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$\mathfrak{S}$–UNIFORM SCALAR INTEGRABILITY AND STRONG LAWS OF LARGE NUMBERS FOR PETTIS INTEGRABLE FUNCTIONS WITH VALUES IN A SEPARABLE LOCALLY CONVEX SPACE

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Abstract. Generalizing techniques developed by Cuesta and Matrán for Bochner integrable random vectors of a separable Banach space, we prove a strong law of large numbers for Pettis integrable random elements of a separable locally convex space E. This result may be seen as a compactness result in a suitable topology on the set of Pettis integrable probabilities on E.

Contents

1. Introduction

In [13] and [14], J. A. Cuesta–Albertos and C. Matrán–Bea proved a very general strong law of large numbers (SLLN) for a sequence of Bochner integrable random elements of a separable Banach space, without geometric condition on the space. Though formulated in a more

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convenient way, their result (*e.g.* [13], Theorem 3) is equivalent to the following:

Theorem . *Let $(X_n)_n$ be a pairwise independent sequence of Bochner integrable random vectors of a separable Banach space E, defined on a probability space $(\Omega, \mathcal{F}, \mu)$, such that*

(*i*) *the sequence $(P_n)_n = (1/n \sum_{i=1}^n P_{X_i})_n$ is tight (we denote by P_X the law of a random variable X),*

(*ii*) *the function $\|.\|$ is uniformly integrable w.r.t. the sequence $(P_n)_n$,* i.e.
$$\lim_{a\to+\infty} \sup_{n\in\mathbb{N}^*} \int_{\{\|.\|>a\}} \|x\| dP_n(x) = 0,$$

(*iii*) *for almost every $\omega \in \Omega$, $\|.\|$ is uniformly integrable w.r.t. the sequence of empirical laws $(1/n \sum_{i=1}^n \delta_{X_i(\omega)})_n$ (where δ_x denotes the Dirac mass at point x),*

Then $(X_n)_n$ satisfies the SLLN, i.e.
$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}X_i) = 0 \ \textit{a.e.}$$

Note that Condition (*ii*) may be written
$$\lim_{a\to+\infty} \sup_{n\in\mathbb{N}^*} \int_{\{\|.\|>a\}} \sup_{x'\in\overline{B}_{E'}} |\langle x',\, x\rangle| dP_n(x) = 0,$$
where $\overline{B}_{E'}$ denotes the closed unit ball of the dual space E'. This condition is stronger than
$$(ii)' \qquad \lim_{a\to+\infty} \sup_{n\in\mathbb{N}^*} \sup_{x'\in\overline{B}_{E'}} \int_{|\langle x',\, .\rangle|>a\}} |\langle x',\, x\rangle| dP_n(x) = 0.$$

Let us define analogously Condition $(iii)'$ for the empirical laws, by interverting an integral and a supremum in Condition (*iii*).

Using some of the ideas of [13, 14], we prove in this paper a SLLN for a sequence $(X_n)_n$ of random elements of a separable locally convex space E, under some hypothesis on E that are satisfied in particular if E is Suslin. In the case when E is a Banach space, our result may be formulated as the above theorem, replacing (*ii*) by $(ii)'$ and (*iii*) by $(iii)'$. The random vector X_n ($n \in \mathbb{N}^*$) need not be Bochner integrable, but simply Pettis integrable.

Our approach is completely different from that of [24, 34, 42], where it is proved that a non–measurable function X with values in a non–separable Banach space "satisfies the SLLN" if and only if the upper integral $\int^* \|X\|$ is finite. On the contrary, we are concerned here with (measurable) separably valued random elements X of E, with possibly $\int \|X\| = +\infty$ for a continuous seminorm $\|.\|$ on E. All random vectors X considered in this paper are measurable (*i.e.* $X^{-1}(B)$ is measurable for every Borel subset B) and have Radon laws.

Note that, if E is Suslin, $(E, \sigma(E, E'))$ is also Suslin, with the same Borel tribe and the same Pettis integrable random vectors as E. Thus our new SLLN also provides a limit theorem in the $\sigma(E, E')$ topology. Another application is given in the case of elements of $\mathrm{L}^1_{E'}[E]$, where E is a separable Banach space (see [25] about the space $\mathrm{L}^1_{E'}[E]$): we obtain a SLLN in the topology of compact convergence on E'.

