\{X_s,\ s \in [0,t]\}$ as $t \to \infty$. Our approach offers an elementary, unifying proof of [4], and it allows to extend the result of [4] to the case when G is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. We will also discuss the cases of subfractional Browian motion and bifractional Brownian motion.

Key words: Parameter estimation, Non-ergodic Gaussian Ornstein-Uhlenbeck process.

1 Introduction

While the statistical inference of Itô type diffusions has a long history, the statistical analysis for equations driven by fractional Gaussian process is obviously more recent. The development of stochastic calculus with respect to the fGp allowed to study such models. We will recall several approaches to estimate the parameters in fractional models but we mention that the below list is not exhaustive:

- The MLE approach in [12], [16]: In general the techniques used to construct maximum likelihood estimators (MLE) for the drift parameter are based on Girsanov transforms for fBm and depend on the properties of the deterministic fractional operators (determined by the Hurst parameter) related to the fBm. In this case, the MLE is not easily computable.
- A least squares approach has been proposed in [10]: The study of the asymptotic properties of the estimator is based on certain criteria formulated in terms of the Malliavin calculus. In the ergodic case, the statistical inference for several fractional Ornstein-Uhlenbeck (fOU) models has been recently developed in the papers [10], [1], [2], [6], [11], [5]. The case of non-ergodic fOU process of the first kind and of the second kind can be found in [4] and [7] respectively.

[1]Laboratoire de Mathématiques Raphaël Salem UMR CNRS 6085, Université de Rouen, France. Email: mohamed.elmachkouri@univ-rouen.fr

[2]National School of Applied Sciences-Marrakesh, Cadi Ayyad University Av. Abdelkrim Khattabi, 40000, Guéliz-Marrakech, Morocco. Email: k.essebaiy@uca.ma

[3]National School of Applied Sciences-Marrakesh, Cadi Ayyad University Av. Abdelkrim Khattabi, 40000, Guéliz-Marrakech, Morocco. Email: ouknine@uca.ma

- Method of moments: A new idea has been provided in [9], to develop the statistical inference for stochastic differential equations related to stationary Gaussian processes by proposing a suitable criteria. This approach is based on ergodic property and Malliavin calculus, and it makes in principle the estimators easier to be simulated. Moreover, the models studied in [10], [1], [2], [6] become particular cases in this approach.

In this paper, we consider the non-ergodic Ornstein-Uhlenbeck process $X = \{X_t, t \geq 0\}$ given by the following linear stochastic differential equation

$$X_0 = 0; \quad dX_t = \theta X_t dt + dG_t, \quad t \geq 0, \tag{1.1}$$

where G is a Gaussian process and $\theta > 0$ is an unknown parameter.
An interesting problem is to estimate the parameter θ when one observes the whole trajectory of X. In the case when the process X has Hölder continuous paths of order $\delta > \frac{1}{2}$ we can consider the following least squares estimator (LSE)

$$\widehat{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}, \quad t \geq 0, \tag{1.2}$$

as estimator of θ, where the integral with respect to X is a Young integral. The estimator $\widehat{\theta}_t$ is obtained by the least squares technique, that is, $\widehat{\theta}_t$ (formally) minimizes

$$\theta \longmapsto \int_0^t \left| \dot{X}_s + \theta X_s \right|^2 ds.$$

Moreover, using the formula (2.5) we can rewrite $\widehat{\theta}_t$ as follows,

$$\widehat{\theta}_t = \frac{X_t^2}{2\int_0^t X_s^2 ds}, \quad t \geq 0. \tag{1.3}$$

Motivated by (1.3) we propose to use, in the general case, the right hand of (1.3) as a statistic to estimate the drift coefficient θ of the equation (1.1). More precisely, we define

$$\widetilde{\theta}_t = \frac{X_t^2}{2\int_0^t X_s^2 ds}, \quad t \geq 0. \tag{1.4}$$

We shall provide sufficient conditions, based the properties of G, under which the estimator $\widetilde{\theta}_t$ is consistent (see Theorem 3.1), and the limit distribution of $\widetilde{\theta}_t$ is a standard Cauchy distribution (see Theorem 3.2). More precisely,

$$\widetilde{\theta}_t \to \theta \text{ almost surely as } t \to \infty,$$

and

$$e^{\theta t} \left(\widetilde{\theta}_t - \theta \right) \overset{\texttt{law}}{\longrightarrow} 2\theta \mathcal{C}(1) \quad \text{as } t \longrightarrow \infty,$$

with $\mathcal{C}(1)$ the standard Cauchy distribution with the probability density function $\frac{1}{\pi(1+x^2)}$; $x \in \mathbb{R}$.

Remark 1.1. *Suppose that G has Hölder continuous paths of order $\delta \in (\frac{1}{2}, 1)$. Then the process X has δ-Hölder continuous paths which implies that the estimator $\widetilde{\theta}_t$ coincides with the LSE $\widehat{\theta}_t$ by using (2.5). This property is satisfied in the cases of fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, sub-fractional Brownian motion with parameter $H > \frac{1}{2}$ and bifractional Brownian motion with parameters $(H, K) \in (0, 1)^2$ such that $HK > \frac{1}{2}$ (see Section 4).*
Indeed, let us prove that X has δ-Hölder continuous paths. From (3.7), it suffices to prove that the process Z given in (3.8) has δ-Hölder continuous paths. Furthermore, Mean Value Theorem and the continuity of G entail that Z has $(1-\varepsilon)$-Hölder continuous paths for all $\varepsilon \in (0, 1)$. Thus, the result is obtained.

Examples of the Gaussian process G.

<u>*Fractional Borwnian motion*</u>:
Suppose that the process G, given in (1.1), is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. By assuming that $H > \frac{1}{2}$, [4] studied the LSE $\widehat{\theta}_t$ which coincides, in this case, with $\widetilde{\theta}_t$ by Remark 1.1. In this paper, we extend the result of [4] to the case $H \in (0, 1)$. Moreover, we offer an elementary proof (see Section 4.1).

<u>*Sub-fractional Borwnian motion*</u>:
Assume that the process G, given in (1.1), is a subfractional Brownian motion with parameter $H \in (0, 1)$. For $H > \frac{1}{2}$, using an idea of [4], [14] established the LSE $\widehat{\theta}_t$ which also coincides with $\widetilde{\theta}_t$. But the proof of Lemma 4.3 in [14] relies on a possibly awed technique, because the passage from line -7 to -6 on page 671 does not allows to obtain the convergence $E\left[\left(e^{-\theta t}\int_0^t e^{\theta s}dS_s^H\right)^2\right]$ as $t \to \infty$. In the present paper, we give a solution of this problem and we extend the result to $H \in (0, 1)$ (see Section 4.2).

