On local linear regression for strongly mixing random fields

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Abstract

We investigate the local linear kernel estimator of the regression function g of a stationary and strongly mixing real random field observed over a general subset of the lattice $\mathbb{Z}^d$. Assuming that g is derivable with derivative g', we provide a new criterion on the mixing coefficients for the consistency and the asymptotic normality of the estimators of g and g' under very mild conditions on the bandwidth parameter. Our results improve the work of Hallin, Lu and Tran (2004) in several directions.

AMS Subject Classifications (2000): 62G05, 60J25, 62G07.
Key words and phrases: Local linear regression estimation, strong mixing, random fields, asymptotic normality.
Short title: Local linear regression for random fields.

1 Introduction and main results

In a variety of fields like soil science, geology, oceanography, econometrics, epidemiology, image processing and many others, the aim of practionners is to handle phenomenons observed on spatial sets. In particular, one of the fundamental question is the understanding of the phenomenon from a set of (dependent) observations based on regression models. In this work, we investigate the problem in the context of strongly mixing spatial processes (or random fields) and we focus on local linear regression estimation. More precisely, let d be a positive integer and let $\{(Y_i, X_i);, i \in \mathbb{Z}^d\}$ be a strictly stationary $\mathbb{R}^2$-valued random field defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The estimation of its regression function g defined by $g(x) = \mathbb{E}(Y_0|X_0 = x)$ for almost all real x is a natural question and a very important task in statistics. The nonspatial case, that is for dependent time series ($d = 1$), has been extensively studied. One can refer for example to Lu and Cheng [15], Masry and Fan [16], Robinson [19], Roussas [21] and many references therein. For $d \geqslant 2$, some contributions were done by Biau and Cadre [1], Carbon, Francq and Tran [2], El Machkouri [6], El Machkouri and Stoica [9], Dabo-Niang and Rachdi [3], Dabo-Niang and Yao [4], Hallin, Lu and Tran [10] and Lu and Chen [13], [14]. Given two σ-algebras $\mathcal{U}$ and $\mathcal{V}$, the α-mixing coefficient introduced by Rosenblatt [20] is defined by

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|\,,\ A \in \mathcal{U},\ B \in \mathcal{V}\}.$$

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Let p be fixed in $[1,\infty]$. The strong mixing coefficients $(\alpha_{1,p}(n))_{n\geqslant 0}$ associated to $\{(Y_i,X_i); i\in\mathbb{Z}^d\}$ are defined by

$$\alpha_{1,p}(n)=\sup\left\{\alpha(\sigma(Y_k,X_k),\mathcal{F}_\Gamma),\ k\in\mathbb{Z}^d,\ \Gamma\subset\mathbb{Z}^d,\ |\Gamma|\leqslant p,\ \rho(\Gamma,\{k\})\geqslant n\right\}$$

where $\mathcal{F}_\Gamma=\sigma(Y_i,X_i\,;\,i\in\Gamma)$, $|\Gamma|$ is the number of element in Γ and the distance ρ is defined for any subsets Γ_1 and Γ_2 of $\mathbb{Z}^d$ by $\rho(\Gamma_1,\Gamma_2)=\min\{|i-j|,\,i\in\Gamma_1,\,j\in\Gamma_2\}$ with $|i-j|=\max_{1\leqslant s\leqslant d}|i_s-j_s|$ for any $i=(i_1,..,i_d)$ and $j=(j_1,..,j_d)$ in $\mathbb{Z}^d$. We say that the random field $(Y_i,X_i)_{i\in\mathbb{Z}^d}$ is strongly mixing if $\lim_{n\to\infty}\alpha_{1,p}(n)=0$. Let x be fixed in $\mathbb{R}$. Following [10], we define the local linear kernel regression estimator of ${}^t(g(x),g'(x))$ by

$$\begin{pmatrix} g_n(x)\\ g_n'(x)\end{pmatrix}=\underset{(s,t)\in\mathbb{R}^2}{\mathrm{Argmin}}\sum_{i\in\Lambda_n}(Y_i-s-t(X_i-x))^2\,\mathrm{K}\left(\frac{X_i-x}{b_n}\right)\tag{1}$$

where b_n is the bandwidth parameter going to zero as n goes to infinity, Λ_n is a finite subset of $\mathbb{Z}^d$ which the number of elements $|\Lambda_n|$ goes to infinity as n goes to infinity and K is a probability kernel, that is a function $K:\mathbb{R}\to\mathbb{R}$ such that $\int_{\mathbb{R}}\mathrm{K}(s)ds=1$. We introduce the following notations:

$$u_{00}(n)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}\mathrm{K}\left(\frac{X_i-x}{b_n}\right),\quad u_{11}(n)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}\left(\frac{X_i-x}{b_n}\right)^2\mathrm{K}\left(\frac{X_i-x}{b_n}\right),$$

$$u_{01}(n)=u_{10}(n)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}\left(\frac{X_i-x}{b_n}\right)\mathrm{K}\left(\frac{X_i-x}{b_n}\right),$$

$$v_0(n)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}Y_i\mathrm{K}\left(\frac{X_i-x}{b_n}\right),\quad v_1(n)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}Y_i\left(\frac{X_i-x}{b_n}\right)\mathrm{K}\left(\frac{X_i-x}{b_n}\right),$$

$$w_0(n)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}Z_i\mathrm{K}\left(\frac{X_i-x}{b_n}\right)\quad\text{and}\quad w_1(n)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}Z_i\left(\frac{X_i-x}{b_n}\right)\mathrm{K}\left(\frac{X_i-x}{b_n}\right)$$

with $Z_i=Y_i-g(x)-g'(x)(X_i-x)$. A straightforward calculation gives

$$\begin{pmatrix} g_n(x)\\ g_n'(x)b_n\end{pmatrix}=U_n^{-1}V_n\quad\text{where}\quad U_n=\begin{pmatrix} u_{00}(n) & u_{10}(n)\\ u_{01}(n) & u_{11}(n)\end{pmatrix}\quad\text{and}\quad V_n=\begin{pmatrix} v_0(n)\\ v_1(n)\end{pmatrix}.$$

Denoting $W_n=V_n-U_n\,{}^t(g(x),g'(x)b_n)={}^t(w_0(n),w_1(n))$, we obtain

$$G(n,x):=\begin{pmatrix} g_n(x)-g(x)\\ (g_n'(x)-g'(x))\,b_n\end{pmatrix}=U_n^{-1}W_n.\tag{2}$$

The main contribution of this paper is to provide sufficient conditions ensuring the consistency (Theorem 1) and the asymptotic normality (Theorem 2) of the estimator defined by (1) under very mild conditions on the bandwidth parameter (see assumption **(A6)**). Our approach is based on the so-called Lindeberg's method (see [7], [8], [9], [12]) instead of the Bernstein's blocking method used in several previous works for proving limit theorems in the random field setting (see [2], [10], [22],...).

Let $\mathrm{K} : \mathbb{R} \to \mathbb{R}$ be a probability kernel. For any $c = (c_0, c_1) \in \mathbb{R}^2$ and any s in $\mathbb{R}$, we define $\mathrm{K}_c(s) = (c_0 + c_1 s)\mathrm{K}(s)$. In the sequel, we consider the following assumptions:

(**A1**) For any c in $\mathbb{R}^2$, we have $\sup_{t\in\mathbb{R}} |\mathrm{K}_c(t)| < \infty$, $\int_{\mathbb{R}} |\mathrm{K}_c(t)|dt < \infty$ and K_c has an integrable second-order radial majorant, that is, the function ψ defined for any real x by $r(x) = \sup_{|t|\geqslant|x|} t^2\mathrm{K}_c(t)$ is integrable.