We begin, in Section 2, with some basic definitions and results: first, we give the topological properties of the space E that will be needed in the sequel, then we prove a Glivenko–Cantelli type theorem, generalizing to Lindelöf spaces that of [13, 14] (given for Polish spaces), but with an additional hypothesis. As in [13, 14], this result is the first step of our proof of the SLLN. We end this section with the definitions and some elementary properties of Pettis integrable laws and $\mathfrak{S}$–uniformly scalarly integrable families of laws.

The second step of the proof of the SLLN is proved in Section 3, where we give a general lemma on narrow convergence, uniform integrability and equicontinuity. This result also yields a Vitali convergence theorem for Pettis integrable laws, through a Vitali convergence theorem of Geitz-Musiał-Castaing [23, 32, 9] and a Skorokhod representation theorem of A. Jakubowski [26]. In particular, this result is a criterion for Pettis integrability. An application of this result to Komlós convergence of Young measures is also given.

We are then ready to state the SLLN: this is done in Section 4. We also give some applications and an example which shows that this result really applies to non necessarily Bochner integrable random vectors.

In Section 5, we define and study a family of semi–distances on the set MU^1_E of $\mathfrak{S}$–uniformly scalarly integrable Radon laws on E. This set contains the Pettis integrable Radon laws on E. Our semi–distances are "weak analogues" of the Lévy-Wasserstein distance, whose properties were used by Cuesta and Matrán [14] to prove the SLLN for Bochner integrable random vectors. As an application, we get another Glivenko–Cantelli type theorem that contains the SLLN proved in Section 4.

Unlike the convergence associated with the Lévy-Wasserstein distance, convergence for this family of semi–distances does not imply narrow convergence. This leads us to define in Section 6 a new topology on MU^1_E, which is the supremum of the topology of narrow convergence and of the topology associated with our semi–distances. The SLLN proved in this paper appears to be a compactness result in this topology, as well as the SLLN in [13, 14] is a compactness result in the topology associated with the Lévy-Wasserstein distance.

2. Basic definitions and results

2.1. **The locally convex space E.** Throughout, E is a Hausdorff locally convex topological vector space, and E' its topological dual. We assume that

- (E_1) E is separable,
- (E_2) there exists a countable set of continuous functions on E which separates the points of E.

Note that Property (E_2) is equivalent to Property (E_2') below:

- (E_2') there exists a countable set of continuous bounded functions on E which separates the points of E.

Indeed, if $(f_n)_n$ is a sequence of continuous functions on E which separates the points of E, let us define, for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$ a bounded continuous function $f_{n,k}$ on E by $f_{n,k}(x) = f_n(x)$ if $-k \leq f_n(x) \leq k$, $f_{n,k}(x) = k$ if $f_n(x) > k$ and $f_{n,k}(x) = -k$ if $f_n(x) < k$. The sequence $(f_{n,k})_{n,k}$ also separates the points of E.

Properties (E_1) and (E_2) are satisfied in particular if E is a Suslin space (see [7, 40]), *i.e.* if E is the continuous image of a Polish space. Here are two examples of non–Suslin Hausdorff locally convex spaces E satisfying (E_1) and (E_2). Let $S = \mathbb{N}^{\mathbb{N}}$, endowed with the product topology (where $\mathbb{N}$ is endowed with the discrete topology). The space S is Polish. Consider the spaces $C_p(S)$ and $C_c(S)$ of continuous functions on S, endowed respectively with the topology of pointwise convergence on S and the topology of uniform convergence on compacta. J.P.R Christensen has proved that $C_c(S)$ is not Suslin (see Corollary of Theorem 0.3 and the more general Theorem 3.7 in [11]). From Theorem 5.7.5 of [31], this implies that also $C_p(S)$ is not Suslin. But, from Corollary 4.2.2 in [31], since S is separable and metrizable, $C_p(S)$ and $C_c(S)$ are separable. Furthermore, let $\mathcal{D}$ be a countable dense subset of S, and define, for each $x \in \mathcal{D}$, a continuous function $\varphi_x : C_p(S) \to \mathbb{R}$ by $\varphi_x(f) = f(x)$. Then the countable set $\{\varphi_x;\, x \in \mathcal{D}\}$ separates the points of $C_p(S)$. Thus $C_p(S)$ and $C_c(S)$ satisfy (E_1) and (E_2).