<u>*Bifractional Borwnian motion*</u>:
To the best of our knowledge there is no study of the problem of estimating the drift of (1.1) in the case when G is a bifractional Borwnian motion with parameters $(H, K) \in (0, 1)^2$. Section 4.3 is devoted to studying this parameter estimation problem.

2 Young integral

For any $\alpha \in [0, 1]$, we denote by $\mathscr{C}^\alpha([0, T])$ the set of α-Hölder continuous functions, that is, the set of functions $f : [0, T] \to \mathbb{R}$ such that

$$|f|_\alpha := \sup_{0 \le s < t \le T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty.$$

(Notice the calligraphic difference between a space $\mathscr{C}$ of Hölder continuous functions, and a space $\mathcal{C}$ of continuously differentiable functions!). We also set $|f|_\infty = \sup_{t\in[0,T]} |f(t)|$, and we equip $\mathscr{C}^\alpha([0, T])$ with the norm

$$\|f\|_\alpha := |f|_\alpha + |f|_\infty.$$

Let $f \in \mathscr{C}^\alpha([0, T])$, and consider the operator $T_f : \mathcal{C}^1([0, T]) \to \mathcal{C}^0([0, T])$ defined as

$$T_f(g)(t) = \int_0^t f(u)g'(u)du, \quad t \in [0, T].$$

It can be shown (see, e.g., [15, Section 3.1]) that, for any $\beta \in (1-\alpha, 1)$, there exists a constant $C_{\alpha,\beta,T} > 0$ depending only on α, β and T such that, for any $g \in \mathscr{C}^\beta([0,T])$,

$$\left\| \int_0^\cdot f(u)g'(u)du \right\|_\beta \leq C_{\alpha,\beta,T} \|f\|_\alpha \|g\|_\beta.$$

We deduce that, for any $\alpha \in (0,1)$, any $f \in \mathscr{C}^\alpha([0,T])$ and any $\beta \in (1-\alpha, 1)$, the linear operator $T_f : \mathcal{C}^1([0,T]) \subset \mathscr{C}^\beta([0,T]) \to \mathscr{C}^\beta([0,T])$, defined as $T_f(g) = \int_0^\cdot f(u)g'(u)du$, is continuous with respect to the norm $\|\cdot\|_\beta$. By density, it extends (in an unique way) to an operator defined on $\mathscr{C}^\beta$. As consequence, if $f \in \mathscr{C}^\alpha([0,T])$, if $g \in \mathscr{C}^\beta([0,T])$ and if $\alpha + \beta > 1$, then the (so-called) Young integral $\int_0^\cdot f(u)dg(u)$ is (well) defined as being $T_f(g)$.

The Young integral obeys the following formula. Let $f \in \mathscr{C}^\alpha([0,T])$ with $\alpha \in (0,1)$ and $g \in \mathscr{C}^\beta([0,T])$ for all $\beta \in (0,1)$. Then $\int_0^\cdot g_u df_u$ and $\int_0^\cdot f_u dg_u$ are well-defined as Young integrals. Moreover, for all $t \in [0,T]$,

$$f_t g_t = f_0 g_0 + \int_0^t g_u df_u + \int_0^t f_u dg_u. \tag{2.5}$$

3 Asymptotic behavior of the estimator

Let $G = (G_t, t \geq 0)$ be a continuous centered Gaussian process, defined on some probability space $(\Omega, \mathcal{F}, P)$. (Here, and throughout the text, we do assume that $\mathcal{F}$ is the sigma-field generated by G.)
The following assumptions are required.

(A1) The process G has Hölder continuous paths of order strictly positive.

(A2) For every $t \geq 0$, $E\left(G_t^2\right) \leq ct^{2\gamma}$ with the constants $c, \gamma > 0$.

3.1 Strong consistency

We will prove that the estimator $\widetilde{\theta}_t$ given by (1.4) is strongly consistent.
It is clear that the linear equation (1.1) has the following explicit solution

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} dG_s, \qquad t \geq 0, \tag{3.6}$$

where the integral in interpreted is the Young sense.
Furthermore, applying the formula (2.5) we can write

$$X_t = G_t - \theta e^{\theta t} Z_t, \qquad t \geq 0, \tag{3.7}$$

where

$$Z_t := \int_0^t e^{-\theta s} G_s ds, \qquad t \geq 0. \tag{3.8}$$

Let us introduce the following process

$$\xi_t := \int_0^t e^{-\theta s} dG_s, \qquad t \geq 0.$$

Thus, we can also write

$$\xi_t = e^{-\theta t} G_t - \theta Z_t, \qquad t \geq 0. \tag{3.9}$$

We will analyze separately the nominator and the denominator in the right hand side of the estimator (1.4). To study these estimations, we need the following lemmas.

Lemma 3.1. *Assume that (A1) and (A2) hold. Let Z be the process defined in (3.8). Then, $Z_\infty = \int_0^\infty e^{-\theta s} G_s ds$ is well defined, and as $t \to \infty$*

$$Z_t \longrightarrow Z_\infty \quad \textit{almost surely and in } L^2(\Omega). \tag{3.10}$$

Thus, also

$$\xi_t \longrightarrow \xi_\infty := -\theta Z_\infty \tag{3.11}$$

almost surely and in $L^2(\Omega)$ as $t \to \infty$.

Lemma 3.2. *Assume that (A1) and (A2) hold. Then, as $t \to \infty$,*

$$e^{-2\theta t} \int_0^t X_s^2 ds = e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds \longrightarrow \frac{\xi_\infty^2}{2\theta} = \frac{\theta}{2} Z_\infty^2 \textit{ almost surely.}$$

Proof of Lemma 3.1. We first notice that the integral $Z_\infty = \int_0^\infty e^{-\theta s} G_s ds$ is well defined because

$$\int_0^\infty e^{-\theta s} E(|G_s|) ds \leq \sqrt{c} \int_0^\infty s^\gamma e^{-\theta s} ds < \infty.$$

Now, we prove (3.10). By using Borel-Cantelli lemma, it is sufficient to prove that, for any $\varepsilon > 0$

$$\sum_{n \geq 0} P\left(\sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} G_s ds \right| > \varepsilon \right) < \infty.$$