(**A2**) g is twice differentiable and g'' is continuous.

(**A3**) There exists a positive constant κ such that $\sup_{k\neq 0} |f_{0,k}(x, y) - f(x)f(y)| \leq \kappa$ for any (x, y) in $\mathbb{R}^2$ where $f_{0,k}$ is the continuous joint density of (X_0, X_k) and f is the continuous marginal density of X_0.

(**A4**) $\mathbb{E}|Y_0|^{2+\delta} < \infty$ for some $\delta > 0$.

(**A5**) $b_n \to 0$ such that $|\Lambda_n|b_n^3 \to \infty$.

(**A6**) $b_n \to 0$ such that $|\Lambda_n|b_n \to \infty$ and $|\Lambda_n|b_n^5 \to 0$.

Our first main result ensures the consistency of the estimator.

Theorem 1 *If* **(A1)**, **(A2)**, **(A3)**, **(A4)** *and* **(A5)** *hold and*

$$\sum_{m=1}^{\infty} m^{\frac{(2d-1)\delta+6d-2}{2+\delta}} \alpha_{1,\infty}^{\frac{\delta}{2+\delta}}(m) < \infty \tag{3}$$

then for any x in $\mathbb{R}$,

$$\frac{G(n,x)}{b_n} \xrightarrow[n\to\infty]{\mathbb{P}} 0 \tag{4}$$

where $G(n, x)$ is defined by (2)*.*

The second main contribution of this paper is the following central limit theorem.

Theorem 2 *If* **(A1)**, **(A2)**, **(A3)**, **(A4)**, **(A6)** *and* (3) *hold then for any x in $\mathbb{R}$ such that $f(x) > 0$,*

$$\sqrt{|\Lambda_n|b_n}\, G(n, x) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, U^{-1}\Sigma\, {}^t(U^{-1})\right)$$

where

$$\Sigma = \mathbb{V}(Y_0/X_0 = x)f(x) \begin{pmatrix} \int_{\mathbb{R}} K^2(t)dt & \int_{\mathbb{R}} tK^2(t)dt \\ \int_{\mathbb{R}} tK^2(t)dt & \int_{\mathbb{R}} t^2K^2(t)dt \end{pmatrix} \tag{5}$$

and

$$U = f(x) \begin{pmatrix} 1 & \int_{\mathbb{R}} tK(t)dt \\ \int_{\mathbb{R}} tK(t)dt & \int_{\mathbb{R}} t^2K(t)\, dt \end{pmatrix}. \tag{6}$$

Remark. Theorem 2 extends results of Hallin, Lu and Tran [10] in several directions. Using our notations, Theorem 3.1 in [10] assumes that $\alpha_{1,\infty}(m) = O\left(m^{-\mu}\right)$ where $\mu > 2(3+\delta)d/\delta$ and this condition is more restrictive than (3). Moreover, the regions Λ_n that we consider in our work are very general and very mild conditions are assumed on the bandwidth parameter b_n. In fact, the condition $|\Lambda_n|b_n^5 \to 0$ in Assumption **(A6)** is assumed only for the cancellation of the bias term in Theorem 2.

2 Proofs

In the sequel, for any sequences $(p_n)_{n\geqslant 1}$ and $(q_n)_{n\geqslant 1}$ of real numbers, we denote $p_n \trianglelefteq q_n$ if and only if there exists $\kappa > 0$ (not depending on n) such that $p_n \leqslant \kappa q_n$. Moreover, proofs of some technical lemmas in this section are postponned to the appendix. Consider the sequence $(m_n)_{n\geqslant 1}$ of positive integers defined by

$$m_n = \max\left\{\tau_n, \left[\left(b_n^{\frac{-3\delta}{4+\delta}} \sum_{|i|>\tau_n} |i|^{\frac{d(4+\delta)}{2+\delta}} \alpha_{1,\infty}^{\frac{\delta}{2+\delta}}(|i|)\right)^{\frac{1}{3d}}\right] + 1\right\} \tag{7}$$

where $\tau_n = \left[b_n^{\frac{-\delta}{2d(4+\delta)}}\right]$ and $[\,.\,]$ denotes the integer part function. The proof of the following lemma is left to the reader (see Lemma 2 in [7]).

Lemma 1 *If* (3) *holds then*

$$m_n \to \infty, \quad m_n^d b_n^{\frac{\delta}{4+\delta}} \to 0 \quad \textit{and} \quad \left(m_n^d b_n^{\frac{\delta}{4+\delta}}\right)^{-\frac{4+\delta}{2+\delta}} \sum_{|i|>m_n} |i|^{\frac{d(4+\delta)}{2+\delta}} \alpha_{1,\infty}^{\frac{\delta}{2+\delta}}\left(|i|\right) \to 0.$$

2.1 Proof of Theorem 1

Let x and $c = (c_0, c_1)$ be fixed in $\mathbb{R}$ and $\mathbb{R}^2$ respectively and denote

$$\eta = \mathbb{V}(Y_0/X_0 = x)f(x)\int_{\mathbb{R}} \mathrm{K}_c^2(t)dt.$$

Lemma 2 $\mathbb{E}\left(cW_n\right) \trianglelefteq b_n^2$ *and* $|\Lambda_n|b_n\mathbb{V}(cW_n) \xrightarrow[n\to\infty]{} \eta$.

Proof of Lemma 2. Let n be a positive integer,

$$\begin{aligned}
\mathbb{E}\left(cW_n\right) &= \frac{1}{b_n}\mathbb{E}\left[Z_0 \mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right] \\
&= \frac{1}{b_n}\mathbb{E}\left[\left[g(X_0)-g(x)-g'(x)(X_0-x)\right]\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right] \\
&= \frac{1}{b_n}\int_{\mathbb{R}}\left[g(u)-g(x)-g'(x)(u-x)\right]\mathrm{K}_c\left(\frac{u-x}{b_n}\right)f(u)du \\
&= \int_{\mathbb{R}}\left[g(x+vb_n)-g(x)-g'(x)vb_n\right]\mathrm{K}_c(v)f(x+vb_n)dv \\
&= \frac{b_n^2}{2}\int_{\mathbb{R}} g''(\theta_n(x,v))v^2\mathrm{K}_c(v)f(x+vb_n)dv
\end{aligned}$$

where $\theta_n(x,v)$ is a real number between x and $x+vb_n$. By the Lesbesgue density theorem (see chapter 2 in [5]), we have

$$\int_{\mathbb{R}} g''(\theta_n(x,v))v^2\mathrm{K}_c(v)f(x+vb_n)dv \xrightarrow[n\to\infty]{} g''(x)f(x)\int_{\mathbb{R}} v^2\mathrm{K}_c(v)dv.$$

So, we obtain $\mathbb{E}\left(cW_n\right) \trianglelefteq b_n^2$. In the other part,

$$\left|\left|\Lambda_n\right|b_n\mathbb{V}(cW_n)-\mathbb{E}(\Delta_0^2)\right)| = \left|\frac{1}{|\Lambda_n|}\mathbb{E}\left(\sum_{i\in\Lambda_n}\Delta_i\right)^2-\mathbb{E}(\Delta_0^2)\right| \leqslant \sum_{j\in\mathbb{Z}^d\backslash\{0\}}|\mathbb{E}(\Delta_0\Delta_j)| \quad (8)$$

where

$$\Delta_i = \frac{Z_i}{\sqrt{b_n}}\mathrm{K}_c\left(\frac{X_i-x}{b_n}\right)-\mathbb{E}\frac{Z_i}{\sqrt{b_n}}\mathrm{K}_c\left(\frac{X_i-x}{b_n}\right). \quad (9)$$