The set of equicontinuous subsets of E' will be denoted by $\mathfrak{S}$. The following topological results about E are proved in [39] and [8]. We recall that the elements of $\mathfrak{S}$ are relatively compact subsets of E' endowed with the topology $\sigma(E', E)$, and that the topology of E is the topology of uniform convergence on the elements of $\mathfrak{S}$. This topology will be denoted by $\tau_{\mathfrak{S}}$. It is also defined by the family of seminorms $N_A(x) = \sup_{x' \in A} |\langle x', x\rangle|$ ($A \in \mathfrak{S}$). Each locally convex topology on E which is consistent with the duality $\langle E, E'\rangle$ is the topology $\tau_{\mathfrak{S}'}$ of uniform convergence on the elements of a set $\mathfrak{S}'$ of $\sigma(E', E)$-relatively compact subsets of E', which covers E'.

2.2. **Probability laws and narrow topology.** The Borel tribe of a topological space T with topology τ will be denoted by $\mathcal{B}_T$ or $\mathcal{B}_\tau$. If (S, Σ, λ) is a probability space, a measurable function $X : (S, \Sigma, \lambda) \to (T, \mathcal{B}_T)$ will be called a *random element* of T, and its law denoted by P_X. A probability measure on $(T, \mathcal{B}_T)$ will also be called a *law* on T.

A set $\mathcal{D}$ of laws on T is said to be (uniformly) *tight* if, for every $\epsilon > 0$, there exists a compact subset K of T such that, for any $P \in \mathcal{D}$, $P(K) \geq 1 - \epsilon$. A similar definition holds for a sequence of laws. A law P on T is *tight* if the set $\{P\}$ is tight.

If P is a tight law on E, then it is a Radon measure, *i.e.* P is inner regular w.r.t. the compact subsets of E. Indeed, from Property (E_2), every compact subset $\mathcal{K}$ of E is metrizable (because its topology coincides with the topology generated by a countable set of continuous functions on E which separates the points of E). From [40], Proposition 6 page 117, $\mathcal{K}$ is a Radon space (*i.e.* every finite Borel measure on $\mathcal{K}$ is Radon). Let $B \in \mathcal{B}_E$ and $\epsilon > 0$. As P is tight, there exists a compact subset $\mathcal{K}$ of E such that $P(B \cap \mathcal{K}) \geq P(B) - \epsilon/2$. But, as the restriction of P on $\mathcal{K}$ is Radon, there exists a compact subset $\mathcal{K}'$ of $\mathcal{K} \cap B$ such that

$$P(B \cap \mathcal{K}') \geq P(B \cap \mathcal{K}) - \epsilon/2 \geq P(B) - \epsilon,$$

which proves that P is Radon. Conversely, if P is Radon, it is clear that P is tight.

Most of the sequences of laws on E we shall consider will be tight. This is why we shall be mainly interested in Radon laws.

If E is Suslin, then E is a Radon space ([40], Theorem 10 page 122) and, for any weaker Hausdorff topology τ on E, the Borel sets (thus also the laws) associated with $\tau_{\mathfrak{S}}$ and τ coincide ([40], Corollary 2 page 101) and (E, τ) is Radon.

Definition 2.1. Let T be a set endowed with a topology τ. We denote by $\mathcal{M}_1((T, \tau))$ or $\mathcal{M}_1(\tau)$ the set of Radon laws on T. The *narrow topology associated to* τ on the space $\mathcal{M}_1(\tau)$ is the coarsest topology for which the functions

$$\begin{cases} \mathcal{M}_1(\tau) & \to & \mathbb{R} \\ P & \mapsto & P(f) \end{cases}$$

are continuous for every bounded continuous $f : (T, \tau) \to \mathbb{R}$.

This topology will be denoted by $\flat(T, \tau)$ or $\flat(\tau)$.

Note that, if T is not completely regular, this topology may be strictly coarser than the weak topology $\mathrm{w}(\tau)$ in the sense of Topsøe [44]. This will not make a big difference for us in this paper: the topology $\flat(\tau)$ is Hausdorff if and only if (T, τ) is an Urysohn space (that is, the set of continuous functions on T separates the points of T); in this case, the topologies $\mathrm{w}(\tau)$ and $\flat(\tau)$ coincide on any subset $\mathcal{K}$ of $\mathcal{M}_1(\tau)$ which is compact for $\mathrm{w}(\tau)$.