On the other hand, for every $\varepsilon > 0$,

$$\begin{aligned} E\left(\sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} G_s ds \right| \right) &\leq E\left(\int_n^\infty e^{-\theta s} \left| G_s \right| ds \right) \\ &\leq \sqrt{c} \int_n^\infty e^{-\theta s} s^\gamma ds \\ &\leq \sqrt{c} e^{-\frac{\theta}{2} n} \int_0^\infty e^{-\frac{\theta}{2} s} s^\gamma ds \\ &= \sqrt{c} \Gamma(1+\gamma) \left(\frac{2}{\theta} \right)^{1+\gamma} e^{-\frac{\theta}{2} n}. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{n \geq 0} P\left(\sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} dG_s \right| > \varepsilon \right) &\leq \varepsilon^{-1} \sum_{n \geq 0} E\left(\sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} dG_s \right| \right) \\ &\leq \varepsilon^{-1} \sqrt{c} \Gamma(1+\gamma) \left(\frac{2}{\theta} \right)^{1+\gamma} \sum_{n \geq 0} e^{-\frac{\theta}{2} n} < \infty, \end{aligned}$$

which imply that $Z_t \longrightarrow Z_\infty$ almost surely as $t \to \infty$. Moreover, since

$$\begin{aligned} E\left[(Z_t - Z_\infty)^2\right] &= \int_t^\infty \int_t^\infty e^{-\theta r} e^{-\theta s} E\left(G_r G_s\right) drds \\ &\leq c \int_t^\infty \int_t^\infty e^{-\theta r} e^{-\theta s} \left(rs\right)^\gamma drds \\ &= c \left(\int_t^\infty e^{-\theta s} s^\gamma ds \right)^2 \\ &\to 0 \text{ as } t \to \infty \end{aligned}$$

the proof of the claim (3.10) is finished. For the convergence (3.11), it is a direct consequence of (3.10) and (3.9). Thus the proof of Lemma 3.1 is done. □

Proof of Lemma 3.2. It follows from (3.11) that $\xi_\infty \backsim \mathcal{N}\left(0, E\left[\xi_\infty^2\right]\right)$, where

$$\begin{aligned} E\left[\xi_\infty^2\right] = \theta^2 E\left[Z_\infty^2\right] &= \theta^2 \int_0^\infty \int_0^\infty e^{-\theta r} e^{-\theta s} E\left(G_r G_s\right) drds \\ &\leq c\theta^2 \int_0^\infty \int_0^\infty e^{-\theta r} e^{-\theta s} \left(rs\right)^\gamma drds \\ &= c \left(\frac{\Gamma(\gamma+1)}{\theta^\gamma} \right)^2 < \infty. \end{aligned}$$

This imply that

$$P(\xi_\infty = 0) = 0. \tag{3.12}$$

The continuity of ξ entails that, for every $t \geq 0$

$$\int_0^t e^{2\theta s} \xi_s^2 ds \geq \int_{\frac{t}{2}}^t e^{2\theta s} \xi_s^2 ds \geq \frac{t}{2} e^{\theta t} \left(\inf_{\frac{t}{2} \leq s \leq t} \xi_s^2 \right) \quad \text{almost surely.} \tag{3.13}$$

Furthermore, the continuity of ξ and (3.11) yield

$$\lim_{t\to\infty} \left(\inf_{\frac{t}{2} \leq s \leq t} \xi_s^2 \right) = \xi_\infty \quad \text{almost surely.}$$

Combining this last convergence with (3.12) and (3.13), we deduce that

$$\lim_{t\to\infty} \int_0^t e^{2\theta s} \xi_s^2 ds = \infty \quad \text{almost surely.}$$

Hence, we can use L'Hôspital's rule and we obtain

$$\lim_{t\to\infty} \frac{\int_0^t e^{2\theta s} \xi_s^2 ds}{e^{2\theta t}} = \lim_{t\to\infty} \frac{\xi_t^2}{2\theta} = \frac{\xi_\infty^2}{2\theta} \qquad \text{almost surely,}$$

which completes the proof of Lemma 3.2. □

The following theorem gives the strong consistency of the estimator $\widetilde{\theta}_t$.

Theorem 3.1. *Assume that (A1) and (A2) hold and let $\widetilde{\theta}_t$ be given by (1.4). Then*

$$\widetilde{\theta}_t \to \theta \text{ almost surely as } t \to \infty. \tag{3.14}$$

Proof. By (3.7), we can write

$$\begin{aligned}
\widetilde{\theta}_t &= \frac{G_t^2 + \theta^2 e^{2\theta t} Z_t^2 - 2\theta e^{\theta t} G_t Z_t}{2\int_0^t e^{2\theta s}\xi_s^2 ds} \\
&= \frac{e^{-2\theta t} G_t^2 + \theta^2 Z_t^2 - 2\theta G_t e^{-\theta t} Z_t}{2e^{-2\theta t}\int_0^t e^{2\theta s}\xi_s^2 ds}.
\end{aligned} \tag{3.15}$$

Moreover, the hypothesis (A2) implies that $G_t e^{-\theta t} \to 0$ almost surely as $t \to \infty$. Combining this with (3.15), Lemma 3.1 and Lemma 3.2, the convergence (3.14) holds true. □

3.2 Asymptotic distribution

This section is devoted to the investigation of asymptotic distribution of the estimator $\widetilde{\theta}_t$ of θ. Then, the following assumptions are required.

(A3) The limiting variance of $e^{-\theta t}\int_0^t e^{\theta s} dG_s$ exists as $t \to \infty$ i.e., there exists a constant $\sigma_G > 0$ such that

$$\lim_{t\to\infty} E\left[\left(e^{-\theta t}\int_0^t e^{\theta s} dG_s\right)^2\right] \longrightarrow \sigma_G.$$

(A4) For all fixed $s \geq 0$

$$\lim_{t\to\infty} E\left(G_s e^{-\theta t}\int_0^t e^{\theta r} dG_r\right) = 0.$$

We start with the following lemma.