Lemma 3 $\mathbb{E}\left(\Delta_0^2\right) \xrightarrow[n\to\infty]{} \eta$ *and moreoever,*

$$\sup_{i\in\mathbb{Z}^d\backslash\{0\}}\mathbb{E}|\Delta_0\Delta_i| \trianglelefteq b_n^{\frac{\delta}{4+\delta}} \quad \textit{and} \quad \left|\mathbb{E}\left(\Delta_0\Delta_i\right)\right| \trianglelefteq b_n^{\frac{-\delta}{2+\delta}}\alpha_{1,1}^{\frac{\delta}{2+\delta}}(|i|) \quad \textit{for any } i\neq 0.$$

Combining Lemma 3 and Lemma 1, we obtain

$$\sum_{j\in\mathbb{Z}^d\backslash\{0\}}|\mathbb{E}(\Delta_0\Delta_j)| \trianglelefteq m_n^d b_n^{\frac{\delta}{4+\delta}} + \left(m_n^d b_n^{\frac{\delta}{4+\delta}}\right)^{-\frac{4+\delta}{2+\delta}}\sum_{\substack{i\in\mathbb{Z}^d\\|i|>m_n}}|i|^{\frac{d(4+\delta)}{2+\delta}}\alpha_{1,\infty}^{\frac{\delta}{2+\delta}}(|i|) \xrightarrow[n\to\infty]{} 0.$$

Using (8) and Lemma 3, we derive $|\Lambda_n|b_n\mathbb{V}(cW_n) \xrightarrow[n\to\infty]{} \eta$. The proof of Lemma 2 is complete.

Lemma 4 $U_n \xrightarrow[n\to\infty]{\mathbb{L}^2} U$ *where* U *is defined by* (6).

Proof of Lemma 4. Let k be fixed in $\{0,1,2\}$ and let x be a real number. Then,

$$\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}\left(\frac{X_i-x}{b_n}\right)^k \mathrm{K}\left(\frac{X_i-x}{b_n}\right)=\frac{1}{|\Lambda_n|\sqrt{b_n}}\sum_{i\in\Lambda_n}\overline{\Delta}_{i,k}+\frac{1}{b_n}\mathbb{E}\left(\frac{X_0-x}{b_n}\right)^k \mathrm{K}\left(\frac{X_0-x}{b_n}\right)$$

where

$$\overline{\Delta}_{i,k}=\frac{1}{\sqrt{b_n}}\left(\frac{X_i-x}{b_n}\right)^k \mathrm{K}\left(\frac{X_i-x}{b_n}\right)-\mathbb{E}\frac{1}{\sqrt{b_n}}\left(\frac{X_0-x}{b_n}\right)^k \mathrm{K}\left(\frac{X_0-x}{b_n}\right).$$

First, using again the Lebesgue density theorem (see chapter 2 in [5]), we have

$$\frac{1}{b_n}\mathbb{E}\left(\frac{X_0-x}{b_n}\right)^k \mathrm{K}\left(\frac{X_0-x}{b_n}\right)=\int_{\mathbb{R}}v^k\mathrm{K}(v)f(x+vb_n)dv\xrightarrow[n\to\infty]{}f(x)\int_{\mathbb{R}}v^k\mathrm{K}(v)dv. \tag{10}$$

In the other part, arguing as in the proof of Lemma 3, we have $\mathbb{E}\left(\overline{\Delta}_{0,k}^2\right)$ converges to $f(x)\int_{\mathbb{R}}t^{2k}K^2(t)dt$ as n goes to infinity and $\sum_{j\in\mathbb{Z}^d\backslash\{0\}}\left|\mathbb{E}\left(\overline{\Delta}_{0,k}\overline{\Delta}_{j,k}\right)\right|$ goes to zero as n goes to infinity. Consequently,

$$\left|\frac{1}{|\Lambda_n|^2b_n}\mathbb{E}\left(\sum_{i\in\Lambda_n}\overline{\Delta}_{i,k}\right)^2-\frac{\mathbb{E}\left(\overline{\Delta}_{0,k}^2\right)}{|\Lambda_n|b_n}\right|\leqslant\frac{1}{|\Lambda_n|b_n}\sum_{j\in\mathbb{Z}^d\backslash\{0\}}\left|\mathbb{E}\left(\overline{\Delta}_{0,k}\overline{\Delta}_{j,k}\right)\right|\xrightarrow[n\to\infty]{}0. \tag{11}$$

Combining (10) and (11) and keeping in mind that $|\Lambda_n|b_n$ goes to infinity as n goes to infinity, we obtain

$$\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}\left(\frac{X_i-x}{b_n}\right)^k \mathrm{K}\left(\frac{X_i-x}{b_n}\right)\xrightarrow[n\to\infty]{\mathbb{L}^2}f(x)\int_{\mathbb{R}}v^kK(v)dv.$$

The proof of Lemma 4 is complete.

Combining (2) with Lemmas 2 and 4 and the fact that

$$G(n,x)=U_n^{-1}\left(W_n-\mathbb{E}W_n\right)+U_n^{-1}\mathbb{E}W_n \tag{12}$$

we obtain (4). The proof of Theorem 1 is complete.

2.2 Proof of Theorem 2

Let $c=(c_0,c_1)$ be fixed in $\mathbb{R}^2$. By Lemma 2 and Assumption **(A6)**, we derive that $\sqrt{|\Lambda_n|b_n}\mathbb{E}(cW_n)$ goes to zero as n goes to infinity. Keeping in mind (12) and using Lemma 4 and Slutsky's lemma, we have only to prove the asymptotic normality of $\sqrt{|\Lambda_n|b_n}\left(W_n-\mathbb{E}W_n\right)$. That is what we establish in the following key result where we recall the notation $\eta=\mathbb{V}(Y_0/X_0=x)f(x)\int_{\mathbb{R}}\mathrm{K}_c^2(t)dt$.

Proposition 1 $\sqrt{|\Lambda_n| b_n}\,(cW_n - \mathbb{E}\,(cW_n)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\,(0,\eta)$.

Proof of Proposition 1. Let $(\xi_i)_{i\in\mathbb{Z}^d}$ be a field of i.i.d. standard normal random variables independent of $(Y_i, X_i)_{i\in\mathbb{Z}^d}$ and denote for all i in $\mathbb{Z}^d$,

$$T_i = \frac{\Delta_i}{|\Lambda_n|^{1/2}} \quad \text{and} \quad \gamma_i = \frac{\sqrt{\eta}\,\xi_i}{|\Lambda_n|^{1/2}}$$

where Δ_i is given by (9). On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i = (i_1, ..., i_d)$ and $j = (j_1, ..., j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{\text{lex}} j$ means that either $i_1 < j_1$ or for some k in $\{2, 3, ..., d\}$, $i_k < j_k$ and $i_l = j_l$ for $1 \leqslant l < k$. Recall that $|\Lambda_n|$ is the number of element in the region Λ_n and let φ be the unique function from $\{1, ..., |\Lambda_n|\}$ to Λ_n such that $\varphi(k) <_{\text{lex}} \varphi(l)$ for $1 \leqslant k < l \leqslant |\Lambda_n|$. For all integer $1 \leqslant k \leqslant |\Lambda_n|$, we put

$$S_{\varphi(k)}(T) = \sum_{i=1}^{k} T_{\varphi(i)} \quad \text{and} \quad S^c_{\varphi(k)}(\gamma) = \sum_{i=k}^{|\Lambda_n|} \gamma_{\varphi(i)}$$

with the convention $S_{\varphi(0)}(T) = S^c_{\varphi(|\Lambda_n|+1)}(\gamma) = 0$. Let h be any measurable function from $\mathbb{R}$ to $\mathbb{R}$. For any $1 \leqslant k \leqslant l \leqslant |\Lambda_n|$, we introduce $h_{k,l} = h(S_{\varphi(k)}(T) + S^c_{\varphi(l)}(\gamma))$. We denote by $B_1^4(\mathbb{R})$ the unit ball of $C_b^4(\mathbb{R})$: h belongs to $B_1^4(\mathbb{R})$ if and only if it belongs to $C^4(\mathbb{R})$ and satisfies $\max_{0\leqslant i\leqslant 4} \|h^{(i)}\|_\infty \leqslant 1$. It suffices to prove that for all h in $B_1^4(\mathbb{R})$,

$$\mathbb{E}\left(h\left(S_{\varphi(|\Lambda_n|)}(T)\right)\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left(h\left(\sqrt{\eta}\xi_0\right)\right).$$