It is clear from the definition that $\flat(\tau)$ is uniformizable, *i.e.* completely regular. Consequently, a subset $\mathcal{D}$ of $\mathcal{M}_1(\tau)$ is *relatively compact for* $\flat(\tau)$ (*i.e.* $\mathcal{D}$ is contained in a $\flat(\tau)$–compact subset of $\mathcal{M}_1(\tau)$) if and only if every net of elements of $\mathcal{D}$ has a subnet which converges to an element of $\mathcal{M}_1(\tau)$. A proof of this characterization of relatively compact subsets of a completely regular space is given in [36]. Definitions and properties of nets and subnets can be found in [28].

A subset $\mathcal{D}$ of $\mathcal{M}_1(\tau)$ is said to be *tight* if, for each $\epsilon > 0$, there exists a compact subset K of (T, τ) such that $P(K) \geq 1-\epsilon$ for every $P \in \mathcal{D}$. If $\mathcal{D}$ is tight, then it is relatively compact for $\flat(\tau)$ (in fact it is relatively compact for $\mathrm{w}(\tau)$, see *e.g.* [44, 40]). When the converse implication holds true, (T, τ) is said to be a *Prokhorov space.* Polish spaces are Prokhorov spaces (see [37, 6, 35, 44, 40]), but not all Lusin spaces: for example, if E is a Fréchet space, the Lusin space $(E, \sigma(E, E')$ is a Prokhorov space if and only if it is nuclear [22] (in particular, if E is a separable Banach space, then $(E, \sigma(E, E'))$ is a Prokhorov space if and only if E is finite dimensional).

According to Corollary 10.3 of [44], since there exists a countable set of continuous bounded functions on E that separates the points of E, each relatively compact subset $\mathcal{D}$ of $(\mathcal{M}_1(E), \flat(\tau_{\mathfrak{S}}))$ is *relatively sequentially compact*, *i.e.* any sequence in $\mathcal{D}$ admits a convergent subsequence.

Definition 2.2. **(**[30]**, page 373)** Let (P_n) and (Q_n) be two sequences in $\mathcal{M}_1(T, \tau)$, where (T, τ) is a topological space. We shall say that (P_n) and (Q_n) are $\flat(\tau)$*–equivalent* (or shortly *equivalent*) if, for every subsequence $(P_{n_k})_k$ of (P_n) which narrowly converges to a limit P, the subsequence $(Q_{n_k})_k$ also converges to P, and, conversely, for every subsequence $(Q_{n_k})_k$ of (Q_n) which narrowly converges to a limit P, the subsequence $(P_{n_k})_k$ also converges to P.

If (P_n) is relatively sequentially compact for $\flat(\tau)$ and if (P_n) and (Q_n) are $\flat(\tau)$–equivalent, then (Q_n) is relatively sequentially compact for $\flat(\tau)$.

Lemma 2.3. *Let T be a topological space such that there exists a countable set $\mathcal{H}$ of continuous bounded functions on T, which separates the points of T. Let (P_n) and (Q_n) be sequences in $\mathcal{M}_1(T)$ such that the set*

$$\mathcal{K} = \{P_n;\, n \in \mathbb{N}\} \cup \{Q_n;\, n \in \mathbb{N}\}$$

is relatively sequentially compact in $(\mathcal{M}_1(T), \flat(T))$. Then (P_n) and (Q_n) are equivalent if and only if

$$\forall f \in \mathcal{H},\ \lim_{n\to+\infty} P_n(f) - Q_n(f) = 0.$$

Proof. Let Let $\flat = \flat(T)$ and let $\flat'$ be the coarsest topology on $\mathcal{M}_1(T)$ such that the mappings

$$\begin{cases} \mathcal{M}_1(T) & \to & \mathbb{R} \\ P & \mapsto & P(f) \end{cases}$$

are continuous for every $f \in \mathcal{H}$. The topology $\flat'$ is Hausdorff and coarser than $\flat$. We only need to prove that, for any sequence (R_n) in $\mathcal{K}$, (R_n) converges for $\flat$ to a law R if and only if it converges to R for $\flat'$. If (R_n) converges to R for $\flat$, then it is clear that it converges to R for $\flat'$. Conversely, assume that (R_n) converges to R for $\flat'$. As $\mathcal{K}$ is relatively sequentially compact for $\flat$, we can extract from any subsequence (R'_n) of (R_n), a subsequence (R''_n) which converges for $\flat$ to a law P, and necessarily $P = R$. Thus (R_n) converges to R for $\flat$. □

The following Glivenko–Cantelli type result is an adaptation of Theorem 2 in [13]. It will be the first main step in the proof of the SLLN. We denote by δ_x the Dirac mass at point x.