Lemma 3.3. *Assume that (A1) holds true. Then, for every $t \geq 0$, we have*

$$\frac{1}{2}X_t^2 = \theta\int_0^t X_s^2 ds - \theta Z_t \int_0^t e^{\theta s} dG_s + R_t,$$

where

$$R_t := \frac{1}{2}G_t^2 - \theta\int_0^t G_s^2 ds - \theta^2\int_0^t ds\int_0^s dr G_s G_r e^{-\theta(s-r)}.$$

Proof. Let $t \geq 0$. Setting $\eta_t = \int_0^t X_s ds$, the equation (1.1) leads to

$$\frac{1}{2}X_t^2 = \frac{1}{2}\theta^2\eta_t^2 + \frac{1}{2}G_t^2 + \theta\eta_t G_t.$$

Moreover, (2.5) and (1.1) entail

$$\begin{aligned}\frac{1}{2}\eta_t^2 &= \int_0^t \eta_s d\eta_s = \int_0^t \eta_s X_s ds \\ &= \theta^{-1}\left(\int_0^t X_s^2 ds - \int_0^t G_s X_s ds\right).\end{aligned}$$

Define $Y_t := \int_0^t e^{\theta s} G_s ds$. Then, by (3.7) and (2.5)

$$\begin{aligned}\int_0^t G_s X_s ds &= \int_0^t e^{\theta s} G_s \left(e^{-\theta s} G_s - \theta Z_s\right) ds \\ &= \int_0^t G_s^2 ds - \theta \int_0^t e^{\theta s} G_s Z_s ds \\ &= \int_0^t G_s^2 ds - \theta \int_0^t Z_s dY_s \\ &= \int_0^t G_s^2 ds - \theta Z_t Y_t + \theta \int_0^t Y_s dZ_s \\ &= \int_0^t G_s^2 ds - \theta Z_t Y_t + \theta \int_0^t ds \int_0^s dr G_s G_r e^{-\theta(s-r)}\end{aligned}$$

Thus, we deduce that

$$\frac{1}{2}X_t^2 = \theta \int_0^t X_s^2 ds + \theta^2 Z_t Y_t + \theta \eta_t G_t + R_t. \tag{3.16}$$

On the other hand, by (1.1) and (3.7) we get

$$\begin{aligned}\theta \eta_t G_t &= G_t \left(X_t - G_t\right) \\ &= -\theta e^{\theta t} G_t Z_t.\end{aligned}$$

This implies that

$$\theta^2 Z_t Y_t + \theta \eta_t G_t = -\theta Z_t (e^{\theta t} G_t - \theta Y_t) \quad = \quad -\theta Z_t \int_0^t e^{\theta s} dG_s.$$

Combining this with (3.16) the proof of Lemma 3.3 is done. □

Lemma 3.4. *Assume that (A1), (A3) and (A4) hold. Let F be any $\sigma\{B\}$-measurable random variable such that $P(F < \infty) = 1$. Then, as $t \to \infty$,*

$$\left(F, e^{-\theta t} \int_0^t e^{\theta s} dG_s\right) \xrightarrow{law} \left(F, \sigma_G N\right),$$

where $N \sim \mathcal{N}(0,1)$ is independent of B.

Proof. For any $d \geq 1$, $s_1 \ldots s_d \in [0, \infty)$, we shall prove that, as $t \to \infty$,

$$\left(B_{s_1}, \ldots, B_{s_d}, e^{-\theta t} \int_0^t e^{\theta s} dG_s\right) \xrightarrow{\text{law}} \left(B_{s_1}, \ldots, B_{s_d}, \sigma_G N\right) \tag{3.17}$$

which is enough to lead to the desired conclusion. Because the left-hand side in the previous convergence is a Gaussian vector (see proof of [Lemma 7, [8]]), to get (3.17) it is sufficient to check the convergence of its covariance matrix. Thus, the assumptions (A3) and (A4) complete the proof. □

Theorem 3.2. *Assume that (A1), (A2), (A3) and (A4) hold. Then, as $t \to \infty$,*

$$e^{\theta t}\left(\widetilde{\theta}_t - \theta\right) \overset{law}{\longrightarrow} 2\theta\mathcal{C}(1), \tag{3.18}$$

with $\mathcal{C}(1)$ the standard Cauchy distribution.

Proof. We can write

$$\begin{aligned} e^{\theta t}\left(\widetilde{\theta}_t - \theta\right) &= \frac{-e^{-\theta t}\int_0^t e^{\theta s} dG_s}{Z_\infty} \times \frac{\theta Z_t Z_\infty}{e^{-2\theta t}\int_0^t X_s^2 ds} + \frac{e^{-\theta t} R_t}{e^{-2\theta t}\int_0^t X_s^2 ds} \\ &:= a_t \times b_t + c_t. \end{aligned}$$

Lemma 3.4 yields, as $t \to \infty$,

$$a_t \overset{\texttt{law}}{\longrightarrow} \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \frac{N}{Z_\infty},$$

where $N \sim \mathcal{N}(0,1)$ is independent of G, whereas Lemmas 3.1 and 3.2 imply that $b_t \longrightarrow 2$ almost surely as $t \to \infty$.

On the other hand , $e^{-\theta t} R_t \longrightarrow 0$ in $L^1(\Omega)$ as $t \to \infty$ because, as $t \to \infty$,

$$\begin{aligned} e^{-\theta t} E\left(G_t^2\right) \leq c t^{2\gamma} e^{-\theta t} \longrightarrow 0, \\ e^{-\theta t}\int_0^t E\left(G_s^2\right) ds \leq c\frac{t^{2\gamma+1}}{2\gamma+1} e^{-\theta t} \longrightarrow 0, \end{aligned}$$

and

$$e^{-\theta t}\int_0^t ds \int_0^s dr E\left|G_s G_r\right| e^{-\theta(s-r)} \leq c e^{-\theta t}\int_0^t ds \int_0^s dr (sr)^\gamma = \frac{t^{2\gamma+2}}{(\gamma+1)(2\gamma+2)} e^{-\theta t} \longrightarrow 0.$$

Combining this with Lemma 3.2 we obtain that $c_t \longrightarrow 0$ in probability as $t \to \infty$.

By plugging all these convergences together we get that, as $t \to \infty$,

$$e^{\theta t}\left(\widetilde{\theta}_t - \theta\right) \overset{\texttt{law}}{\longrightarrow} 2\frac{\sqrt{H\Gamma(2H)}}{\theta^H} \frac{N}{Z_\infty}.$$

Moreover, $Z_\infty \sim \mathcal{N}\left(0, E\left[Z_\infty^2\right]\right)$ which completes the proof. □

4 Applications to fractional Gaussian processes

This section is devoted to some examples of the Gaussian process G given in (1.1). We will need the following technical lemmas.