We use Lindeberg's decomposition:

$$\mathbb{E}\left(h\left(S_{\varphi(|\Lambda_n|)}(T)\right) - h\left(\sqrt{\eta}\xi_0\right)\right) = \sum_{k=1}^{|\Lambda_n|} \mathbb{E}\left(h_{k,k+1} - h_{k-1,k}\right).$$

Now, we have $h_{k,k+1} - h_{k-1,k} = h_{k,k+1} - h_{k-1,k+1} + h_{k-1,k+1} - h_{k-1,k}$ and by Taylor's formula we obtain

$$\begin{aligned} h_{k,k+1} - h_{k-1,k+1} &= T_{\varphi(k)} h'_{k-1,k+1} + \frac{1}{2} T^2_{\varphi(k)} h''_{k-1,k+1} + R_k \\ h_{k-1,k+1} - h_{k-1,k} &= -\gamma_{\varphi(k)} h'_{k-1,k+1} - \frac{1}{2} \gamma^2_{\varphi(k)} h''_{k-1,k+1} + r_k \end{aligned}$$

where $|R_k| \leqslant T^2_{\varphi(k)}(1 \wedge |T_{\varphi(k)}|)$ and $|r_k| \leqslant \gamma^2_{\varphi(k)}(1 \wedge |\gamma_{\varphi(k)}|)$. Since $(T, \xi_i)_{i\neq\varphi(k)}$ is independent of $\xi_{\varphi(k)}$, it follows that

$$\mathbb{E}\left(\gamma_{\varphi(k)} h'_{k-1,k+1}\right) = 0 \quad \text{and} \quad \mathbb{E}\left(\gamma^2_{\varphi(k)} h''_{k-1,k+1}\right) = \mathbb{E}\left(\frac{\eta}{|\Lambda_n|} h''_{k-1,k+1}\right)$$

Hence, we obtain

$$\begin{aligned}\mathbb{E}\left(h(S_{\varphi(|\Lambda_n|)}(T)) - h\left(\sqrt{\eta}\xi_0\right)\right) = &\sum_{k=1}^{|\Lambda_n|}\mathbb{E}(T_{\varphi(k)}h^{'}_{k-1,k+1}) \\ &+ \sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left(\left(T^2_{\varphi(k)} - \frac{\eta}{|\Lambda_n|}\right)\frac{h^{''}_{k-1,k+1}}{2}\right) \\ &+ \sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left(R_k + r_k\right).\end{aligned}$$

Let L be a positive real number.

$$\begin{aligned}\sum_{k=1}^{|\Lambda_n|}\mathbb{E}|R_k| &\leqslant \mathbb{E}\left(\Delta_0^2\left(1\wedge\frac{|\Delta_0|}{|\Lambda_n|^{1/2}}\right)\right) \\ &\trianglelefteq \mathbb{E}\left[\frac{Z_0^2}{b_n}\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right)\left(1\wedge\frac{|Z_0|}{\sqrt{|\Lambda_n|b_n}}\left|\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right|\right)\right] \\ &\trianglelefteq \mathbb{E}\left[\frac{Z_0^2}{b_n}\mathbb{1}_{|Z_0|\leqslant L}\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right)\left(1\wedge\frac{|Z_0|}{\sqrt{|\Lambda_n|b_n}}\left|\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right|\right)\right] \\ &\quad + \mathbb{E}\left[\frac{Z_0^2}{b_n}\mathbb{1}_{|Z_0|>L}\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right)\left(1\wedge\frac{|Z_0|}{\sqrt{|\Lambda_n|b_n}}\left|\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right|\right)\right] \\ &\trianglelefteq \frac{L^3}{\sqrt{|\Lambda_n|}b_n^{3/2}}\mathbb{E}\left[\left|\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right|^3\right] + L^{-\delta}\mathbb{E}\left[\frac{|Z_0|^{2+\delta}}{b_n}\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right)\right] \\ &\trianglelefteq \frac{L^3}{\sqrt{|\Lambda_n|b_n}}\int_{\mathbb{R}}|\mathrm{K}_c(v)|^3 f(x+vb_n)dv \\ &\quad + L^{-\delta}\int_{\mathbb{R}}\mathbb{E}\left(|Z_0|^{2+\delta}/X_0 = x+vb_n\right)|\mathrm{K}_c(v)|^3 f(x+vb_n)dv.\end{aligned}$$

By the Lebesgue density theorem (see chapter 2 in [5]), we have

$$\int_{\mathbb{R}}\mathbb{E}\left(|Z_0|^{2+\delta}/X_0 = x+vb_n\right)|\mathrm{K}_c(v)|^3 f(x+vb_n)dv \xrightarrow[n\to\infty]{} f(x)\mathbb{E}\left(|Z_0|^{2+\delta}/X_0 = x\right)\int_{\mathbb{R}}|\mathrm{K}_c(v)|^3 dv$$

and

$$\int_{\mathbb{R}}|\mathrm{K}_c(v)|^3 f(x+vb_n)dv \xrightarrow[n\to\infty]{} f(x)\int_{\mathbb{R}}|\mathrm{K}_c(v)|^3 dv.$$

Consequently, we obtain

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}|R_k| \trianglelefteq \left(\frac{L^3}{\sqrt{|\Lambda_n|b_n}} + L^{-\delta}\right).$$

Choosing $L=(|\Lambda_n|b_n)^{\frac{1}{2(3+\delta)}}$, we obtain

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}|R_k|\trianglelefteq(|\Lambda_n|b_n)^{\frac{-\delta}{2(3+\delta)}}\xrightarrow[n\to\infty]{}0.$$

Moreover,

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}|r_k|\leqslant\frac{\eta^{3/2}\mathbb{E}|\xi_0|^3}{\sqrt{|\Lambda_n|}}\xrightarrow[n\to\infty]{}0.$$

So, it is sufficient to show

$$\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\left(\mathbb{E}(T_{\varphi(k)}h^{'}_{k-1,k+1})+\mathbb{E}\left(\left(T^2_{\varphi(k)}-\frac{\eta}{|\Lambda_n|}\right)\frac{h^{''}_{k-1,k+1}}{2}\right)\right)=0. \tag{13}$$