Theorem 2.4 (Glivenko–Cantelli Theorem). *Let T be a topological space such that there exists a countable set $\mathcal{H}$ of continuous bounded functions on T, which separates the points of T. Let (X_n) be a sequence of pairwise independent random elements of T, defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Assume that the sequence $(P_n)_n = (1/n \sum_{i=1}^n P_{X_i})_n$ is tight. Then, μ–almost everywhere, the sequence $(Q_n^\omega)_n = (1/n \sum_{i=1}^n \delta_{X_i(\omega)})_n$ is tight and $(P_n)_n$ and $(Q_n^\omega)_n$ are $\flat(T)$–equivalent.*

Proof. For any bounded (borel) measurable function $f : T \to \mathbb{R}$, we have, for almost every $\omega \in \Omega$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} (f(X_i(\omega)) - \mathbb{E}f(X_i)) = 0 \quad \text{a.e.} \tag{2.1}$$

Indeed, this is a consequence of the SLLN proved by Csörgo, Tandori and Totik ([12], Theorem 1), or of the similar one proved by Etemadi ([19], Corollary 1). As $\mathcal{H}$ is countable, we have, for almost every $\omega \in \Omega$,

$$\forall f \in \mathcal{H}, \quad \lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} (f(X_i(\omega)) - \mathbb{E}f(X_i)) = 0.$$

From Lemma 2.3, as $\{P_n;\, n \in \mathbb{N}\}$ is relatively compact, we only need to show that, for almost every $\omega \in \Omega$, $\{Q_n^\omega;\, n \in \mathbb{N}\}$ is relatively compact.

For each $m \in \mathbb{N}^*$, there exists a compact subset K_m of T such that, for any $n \in \mathbb{N}^*$, $P_n(K_m) > 1 - 1/m$. Application of formula (2.1) to the function $f = \mathbb{1}_{K_m}$ gives

$$\liminf Q_n^\omega(K_m) \geq 1 - \frac{1}{m} \text{ a.e.},$$

i.e. there exists for μ-almost every $\omega \in \Omega$ an integer $n(\omega, m)$ such that

$$n \geq n(\omega, m) \Rightarrow Q_n^\omega(K_m) \geq 1 - \frac{2}{m}.$$

Let

$$K(\omega, m) = K_m \cup \{X_i(\omega);\, 1 \leq i \leq n(\omega, m)\}.$$

Then $K(\omega, m)$ is compact and, for any $n \in \mathbb{N}^*$,

$$Q_n^\omega(K(\omega, m)) \geq 1 - \frac{2}{m}.$$

So, there exists $\Omega' \in \mathcal{F}$ with $\mu(\Omega') = 1$ such that, for every $\omega \in \Omega'$, $(Q_n^\omega)_n$ is tight, thus relatively compact. □

2.3. **Pettis integrability and $\mathfrak{S}$–uniform scalar integrability.** A law P on E is said to be *scalarly integrable* if , for every $x' \in E'$, $\int |x'|\, dP$ is finite. A scalarly integrable law P is *Pettis integrable* if, for each $B \in \mathcal{B}_E$, there exists an element of E, denoted by $\int_B x\, dP(x)$, such that, for every $x' \in E'$,

$$\langle x', \int_B x\, dP(x) \rangle = \int_B \langle x', x \rangle\, dP(x).$$

A random element of E is *scalarly integrable* (respectively *Pettis integrable*) if its law is scalarly integrable (respectively Pettis integrable). If $X : (\Omega, \mathcal{F}, \mu) \to E$ is Pettis integrable, we use the notations

$$\mathbb{E}X = \int X\, d\mu = \int_E x\, dP_X(x).$$

This definition of Pettis integrability of a random element seems somewhat weaker than the usual one: if X is Pettis integrable in our sense, then, for each $B \in \mathcal{F}$ of the form $B = X^{-1}(\tilde{B})$ ($\tilde{B} \in \mathcal{B}_E$), there exists an element $\int_B X\, d\mu$ of E, such that, for every $x' \in E'$,

$$\langle x',\, \int_B X\, d\mu\rangle = \int_B \langle x',\, X\rangle\, d\mu.$$

But, for X to be Pettis integrable in the usual sense (*e.g.* [43]), we should have the same property for any $B \in \mathcal{F}$. We shall see later in Proposition 2.11 that both definitions coincide if E is quasi–complete (*i.e.* if every bounded closed subset of E is complete).