Lemma 4.1. *Let $g : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a symmetric function such that its first order partial derivatives and $\frac{\partial^2 g}{\partial s \partial r}(s,r)$ are Hölder continuous functions of orders strictly positives. Then, for every $t \geq 0$,*

$$\begin{aligned}\Delta_g(t) &:= g(t,t) - 2\theta e^{-\theta t}\int_0^t g(s,t)e^{\theta s}ds + \theta^2 e^{-2\theta t}\int_0^t\int_0^t g(s,r)e^{\theta(s+r)}drds \\ &= 2e^{-2\theta t}\int_0^t e^{\theta s}\frac{\partial g}{\partial s}(s,0)ds + 2e^{-2\theta t}\int_0^t dse^{\theta s}\int_0^s dr\frac{\partial^2 g}{\partial s\partial r}(s,r)e^{\theta r}. \qquad (4.19)\end{aligned}$$

Proof. Set $h(s) := \int_0^s g(s,r)e^{\theta r}dr$. Combining (2.5) together with

$$\frac{\partial h}{\partial s}(s) = e^{\theta s}g(s,s) + \int_0^s \frac{\partial g}{\partial s}(s,r)e^{\theta r}dr$$

we obtain

$$\begin{aligned}\Delta_g(t) &= g(t,t) - 2\theta e^{-2\theta t}\int_0^t g(s,s)e^{2\theta s}ds - 2\theta e^{-2\theta t}\int_0^t dse^{\theta s}\int_0^s dr\frac{\partial g}{\partial s}(s,r)e^{\theta r} \\ &= e^{-2\theta t}\int_0^t \frac{\partial g(s,s)}{\partial s}(s)e^{2\theta s}ds - 2\theta e^{-2\theta t}\int_0^t dse^{\theta s}\int_0^s dr\frac{\partial g}{\partial s}(s,r)e^{\theta r}.\end{aligned}$$

Since g is symmetric we have for $r = s$, $2\frac{\partial g}{\partial s}(s,r) = \frac{\partial g(s,s)}{\partial s}(s)$. Thus by using again (2.5), the claim (4.19) is obtained. □

Lemma 4.2. *Let $\lambda > -1$. Define*

$$J_\lambda(t) := e^{-2\theta t}\int_0^t\int_0^t e^{\theta s}e^{\theta r}|s-r|^\lambda drds; \quad I_\lambda(t) := e^{-\theta t}\int_0^t e^{\theta s}(t-s)^\lambda drds.$$

Then

$$\lim_{t\to\infty} J_\lambda(t) = \lim_{t\to\infty}\left(\frac{1}{\theta}I_\lambda(t)\right) = \frac{\Gamma(\lambda+1)}{\theta^{\lambda+2}}. \qquad (4.20)$$

Proof. Let $t \geq 0$. We have

$$\begin{aligned}J_\lambda(t) &= 2e^{-2\theta t}\int_0^t dse^{\theta s}\int_0^s dre^{\theta r}(s-r)^\lambda \\ &= 2e^{-2\theta t}\int_0^t dse^{2\theta s}\int_0^s dre^{-\theta u}u^\lambda \\ &= 2e^{-2\theta t}\int_0^t due^{-\theta u}u^\lambda\int_u^t dse^{2\theta s} \\ &= \frac{1}{\theta}\left(\int_0^t u^\lambda e^{-\theta u}du - e^{-2\theta t}\int_0^t u^\lambda e^{\theta u}du\right) \\ &\to \frac{\Gamma(\lambda+1)}{\theta^{\lambda+2}} \text{ as } t\to\infty,\end{aligned}$$

which proves (4.20). □

4.1 Fractional Browian motion

The fractional Brownian motion (fBm) $B^H = \left(B_t^H, t \geq 0\right)$ with Hurst parameter $H \in (0,1)$, is defined as a centered Gaussian process starting from zero with covariance

$$E\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Note that, when $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is a standard Brownian motion.

Proposition 4.1. *Suppose that, in (1.1), the process G is the fBm B^H. Then for all fixed $H \in (0,1)$ the convergences (3.14) and (3.18) hold.*

Proof. By Kolmogorov's continuity criterion and the fact

$$E\left(B_t^H - B_s^H\right)^2 = |s-t|^{2H};\ s,\ t \geq\ 0,$$

we deduce that B^H has Hölder continuous paths of order $H - \varepsilon$, for all $\varepsilon \in (0, H)$. So, the process B^H satisfies the assumptions (A1) and (A2) in the case $G = B^H$. Thus, by Theorem 3.1 the convergence (3.14) is obtained.
For the convergence (3.18), it suffices to check (A3) and (A4). Let us first compute the limiting variance of $e^{-\theta t}\int_0^t e^{\theta s} dB_s^H$ as $t \to \infty$. We have

$$\begin{aligned}
& E\left[\left(e^{-\theta t}\int_0^t e^{\theta s} dB_s^H\right)^2\right] \\
= & E\left[\left(e^{-\theta t}\left(e^{\theta t}B_t^H - \theta\int_0^t e^{\theta s}B_s^H ds\right)\right)^2\right] \\
= & t^{2H} - 2\theta e^{-\theta t}\int_0^t e^{\theta s}E(B_s^H B_t^H)ds + \theta^2 e^{-2\theta t}\int_0^t\int_0^t e^{\theta s}e^{\theta r}E(B_s^H B_r)dsdr \\
= & t^{2H} - \theta e^{-\theta t}\int_0^t e^{\theta s}\left(s^{2H} + t^{2H} - (t-s)^{2H}\right)ds \\
& + \frac{1}{2}\theta^2 e^{-2\theta t}\int_0^t\int_0^t e^{\theta s}e^{\theta r}\left(s^{2H} + r^{2H} - |r-s|^{2H}\right)dsdr \\
= & \Delta_{g_{B^H}}(t) + \theta I_{2H}(t) - \frac{\theta^2}{2}J_{2H}(t), \qquad (4.21)
\end{aligned}$$

where $g_{B^H}(s,r) = \frac{1}{2}\left(s^{2H} + t^{2H}\right)$.
On the other hand, (4.19) implies that

$$\Delta_{g_{B^H}}(t) = 2He^{-2\theta t}\int_0^t s^{2H-1}e^{\theta s}ds \quad\to\quad 0 \text{ as } t \to \infty. \qquad (4.22)$$

Combining (4.21), (4.22) and (4.20) we get for every $H \in (0,1)$

$$E\left[\left(e^{-\theta t}\int_0^t e^{\theta s} dB_s^H\right)^2\right] \quad\longrightarrow\quad \frac{H\Gamma(2H)}{\theta^{2H}} \quad \text{as } t \to \infty.$$

Hence, to finish the proof it remains to check that, for all fixed $s \geq 0$

$$\lim_{t\to\infty} E\left(B_s^H e^{-\theta t}\int_0^t e^{\theta r} dB_r^H\right) = 0.$$