For any i in $\mathbb{Z}^d$ and any integer $k\geqslant 1$, we define $V_i^k=\{j\in\mathbb{Z}^d\,/\,j<_{\text{lex}}i \text{ and } |i-j|\geqslant k\}$. For all integer $n\geqslant 1$ and all integer $1\leqslant k\leqslant|\Lambda_n|$, we denote

$$\mathrm{E}_k^{(n)}=\varphi(\{1,..,k\})\cap V_{\varphi(k)}^{m_n}\quad\text{and}\quad S_{\varphi(k)}^{(m_n)}(T)=\sum_{i\in\mathrm{E}_k^{(n)}}T_i$$

where m_n is defined by (7). In the sequel, for all function ψ from $\mathbb{R}$ to $\mathbb{R}$, we adopt the notation $\psi_{k-1,l}^{(m_n)}=\psi\left(S_{\varphi(k)}^{(m_n)}(T)+S^c_{\varphi(l)}(\gamma)\right)$. More precisely, we are going to use this notation with ψ equals to $h^{'}$ or $h^{''}$. Our aim is to show that

$$\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left(T_{\varphi(k)}h^{'}_{k-1,k+1}-T_{\varphi(k)}\left(S_{\varphi(k-1)}(T)-S_{\varphi(k)}^{(m_n)}(T)\right)h^{''}_{k-1,k+1}\right)=0. \tag{14}$$

First, we use the decomposition

$$T_{\varphi(k)}h^{'}_{k-1,k+1}=T_{\varphi(k)}h^{'(m_n)}_{k-1,k+1}+T_{\varphi(k)}\left(h^{'}_{k-1,k+1}-h^{'(m_n)}_{k-1,k+1}\right).$$

Since γ is independent of T, we have $\mathbb{E}\left(T_{\varphi(k)}h^{'}\left(S^c_{\varphi(k+1)}(\gamma)\right)\right)=0$ and consequently,

$$\left|\mathbb{E}\left(T_{\varphi(k)}h^{'(m_n)}_{k-1,k+1}\right)\right|=\left|\mathbb{E}\left(T_{\varphi(k)}\left(h^{'(m_n)}_{k-1,k+1}-h^{'}\left(S^c_{\varphi(k+1)}(\gamma)\right)\right)\right)\right|\leqslant\mathbb{E}\left|T_{\varphi(k)}S_{\varphi(k)}^{(m_n)}(T)\right|.$$

Moreover,

$$\mathbb{E}\left|T_{\varphi(k)}S_{\varphi(k)}^{(m_n)}(T)\right|\leqslant\frac{1}{|\Lambda_n|}\sum_{i\in E_k^{(n)}}\mathbb{E}|\Delta_{\varphi(k)}\Delta_i|\leqslant\sup_{j\in\mathbb{Z}^d\backslash\{0\}}\mathbb{E}|\Delta_0\Delta_j|.$$

So, by Lemma 3, we obtain

$$\left|\mathbb{E}\left(T_{\varphi(k)}h^{'(m_n)}_{k-1,k+1}\right)\right|\trianglelefteq b_n^{\frac{\delta}{4+\delta}}\xrightarrow[n\to\infty]{}0.$$

Applying again Taylor's formula,

$$T_{\varphi(k)}(h'_{k-1,k+1} - h'^{(m_n)}_{k-1,k+1}) = T_{\varphi(k)}\left(S_{\varphi(k-1)}(T) - S^{(m_n)}_{\varphi(k)}(T)\right) h''_{k-1,k+1} + R'_k,$$

where

$$|R'_k| \leqslant 2\left|T_{\varphi(k)}\left(S_{\varphi(k-1)}(T) - S^{(m_n)}_{\varphi(k)}(T)\right)\left(1 \wedge |S_{\varphi(k-1)}(T) - S^{(m_n)}_{\varphi(k)}(T)|\right)\right|.$$

Using Lemma 1 and Lemma 3, it follows that

$$\begin{aligned}
\sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| &\leqslant 2\mathbb{E}\left(|\Delta_0|\left(\sum_{\substack{|i|\leqslant m_n\\ i\neq 0}} |\Delta_i|\right)\left(1 \wedge \frac{1}{\sqrt{|\Lambda_n|}}\sum_{\substack{|i|\leqslant m_n\\ i\neq 0}} |\Delta_i|\right)\right)\\
&\leqslant 2\sum_{\substack{|i|\leqslant m_n\\ i\neq 0}} \mathbb{E}|\Delta_0\Delta_i| \leqslant 2m_n^d \sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\Delta_0\Delta_i|\\
&\trianglelefteq m_n^d b_n^{\frac{\delta}{4+\delta}} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

So, we obtain (14). In order to derive (13) it remains to control

$$\mathrm{F}_1 := \mathbb{E}\left(\sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1}\left(\frac{T^2_{\varphi(k)}}{2} + T_{\varphi(k)}\left(S_{\varphi(k-1)}(T) - S^{(m_n)}_{\varphi(k)}(T)\right) - \frac{\eta}{2|\Lambda_n|}\right)\right).$$

We have

$$\mathrm{F}_1 \leqslant \left|\mathbb{E}\left(\frac{1}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1}\left(\Delta^2_{\varphi(k)} - \mathbb{E}(\Delta_0^2)\right)\right)\right| + \left|\eta - \mathbb{E}\left(\Delta_0^2\right)\right| + 2\sum_{\substack{|j|\leqslant m_n\\ j\neq 0}} \mathbb{E}|\Delta_0\Delta_j|.$$

By Lemma 3, we know that

$$\mathbb{E}\left(\Delta_0^2\right) \xrightarrow[n\to\infty]{} \eta \quad \text{and} \quad \sum_{\substack{|j|\leqslant m_n\\ j\neq 0}} \mathbb{E}|\Delta_0\Delta_j| \trianglelefteq m_n^d b_n^{\frac{\delta}{4+\delta}} \xrightarrow[n\to\infty]{} 0.$$

So, it suffices to prove

$$\mathrm{F}_2 := \left|\mathbb{E}\left(\frac{1}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1}\left(\Delta^2_{\varphi(k)} - \mathbb{E}(\Delta_0^2)\right)\right)\right| \xrightarrow[n\to\infty]{} 0. \tag{15}$$

Let M be a positive constant and denote $\mathbb{E}_M\left(\Delta^2_{\varphi(k)}\right) = \mathbb{E}\left(\Delta^2_{\varphi(k)}/\mathcal{F}_{V^M_{\varphi(k)}}\right)$ where $\mathcal{F}_{V^M_{\varphi(k)}}$ is the σ-algebra generated by (X_s, Y_s) for s in $V^M_{\varphi(k)}$. We have $F_2 \leqslant F_2' + F_2''$ where

$$F_2' := \left|\mathbb{E}\left(\frac{1}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1}\left(\Delta^2_{\varphi(k)} - \mathbb{E}_M\left(\Delta^2_{\varphi(k)}\right)\right)\right)\right|$$

and

$$F_2'' := \left| \mathbb{E}\left(\frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h_{k-1,k+1}^{''} \left(\mathbb{E}_M\left(\Delta_{\varphi(k)}^2\right) - \mathbb{E}(\Delta_0^2) \right) \right) \right|.$$

The following technical lemma is proved in the appendix.

Lemma 5 $\|\Delta_0\|_{2+\delta}^2 \trianglelefteq b_n^{\frac{-\delta}{2+\delta}}$.

The next result can be found in [17].