Let us consider $s < t$. Setting $f_{B^H}(s,r) = E(B_s^H B_r^H)$, it follows from (2.5) that

$$\begin{aligned}
E\left(B_s^H e^{-\theta t}\int_0^t e^{\theta r} dB_r^H\right) &= f_{B^H}(s,t) - \theta e^{-\theta t}\int_0^t e^{\theta r} f_{B^H}(s,r)dr \\
&= f_{B^H}(s,t) - \theta e^{-\theta t}\int_s^t e^{\theta r} f_{B^H}(s,r)dr - \theta e^{-\theta t}\int_0^s e^{\theta r} f_{B^H}(s,r)dr \\
&= e^{-\theta(t-s)} f_{B^H}(s,s) + e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial f_{B^H}}{\partial r}(s,r)dr - \theta e^{-\theta t}\int_0^s e^{\theta r} f_{B^H}(s,r)dr
\end{aligned}$$

It is clear that $e^{-\theta(t-s)} f_{B^H}(s,s) - \theta e^{-\theta t}\int_0^s e^{\theta r} f_{B^H}(s,r)dr \longrightarrow 0$ as $t \to \infty$.
Furthermore, if $H = \frac{1}{2}$, $\frac{\partial f_{B^H}}{\partial r}(s,r) = 0$ for every $r > s$. Then, for $H = \frac{1}{2}$

$$e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial f_{B^H}}{\partial r}(s,r)dr = 0.$$

Now, suppose that $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Since

$$\begin{aligned}
\int_s^t e^{\theta r}\left|r^{2H-1} - (r-s)^{2H-1}\right| dr &\geq |2H-1|s\int_s^t e^{\theta r} r^{2H-2} dr \\
&\geq |2H-1|st^{2H-2}\int_s^t e^{\theta r} dr \\
&\to \infty \text{ as } t \to \infty,
\end{aligned}$$

we can apply L'Hôspital's rule and we obtain

$$\begin{aligned}
\lim_{t\to\infty}\left|e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial f_{B^H}}{\partial r}(s,r)dr\right| &= \lim_{t\to\infty}\left|He^{-\theta t}\int_s^t e^{\theta r}\left(r^{2H-1} - (r-s)^{2H-1}\right) dr\right| \\
&\leq \lim_{t\to\infty}\left(He^{-\theta t}\int_s^t e^{\theta r}\left|r^{2H-1} - (r-s)^{2H-1}\right| dr\right) \\
&= \lim_{t\to\infty}\left(\frac{H}{\theta}\left|t^{2H-1} - (t-s)^{2H-1}\right|\right) \\
&\leq \lim_{t\to\infty}\left(\frac{H|2H-1|}{\theta}t^{2H-2}\right) \\
&\to 0 \text{ as } t \to \infty
\end{aligned} \tag{4.23}$$

which finishes the proof of Proposition 4.1. □

4.2 Sub-fractional Browian motion

The sub-fractional Brownian motion (sfBm) S^H with parameter $H \in (0,1)$ is a centred Gaussian process with covariance function

$$E\left(S_t^H S_s^H\right) = t^{2H} + s^{2H} - \frac{1}{2}\left((t+s)^{2H} + |t-s|^{2H}\right).$$

Note that, when $H = \frac{1}{2}$, $S^{\frac{1}{2}}$ is a standard Brownian motion.

Proposition 4.2. *Suppose that, in (1.1), the process G is the sfBm S^H. Then for all fixed $H \in (0,1)$ the convergences (3.14) and (3.18) hold.*

Proof. By Kolmogorov's continuity criterion and the fact

$$E\left(S_t^H - S_s^H\right)^2 \leq (2 - 2^{2H-1})|s-t|^{2H}; \ s, \ t \geq \ 0,$$

we deduce that S^H has Hölder continuous paths of order $H-\varepsilon$, for all $\varepsilon \in (0,H)$. So, the process S^H satisfies the assumptions (A1) and (A2). Thus, by Theorem 3.1 the convergence (3.14) is obtained.

To prove (3.18), it suffices to check (A3) and (A4). The case $H = \frac{1}{2}$ has already been established above. Suppose now that $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Using the same argument as in (4.21), we get

$$E\left[\left(e^{-\theta t}\int_0^t e^{\theta s} dS_s^H\right)^2\right] = \Delta_{g_{S^H}}(t) + \theta I_{2H}(t) - \frac{\theta^2}{2} J_{2H}(t), \tag{4.24}$$

where $g_{S^H}(s,r) = s^{2H} + t^{2H} - \frac{1}{2}(s+t)^{2H}$.

Moreover, we have

$$\Delta_{g_{S^H}}(t) = 2He^{-2\theta t}\int_0^t s^{2H-1}e^{\theta s}ds - 2H(2H-1)e^{-2\theta t}\int_0^t ds e^{\theta s}\int_0^s dr e^{\theta r}(s+r)^{2H-2}.$$

It is easy to see that $2He^{-2\theta t}\int_0^t s^{2H-1}e^{\theta s}ds \to 0$ as $t \to \infty$.

Furthermore, using the fact that

$$\begin{aligned}
\int_0^t ds e^{\theta s}\int_0^s dr e^{\theta r}(s+r)^{2H-2} &\geq (2t)^{2H-2}\int_0^t ds e^{\theta s}\int_0^s dr e^{\theta r} \\
&= \frac{(2t)^{2H-2}}{2}\left(\int_0^t e^{2\theta s} ds\right)^2 \\
&\to \infty \text{ as } t \to \infty,
\end{aligned}$$

L'Hôspital's rule entails

$$\begin{aligned}
\lim_{t\to\infty}\left(e^{-2\theta t}\int_0^t ds e^{\theta s}\int_0^s dr e^{\theta r}(s+r)^{2H-2}\right) &= \lim_{t\to\infty}\left(\frac{1}{2\theta}e^{-\theta t}\int_0^t e^{\theta r}(t+r)^{2H-2}dr\right) \\
&\leq \lim_{t\to\infty}\left(\frac{t^{2H-2}}{2\theta}e^{-\theta t}\int_0^t e^{\theta r}dr\right) \\
&\to 0 \quad \text{as } t \to \infty.
\end{aligned}$$

Thus, we deduce that

$$\Delta_{g_{S^H}}(t) \ \to \ 0 \text{ as } t \to \infty. \tag{4.25}$$

Combining (4.24), (4.25) and (4.20) we get

$$E\left[\left(e^{-\theta t}\int_0^t e^{\theta s} dS_s^H\right)^2\right] \longrightarrow \frac{H\Gamma(2H)}{\theta^{2H}} \quad \text{as } t \to \infty.$$

Hence, to finish the proof it remains to check that, for all fixed $s \geq 0$

$$\lim_{t\to\infty} E\left(S_s^H e^{-\theta t}\int_0^t e^{\theta r} dS_r^H\right) = 0.$$

Let us consider $s < t$ and let $f_{S^H}(s,r) = E(S_s^H S_r^H)$. Then, as in the fBm case, we can write

$$E\left(S_s^H e^{-\theta t}\int_0^t e^{\theta r} dS_r^H\right) = e^{-\theta(t-s)} f(s,s) + e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial f_{S^H}}{\partial r}(s,r)dr - \theta e^{-\theta t}\int_0^s e^{\theta r} f_{S^H}(s,r)dr.$$

It is clear that $e^{-\theta(t-s)} f_{S^H}(s,s) - \theta e^{-\theta t}\int_0^s e^{\theta r} f_{S^H}(s,r)dr \longrightarrow 0$ as $t \to \infty$.
On the other hand, since

$$e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial f_{S^H}}{\partial r}(s,r)dr = \frac{H}{2}e^{-\theta t}\int_s^t e^{\theta r}\left(2r^{2H-1} - (r+s)^{2H-1} - (r-s)^{2H-1}\right)dr,$$

the same argument as in (4.23) leads to

$$e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial f_{S^H}}{\partial r}(s,r)dr \quad\longrightarrow\quad 0 \qquad \text{as } t \to \infty.$$

$\square$

4.3 Bifractional Browian motion

Let $B^{H,K} = \left(B_t^{H,K}, t \geq 0\right)$ be a bifractional Brownian motion (bifBm) with parameters $H \in (0,1)$ and $K \in (0,1]$. This means that $B^{H,K}$ is a centered Gaussian process with the covariance function

$$E(B_s^{H,K} B_t^{H,K}) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right).$$

The case $K = 1$ corresponds to the fractional Brownian motion (fBm) with Hurst parameter H. The process $B^{H,K}$ verifies,

$$E\left(\left|B_t^{H,K} - B_s^{H,K}\right|^2\right) \leq 2^{1-K}|t-s|^{2HK},$$

so $B^{H,K}$ has $(HK - \varepsilon)-$Hölder continuous paths for any $\varepsilon \in (0, HK)$ thanks to Kolmogorov's continuity criterion. The bifBm $B^{H,K}$ can be extended for $1 < K < 2$ with $H \in (0,1)$ and $HK \in (0,1)$ (see [3] and [13]).

Proposition 4.3. *Suppose that, in (1.1), the process G is the bifBm $B^{H,K}$. Then the convergences (3.14) and (3.18) hold true for all fixed $(H,K) \in (0,1)^2$.*

Proof. Since $B^{H,K}$ has Hölder continuous paths of order $HK - \varepsilon$, for all $\varepsilon \in (0, HK)$, it satisfies the assumptions (A1) and (A2). Thanks to Theorem 3.1, the convergence (3.14) is obtained.
To prove (3.18), it suffices to check (A3) and (A4). Using the same argument as in (4.21), we have

$$E\left[\left(e^{-\theta t}\int_0^t e^{\theta s} dB_s^{H,K}\right)^2\right] = \Delta_{g_{B^{H,K}}}(t) + 2^{1-K}\theta I_{2HK}(t) - 2^{-K}\theta^2 J_{2HK}(t), \qquad (4.26)$$

where $g_{B^{H,K}}(s,r) = \frac{1}{2^K}\left(s^{2H}+r^{2H}\right)^K$.
On the other hand,

$$\begin{aligned}\Delta_{g_{B^{H,K}}}(t) &= 2^{2-K}HKe^{-2\theta t}\int_0^t s^{2HK-1}e^{\theta s}ds \\ &\quad -2^{3-K}H^2K(K-1)e^{-2\theta t}\int_0^t dse^{\theta s}\int_0^s dre^{\theta r}(sr)^{2H-1}\left(s^{2H}+r^{2H}\right)^{K-2}\end{aligned}$$

The convergence $2^{2-K}HKe^{-2\theta t}\int_0^t s^{2HK-1}e^{\theta s}ds \to 0$ as $t\to\infty$ is immediate.
Also, it is straightforward to check that there exist a constant $C_{H,K}$ depending on H,K such that

$$\begin{aligned}\int_0^t dse^{\theta s}\int_0^s dre^{\theta r}(sr)^{2H-1}\left(s^{2H}+r^{2H}\right)^{K-2} &\geq C_{H,K}t^{2HK-2}\int_{\frac{t}{2}}^t dse^{\theta s}\int_{\frac{s}{2}}^s dre^{\theta r} \\ &\geq \frac{C_{H,K}}{2}t^{2HK-2}\int_{\frac{t}{2}}^t dsse^{\frac{3\theta}{2}s} \\ &\to\infty \quad \text{as } t\to\infty.\end{aligned}$$

So, we can apply L'Hôspital's rule and we obtain

$$\begin{aligned}&\lim_{t\to\infty}\left(e^{-2\theta t}\int_0^t dse^{\theta s}\int_0^s dre^{\theta r}(sr)^{2H-1}\left(s^{2H}+r^{2H}\right)^{K-2}\right) \\ =\ &\lim_{t\to\infty}\left(\frac{e^{-\theta t}}{2\theta}\int_0^t e^{\theta r}(tr)^{2H-1}\left(t^{2H}+r^{2H}\right)^{K-2}dr\right) \\ \leq\ &\lim_{t\to\infty}\left(\frac{2^{K-3}e^{-\theta t}}{\theta}\int_0^t e^{\theta r}(tr)^{HK-1}dr\right) \\ \leq\ &\lim_{t\to\infty}\left(\frac{2^{K-3}t^{2HK-2}}{\theta}e^{-\theta t}\int_0^t e^{\theta r}dr\right) \\ \to 0\ &\quad \text{as } t\to\infty.\end{aligned}$$

Hence, for every $(H,K)\in(0,1)^2$

$$\Delta_{g_{B^{H,K}}}(t) \ \to\ 0 \text{ as } t\to\infty. \tag{4.27}$$

Consequently, (4.26), (4.27) and (4.20) imply

$$E\left[\left(e^{-\theta t}\int_0^t e^{\theta s}dB_s^{H,K}\right)^2\right] \longrightarrow \frac{HK\Gamma(2HK)}{\theta^{2HK}} \quad \text{as } t\to\infty.$$

Hence, to finish the proof it remains to check that, for all fixed $s\geq 0$

$$\lim_{t\to\infty}E\left(B_s^{H,K}e^{-\theta t}\int_0^t e^{\theta r}dB_r^{H,K}\right)=0.$$