Lemma 6 *Let $\mathcal{U}$ and $\mathcal{V}$ be two σ-algebras and let X be a random variable measurable with respect to $\mathcal{U}$. If $1 \leqslant p \leqslant r \leqslant \infty$ then*

$$\|\mathbb{E}(X|\mathcal{V}) - \mathbb{E}(X)\|_p \leqslant 2(2^{1/p}+1)\left(\alpha(\mathcal{U},\mathcal{V})\right)^{\frac{1}{p}-\frac{1}{r}} \|X\|_r.$$

Using Lemma 5 and Lemma 6 with $p = 1$ and $r = (2+\delta)/2$, we derive

$$F_2^{''} \leqslant \|\mathbb{E}_M\left(\Delta_0^2\right) - \mathbb{E}(\Delta_0^2)\|_1 \leqslant 6\|\Delta_0\|_{2+\delta}^2\, \alpha_{1,\infty}^{\frac{\delta}{2+\delta}}(M) \leqslant 6\, b_n^{\frac{-\delta}{2+\delta}}\, \alpha_{1,\infty}^{\frac{\delta}{2+\delta}}(M). \tag{16}$$

In the other part,

$$F_2^{'} \leqslant \frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} \left(J_k^1(M) + J_k^2(M) \right)$$

where

$$J_k^1(M) = \left| \mathbb{E}\left(h_{k-1,k+1}^{''(M)} \left(\Delta_{\varphi(k)}^2 - \mathbb{E}_M\left(\Delta_{\varphi(k)}^2\right) \right) \right) \right|$$

and

$$J_k^2(M) = \left| \mathbb{E}\left(\left(h_{k-1,k+1}^{''} - h_{k-1,k+1}^{''(M)} \right) \left(\Delta_{\varphi(k)}^2 - \mathbb{E}_M\left(\Delta_{\varphi(k)}^2\right) \right) \right) \right|.$$

Since $h_{k-1,k+1}^{''(M)}$ is $\sigma(\gamma_i\,;\, i \in \mathbb{Z}^d) \vee \mathcal{F}_{V_{\varphi(k)}^M}$-measurable and $(\gamma_i)_{i\in\mathbb{Z}^d}$ is independent of $(Y_i, X_i)_{i\in\mathbb{Z}^d}$ then $J_k^1(M) = 0$. Moreover, if L is a positive real number then

$$\begin{aligned} J_k^2(M) &\leqslant \mathbb{E}\left(\left(2 \wedge \sum_{\substack{|i|<M \\ i\neq 0}} \frac{|\Delta_i|}{\sqrt{|\Lambda_n|}} \right) \Delta_0^2 \right) \leqslant \frac{L}{\sqrt{|\Lambda_n|}} \sum_{\substack{|i|<M \\ i\neq 0}} \mathbb{E}|\Delta_0\Delta_i| + 2\mathbb{E}\left(\Delta_0^2\, \mathbb{1}_{|\Delta_0|>L}\right) \\ &\leqslant \frac{M^d L}{\sqrt{|\Lambda_n|}} \sup_{i\in\mathbb{Z}^d\setminus\{0\}} \mathbb{E}|\Delta_0\Delta_i| + 2L^{-\delta}\mathbb{E}\left(|\Delta_0|^{2+\delta}\right). \end{aligned}$$

Applying again Lemma 3 and Lemma 5, we derive

$$J_k^2(M) \leqslant \frac{LM^d b_n^{\frac{\delta}{4+\delta}}}{\sqrt{|\Lambda_n|}} + 2L^{-\delta} b_n^{-\delta/2}.$$

In particular, for

$$L = \frac{|\Lambda_n|^{\frac{1}{2(1+\delta)}}}{M^{\frac{d}{1+\delta}} b_n^{\frac{\delta^2+6\delta}{2(4+\delta)(1+\delta)}}}$$

we obtain

$$J_k^2(M) \trianglelefteq \frac{M^{\frac{d\delta}{1+\delta}}}{|\Lambda_n|^{\frac{\delta}{2(1+\delta)}} b_n^{\frac{-\delta^2+4\delta}{2(4+\delta)(1+\delta)}}}.$$

Now, choosing M such that $M^{\frac{(2d-1)\delta+6d-2}{2+\delta}} = b_n^{\frac{-\delta}{2+\delta}}$ then

$$J_k^2(M) \trianglelefteq \frac{b_n^{\theta}}{(|\Lambda_n| b_n)^{\frac{\delta}{2(1+\delta)}}}$$

where

$$\theta = \frac{d\delta^3(4+\delta) + \delta\tau^2}{(1+\delta)(4+\delta)\tau^2} \quad \text{and} \quad \tau = (2d-1)\delta + 6d - 2.$$

So, we obtain $F_1' \xrightarrow[n\to\infty]{} 0$. Using (3) and (16), we derive

$$F_2'' \trianglelefteq M^{\frac{(2d-1)\delta+6d-2}{2+\delta}} \alpha_{1,\infty}^{\frac{\delta}{2+\delta}}(M) \xrightarrow[n\to\infty]{} 0.$$

Consequently, we obtain (15). The proof of Proposition 1 is complete.

Combining (2) with Lemmas 2 and 4 and Proposition 1, we derive Theorem 2.

3 Numerical results

In this section, we consider the autoregressive random field $(X_{i,j})_{(i,j)\in\mathbb{Z}^2}$ defined by

$$X_{i,j} = 0.75 X_{i-1,j} + 0.2 X_{i,j-1} + \varepsilon_{i,j} \tag{17}$$

where $(\varepsilon_{i,j})_{(i,j)\in\mathbb{Z}^2}$ are iid random variables with standard normal law. From [11], we know that (17) has a stationary solution $X_{i,j}$ given by

$$X_{i,j} = \sum_{k_1\geqslant 0}\sum_{k_2\geqslant 0} \binom{k_1+k_2}{k_1} (0.75)^{k_1} (0.2)^{k_2} \varepsilon_{i-k_1,j-k_2} \tag{18}$$

and one can check that $X_{i,j}$ has a normal law with zero mean and variance $\sigma^2 = 3.8346$. If we denote by f the density of $X_{i,j}$ then $f(0) = 0.2037$. Let s be a positive integer. We simulate the $\varepsilon_{i,j}$'s over the grid $[0,2s]^2 \cap \mathbb{Z}^2$ and we obtain the data $X_{i,j}$'s for (i,j) in $\Lambda_s = [s+1,2s]^2 \cap \mathbb{Z}^2$ following (18). Thus, we construct

$$u_{00}(s) = \frac{1}{s^2 b_s} \sum_{(i,j)\in\Lambda_s} \mathrm{K}\left(\frac{X_{i,j}}{b_s}\right), \quad u_{11}(s) = \frac{1}{s^2 b_s} \sum_{(i,j)\in\Lambda_s} \left(\frac{X_{i,j}}{b_s}\right)^2 \mathrm{K}\left(\frac{X_{i,j}}{b_s}\right)$$

and

$$u_{01}(s) = u_{10}(s) = \frac{1}{s^2 b_s} \sum_{(i,j)\in\Lambda_s} \left(\frac{X_{i,j}}{b_s}\right) \mathrm{K}\left(\frac{X_{i,j}}{b_s}\right)$$

where K is the gaussian kernel defined for any real u by $\mathrm{K}(u) = \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)$. From the data set

$$Y_{i,j} = \frac{2}{1 + X_{i,j}^2} + \varepsilon_{i,j}$$

(that is, $g(u) = 2/(1 + u^2)$ for any real u), we define also

$$w_0(s) = \frac{1}{s^2 b_s} \sum_{(i,j)\in\Lambda_s} Z_{i,j} \mathrm{K}\left(\frac{X_{i,j}}{b_s}\right) \quad \text{and} \quad w_1(s) = \frac{1}{s^2 b_s} \sum_{(i,j)\in\Lambda_s} Z_{i,j} \left(\frac{X_{i,j}}{b_s}\right) \mathrm{K}\left(\frac{X_{i,j}}{b_s}\right)$$

with $Z_{i,j} = g(X_{i,j}) + \varepsilon_{i,j} - 2$. The local linear estimator $\hat{G}(s, 0)$ of the regression function g at the point $x = 0$ is given by

$$\hat{G}(s, 0) = \begin{pmatrix} u_{00}(s) & u_{10}(s) \\ u_{01}(s) & u_{11}(s) \end{pmatrix}^{-1} \begin{pmatrix} w_0(s) \\ w_1(s) \end{pmatrix} =: \begin{pmatrix} \hat{\tau}_0(s) \\ \hat{\tau}_1(s) \end{pmatrix}$$