Let us consider $s<t$ and let $f_{B^{H,K}}(s,r)=E(B_s^{H,K}B_r^{H,K})$. Then, as in the fBm case, we can write

$$\begin{aligned}&E\left(B_s^{H,K}e^{-\theta t}\int_0^t e^{\theta r}dB_r^{H,K}\right) \\ =\ &e^{-\theta(t-s)}f_{B^{H,K}}(s,s)+e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial f_{B^{H,K}}}{\partial r}(s,r)dr-\theta e^{-\theta t}\int_0^s e^{\theta r}f_{B^{H,K}}(s,r)dr.\end{aligned}$$

We have $e^{-\theta(t-s)} f_{B^{H,K}}(s,s) - \theta e^{-\theta t} \int_0^s e^{\theta r} f_{B^{H,K}}(s,r)dr \longrightarrow 0$ as $t \to \infty$.
Also,

$$e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B^{H,K}}}{\partial r}(s,r)dr = 2^{1-K} HK e^{-\theta t} \int_s^t e^{\theta r} \left(r^{2H-1} \left(s^{2H} + r^{2H}\right)^{K-1} - (r-s)^{2HK-1} \right) dr.$$

Hence, if $HK < \frac{1}{2}$, L'Hôspital's rule leads to

$$\begin{aligned}
\left| e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B^{H,K}}}{\partial r}(s,r)dr \right| &\leq 2^{1-K} HK e^{-\theta t} \int_s^t e^{\theta r} \left(r^{2HK-1} + (r-s)^{2HK-1} \right) dr \\
&\leq 2^{2-K} HK e^{-\theta t} \int_s^t e^{\theta r} (r-s)^{2HK-1} dr \\
&\longrightarrow \lim_{t\to\infty} \left(\frac{2^{2-K} HK}{\theta} (t-s)^{2HK-1} \right) = 0 \quad \text{as } t \to \infty.
\end{aligned}$$

If $HK = \frac{1}{2}$,

$$\begin{aligned}
\left| e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B^{H,K}}}{\partial r}(s,r)dr \right| &= 2^{-K} e^{-\theta t} \int_s^t e^{\theta r} \left(1 - \left(1 + \left(\frac{s}{r}\right)^{2H} \right)^{K-1} \right) dr \\
&\leq 2^{-K} e^{-\theta t} \left| \int_s^t e^{\theta r} \left(\left(1 + \left(\frac{s}{r}\right)^{2H} \right)^{K-1} - 1 \right) dr \right| \\
&\longrightarrow 0 \quad \text{as } t \to \infty.
\end{aligned}$$

The last convergence comes from $H > \frac{1}{2}$, the fact that for r large,

$$1 - \left(1 + \left(\frac{s}{r}\right)^{2H} \right)^{K-1} \leq 1 - \left(1 + \frac{s}{r} \right)^{K-1} \sim (1-K)\frac{s}{r},$$

and L'Hôspital's rule. Similarly, if $HK > \frac{1}{2}$,

$$\begin{aligned}
&\left| e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B^{H,K}}}{\partial r}(s,r)dr \right| \\
\leq\ & 2^{1-K} HK e^{-\theta t} \int_s^t e^{\theta r} r^{2HK-1} \left| \left(1 + \left(\frac{s}{r}\right)^{2H} \right)^{K-1} - \left(1 - \frac{s}{r} \right)^{2HK-1} \right| dr \\
\longrightarrow\ & 0 \quad \text{as } t \to \infty
\end{aligned}$$

which completes the proof. □

References

[1] Azmoodeh, E., Morlanes, I. (2013). Drift parameter estimation for fractional Ornstein-Uhlenbeck process of the Second Kind. Statistics: A Journal of Theoretical and Applied Statistics. DOI: 10.1080/02331888.2013.863888.

[2] Azmoodeh, E. and Viitasaari, L. (2013). Parameter estimation based on discrete observations of fractional Ornstein-Uhlenbeck process of the second kind. http://arxiv.org/abs/1304.2466.

[3] Bardina, X. and Es-Sebaiy, K. (2011). An extension of bifractional Brownian motion, Commun. Stochast. Analysis 5, pp. 333-340.

[4] Belfadli, R., Es-Sebaiy, K. and Ouknine, Y. (2011). Parameter Estimation for Fractional Ornstein-Uhlenbeck Processes: Non-Ergodic Case. *Frontiers in Science and Engineering* (An International Journal Edited by Hassan II Academy of Science and Technology). **1**, no. 1, 1-16.

[5] Cénac, P. and Es-Sebaiy, K. (2015). *Almost sure central limit theorems for random ratios and applications to LSE for fractional Ornstein-Uhlenbeck processes, to appear.*

[6] El Onsy, B., Es-Sebaiy, K. and Viens, F. (2014). Parameter Estimation for a partially observed Ornstein-Uhlenbeck process with long-memory noise. Preprint.

[7] El Onsy, B., Es-Sebaiy, K. and Tudor, C. (2014). Statistical analysis of the non-ergodic fractional Ornstein-Uhlenbeck process of the second kind. Preprint.

[8] Es-Sebaiy, K. and Nourdin, I. (2013). Parameter estimation for α-fractional bridges. Springer Proceedings in Mathematics and Statistics Volume 34, pp 385-412.

[9] Es-Sebaiy, K. and Viens, F. (2015). Parameter estimation for SDEs related to stationary Gaussian processes. Preprint.

[10] Hu, Y. and Nualart, D. (2010). Parameter estimation for fractional Ornstein-Uhlenbeck processes. Statistics and Probability Letters 80, 1030-1038.

[11] Hu, Y. and Song, J. (2013). Parameter estimation for fractional Ornstein- Uhlenbeck processes with discrete observations. F. Viens et al (eds), Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart, Springer Proceedings in Mathematics and Statistics **34**, 427-442.

[12] Kleptsyna, M. and Le Breton, A. (2002). Statistical analysis of the fractional Ornstein-Uhlenbeck type process. Statistical Inference for Stochastic Processes **5**, 229-241.

[13] Lifshits, M. and Volkova, K. (2015). Bifractional Brownian Motion: Existence and Border Cases. Preprint, http://arxiv.org/pdf/1502.02217.pdf.

[14] Mendy, I. (2013). Parametric estimation for sub-fractional Ornstein-Uhlenbeck process. Journal of Statistical Planning and Inference 143, 663-674.

[15] Nourdin, I. (2012). Selected aspects of fractional Brownian motion. Bocconi & Springer Series 4. Springer, Milan; Bocconi University Press, Milan.

[16] Tudor, C. and Viens, F. (2007). Statistical aspects of the fractional stochastic calculus. Ann. Statist. **35**, no. 3, 1183-1212.

[17] Young, L. C. (1936). An inequality of the Hölder type connected with Stieltjes integration. Acta Math. 67, 251-282.