For $s \in \{10, 20, 30, 40\}$ and $b_s = |\Lambda_s|^{-1/3}$, we take the arithmetic mean value $\hat{m}(s)$ of 300 replications of

$$\frac{\hat{\tau}_0(s) + \hat{\tau}_1(s)}{b_s}$$

and the following table

s	$\lvert\Lambda_s\rvert = s^2$	$b_s = \lvert\Lambda_s\rvert^{-1/3}$	$\hat{m}(s)$
10	100	0.215	-0.408
20	400	0.136	-0.309
30	600	0.104	-0.271
40	1600	0.085	-0.114

put on light that $\hat{m}(s)$ decreases to zero when s increases. In order to illustrate the asymptotic normality of the estimator, we consider 300 replications of

$$\frac{2 \times \pi^{1/4} \sqrt{|\Lambda_s| b_s}\, (\hat{\tau}_0(s) + \hat{\tau}_1(s))}{\sqrt{3 \times f(0) \times \mathbb{V}(Y_0/X_0 = 0)}} = 1.703 \times \sqrt{s^2 \times b_s}\, (\hat{\tau}_0(s) + \hat{\tau}_1(s))$$

for $(s, b_s) \in \{(10, 0.215); (20, 0.136); (30, 0.104); (40, 0.085)\}$ and we obtain histograms (see figure 1) which fit well to the standard normal law $\mathcal{N}(0, 1)$.

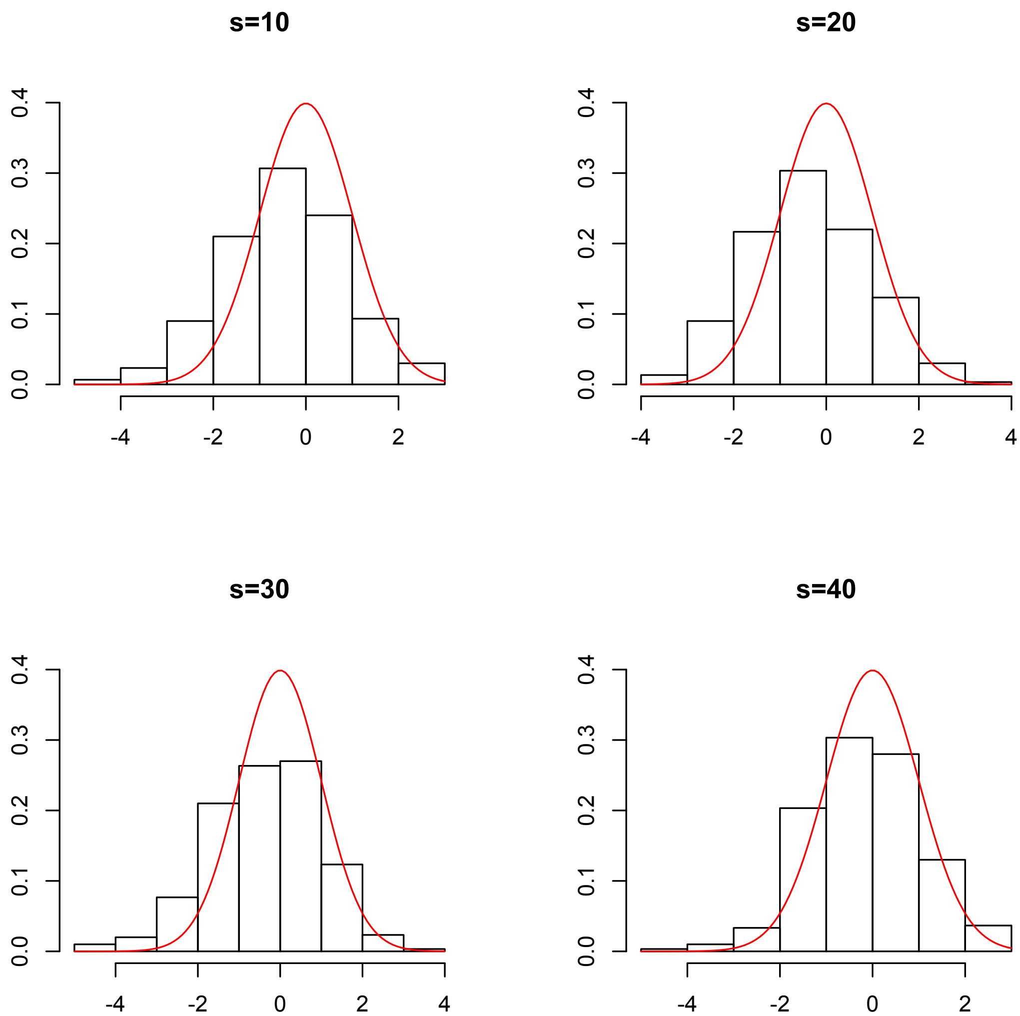

Figure 1: Asymptotic normality of the local linear estimator.

4 Appendix

Proof of Lemma 3. Let i in $\mathbb{Z}^d \backslash \{0\}$ be fixed. Applying Rio's covariance inequality ([18], Theorem 1.1), we obtain

$$|\mathbb{E}\left(\Delta_0 \Delta_i\right)| = |\mathrm{Cov}(\Delta_0, \Delta_i)| \leqslant 4 \int_0^{\alpha_{1,1}(|i|)} Q^2_{\Delta_0}(u) du$$

where Q_{Δ_0} is defined by $Q_{\Delta_0}(u) = \inf\{t \geqslant 0\,; \mathbb{P}(|\Delta_0| > t) \leqslant u\}$ for any u in $[0,1]$. Since $Q_{\Delta_0}(u) \leqslant u^{\frac{-1}{2+\delta}} \|\Delta_0\|_{2+\delta}$, using Lemma 5, we derive

$$|\mathbb{E}\left(\Delta_0\Delta_i\right)| \trianglelefteq b_n^{\frac{-\delta}{2+\delta}}\, \alpha_{1,1}^{\frac{\delta}{2+\delta}}(|i|).$$

In the other part, let $L \geqslant 1$ be a fixed real number. Arguing as in the proof of Lemma 5.2 in [10], we denote $\Delta_j^{(1)} = \Delta_j\, \mathbb{1}_{|\Delta_j|\leqslant L}$ and $\Delta_j^{(2)} = \Delta_j\, \mathbb{1}_{|\Delta_j|\geqslant L}$ for any j in $\mathbb{Z}^d$. So, we have

$$\mathbb{E}|\Delta_0\Delta_i| \leqslant \mathbb{E}|\Delta_0^{(1)}\Delta_i^{(1)}| + \mathbb{E}|\Delta_0^{(1)}\Delta_i^{(2)}| + \mathbb{E}|\Delta_0^{(2)}\Delta_i^{(1)}| + \mathbb{E}|\Delta_0^{(2)}\Delta_i^{(2)}|.$$

Moreover,

$$\begin{aligned}
\sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\Delta_0^{(1)}\Delta_i^{(2)}| &\trianglelefteq \sqrt{\mathbb{E}\left(\frac{Z_0^2}{b_n}\,\mathbb{1}_{|Z_0|\leqslant L}\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right)\right)} \times \sqrt{\mathbb{E}\left(\frac{Z_0^2}{b_n}\,\mathbb{1}_{|Z_0|> L}\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right)\right)} \\
&\trianglelefteq L^{-\delta/2}\sqrt{\mathbb{E}\left(\frac{Z_0^2}{b_n}\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right)\right)} \times \sqrt{\mathbb{E}\left(\frac{|Z_0|^{2+\delta}}{b_n}\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right)\right)} \\
&\trianglelefteq L^{-\delta/2}\sqrt{\int_{\mathbb{R}} \mathbb{E}\left(|Z_0|^2/X_0 = x+vb_n\right)\mathrm{K}_c^2(v) f(x+vb_n)dv} \\
&\qquad\times \sqrt{\int_{\mathbb{R}} \mathbb{E}\left(|Z_0|^{2+\delta}/X_0 = x+vb_n\right)\mathrm{K}_c^2(v) f(x+vb_n)dv}.
\end{aligned}$$

Applying again the Lebesgue density theorem (see chapter 2 in [5]), we derive

$$\sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\Delta_0^{(1)}\Delta_i^{(2)}| \trianglelefteq L^{-\delta/2}.$$

Similarly, $\sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\Delta_0^{(2)}\Delta_i^{(1)}| \trianglelefteq L^{-\delta/2}$ and $\sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\Delta_0^{(2)}\Delta_i^{(2)}| \trianglelefteq L^{-\delta}$. Finally,

$$\begin{aligned}
\sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\Delta_0^{(1)}\Delta_i^{(1)}| &\leqslant \mathbb{E}\left(\frac{|Z_0|}{\sqrt{b_n}}\,\mathbb{1}_{|Z_0|\leqslant L}\left|\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right| \times \frac{|Z_i|}{\sqrt{b_n}}\,\mathbb{1}_{|Z_i|\leqslant L}\left|\mathrm{K}_c\left(\frac{X_i-x}{b_n}\right)\right|\right) \\
&\quad + 3\left(\mathbb{E}\frac{|Z_0|}{\sqrt{b_n}}\,\mathbb{1}_{|Z_0|\leqslant L}\left|\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right|\right)^2 \\
&\leqslant \frac{L^2}{b_n}\mathbb{E}\left|\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\mathrm{K}_c\left(\frac{X_i-x}{b_n}\right)\right| + 3L^2 b_n\left(\int_{\mathbb{R}} |\mathrm{K}_c(v)| f(x+vb_n)\,dv\right)^2 \\
&\trianglelefteq \frac{L^2}{b_n}\int_{\mathbb{R}}\left|\mathrm{K}_c\left(\frac{u-x}{b_n}\right)\mathrm{K}_c\left(\frac{v-x}{b_n}\right)\right| |f_{0,i}(u,v) - f(u)f(v)|dudv \\
&\quad + \frac{L^2}{b_n}\int_{\mathbb{R}}\left|\mathrm{K}_c\left(\frac{u-x}{b_n}\right)\mathrm{K}_c\left(\frac{v-x}{b_n}\right)\right| f(u)f(v)dudv \\
&\quad + L^2 b_n\left(\int_{\mathbb{R}} |\mathrm{K}_c(v)| f(x+vb_n)\,dv\right)^2.
\end{aligned}$$

Using Assumption **(A3)** and the Lebesgue density theorem (see chapter 2 in [5]), we derive

$$\sup_{i\in\mathbb{Z}^d\setminus\{0\}} \mathbb{E}|\Delta_0^{(1)}\Delta_i^{(1)}| \trianglelefteq \frac{L^2}{b_n}\left(\int_{\mathbb{R}} |\mathrm{K}_c(v)|b_n dv\right)^2 + \frac{L^2}{b_n}\left(\int_{\mathbb{R}} |\mathrm{K}_c(v)|f(x+vb_n)b_n dv\right)^2 + L^2 b_n \trianglelefteq L^2 b_n.$$

Consequently, we obtain $\sup_{i\in\mathbb{Z}^d\setminus\{0\}} \mathbb{E}|\Delta_0\Delta_i| \trianglelefteq \left(L^{-\delta/2} + L^2 b_n\right)$. Choosing $L = b_n^{\frac{-2}{4+\delta}}$, it follows that

$$\sup_{i\in\mathbb{Z}^d\setminus\{0\}} \mathbb{E}|\Delta_0\Delta_i| \trianglelefteq b_n^{\frac{\delta}{4+\delta}}.$$

In the other part, we have

$$\mathbb{E}(\Delta_0^2) = \frac{1}{b_n}\left(\mathbb{E}Z_0^2\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right) - \left(\mathbb{E}Z_0\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right)^2\right).$$

Moreover,

$$\begin{aligned}
\mathbb{E}Z_0\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right) &= \mathbb{E}\left(Y_0 - g(x) - g'(x)(X_0-x)\right)\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\\
&= \mathbb{E}\left(g(X_0) - g(x) - g'(x)(X_0-x)\right)\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\\
&= \int_{\mathbb{R}} \left(g(u) - g(x) - g'(x)(u-x)\right)\mathrm{K}_c\left(\frac{u-x}{b_n}\right) f(u)\,du\\
&= \frac{1}{2}\int_{\mathbb{R}} (u-x)^2 g''(x+\theta(u-x))\mathrm{K}_c\left(\frac{u-x}{b_n}\right) f(u)du \quad \text{avec } |\theta|<1\\
&= \frac{b_n^3}{2}\int_{\mathbb{R}} g''(x+\theta v b_n)v^2\mathrm{K}_c(v)f(x+vb_n)dv.
\end{aligned}$$

Using the Lebesgue density theorem (see chapter 2 in [5]), we obtain

$$\mathbb{E}Z_0\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right) \trianglelefteq b_n.$$

In the other part,

$$\frac{1}{b_n}\mathbb{E}Z_0^2\mathrm{K}_c^2\left(\frac{X_0-x}{b_n}\right) = \int_{\mathbb{R}} \mathbb{E}(Z_0^2/X_0 = x+vb_n)\mathrm{K}_c^2(v)f(x+vb_n)dv.$$

Noting that

$$\mathbb{E}(Z_0^2/X_0 = x+vb_n) = \mathbb{V}(Y_0/X_0 = x+vb_n) \xrightarrow[n\to\infty]{} \mathbb{V}(Y_0/X_0 = x)$$

and applying again the Lebesgue density theorem (see chapter 2 in [5]), we obtain $\mathbb{E}\Delta_0^2 \xrightarrow[n\to\infty]{} \eta$. The proof of Lemma 3 is complete.

Proof of Lemma 5.

$$\begin{aligned}
\|\Delta_0\|_{2+\delta}^2 &\trianglelefteq \left(\mathbb{E}\left|\frac{Z_0}{\sqrt{b_n}}\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right|^{2+\delta}\right)^{\frac{2}{2+\delta}} \\
&\trianglelefteq b_n^{-1}\left(\mathbb{E}\left[\mathbb{E}\left(|Z_0|^{2+\delta}/X_0\right)\left|\mathrm{K}_c\left(\frac{X_0-x}{b_n}\right)\right|^{2+\delta}\right]\right)^{\frac{2}{2+\delta}} \\
&\trianglelefteq b_n^{\frac{-\delta}{2+\delta}}\left(\int_{\mathbb{R}}\mathbb{E}\left(|Z_0|^{2+\delta}/X_0=x+vb_n\right)|\mathrm{K}_c(v)|^{2+\delta}f(x+vb_n)dv\right)^{\frac{2}{2+\delta}}.
\end{aligned}$$

By the Lebesgue density theorem (see chapter 2 in [5]), we obtain

$$\|\Delta_0\|_{2+\delta}^2 \trianglelefteq b_n^{\frac{-\delta}{2+\delta}}.$$

The proof of Lemma 5 is complete.

Acknowledgments. The first author was supported by Erasmus Mundus (Battuta project) funded from the European commission. He would like to thank the hospitality of Cadi Ayyad University of Marrakesh where a large part of this work was done.

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