Let us first investigate some links between Pettis integrability and uniform integrability.

Definition 2.5. Let $\mathcal{D}$ be a set of laws on E and $\mathcal{H}$ a set of measurable functions defined on E with values in $\mathbb{R}$. We shall say that $\mathcal{H}$ is *uniformly integrable w.r.t.* $\mathcal{D}$ if

$$\lim_{a\to+\infty} \sup_{f\in\mathcal{H},\, P\in\mathcal{D}} \int_{|f|\geq a} |f|\, dP = 0. \tag{2.2}$$

If $\mathcal{H}$ is a family $(f_i)_{i\in I}$, we shall say that $\mathcal{H}$ is uniformly integrable w.r.t. $\mathcal{D}$ if $\{f_i;\, i \in I\}$ is uniformly integrable w.r.t. $\mathcal{D}$. We shall make the analogous convention for $\mathcal{D}$.

Equation (2.2) implies in particular that each $f \in \mathcal{H}$ is integrable w.r.t. each $P \in \mathcal{D}$.

It is easy to extend de la Vallée Poussin's criterion to this definition. We recall [29] that an N-function is a (necessarily continuous) convex even function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$ and $\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty$.

Lemma 2.6 (**Generalized de la Vallée Poussin's criterion**)**.** *Let $\mathcal{D}$ and $\mathcal{H}$ be as in Definition 2.5. The following are equivalent:*

(i) *$\mathcal{H}$ is uniformly integrable w.r.t. $\mathcal{D}$.*
(ii) *There exists a convex even function $\varphi : \mathbb{R} \to [0, +\infty[$ such that $\lim_{t\to+\infty} \varphi(t)/t = +\infty$ and*

$$\sup_{f\in\mathcal{H},\, P\in\mathcal{D}} \int \varphi \circ f\, dP < +\infty. \tag{2.3}$$

(iii) *There exists an* N*-function* φ *satisfying (2.3).*

Proof. Let X_P ($P \in \mathcal{D}$) be the coordinate mappings on $E^{\mathcal{D}}$ and consider the probability space

$$(S, \Sigma, \lambda) = (E^{\mathcal{D}}, \underset{P\in\mathcal{D}}{\otimes} \mathcal{B}_E, \underset{P\in\mathcal{D}}{\otimes} P).$$

Then, for each P, the law of X_P is P, and we have, for every $f \in \mathcal{H}$ and every Borel subset A of E,

$$\int_A f\, dP = \int_{X_P^{-1}(A)} f \circ X_P\, d\lambda.$$

Thus, Condition (2.2) may be written

$$\lim_{a\to+\infty} \sup_{f\in\mathcal{H},\, P\in\mathcal{D}} \int_{|f\circ X_P|\geq a} |f \circ X_P|\, d\lambda = 0.$$

Analogously, Inequality (2.3) may be written

$$\sup_{f\in\mathcal{H},\, P\in\mathcal{D}} \int \varphi(|f \circ X_P|)\, d\lambda < +\infty.$$

The conclusion follows from the usual de la Vallée Poussin's criterion (*e.g.* [16, 33], see also [1] for a sophisticated version), applied to the set $\{f \circ X_P;\ f \in \mathcal{H}, P \in \mathcal{D}\}$. $\square$

Definition 2.7. We shall say that a set $\mathcal{D}$ of laws on E is $\mathfrak{S}$*–uniformly scalarly integrable* if every element of $\mathfrak{S}$ is uniformly integrable w.r.t. $\mathcal{D}$.

A set $\mathcal{H}$ of random elements of E defined on a probability space (S, Σ, λ) is $\mathfrak{S}$*–uniformly scalarly integrable* if the set $\{P_X;\ X \in \mathcal{H}\}$ is $\mathfrak{S}$–uniformly scalarly integrable.

Our definition of $\mathfrak{S}$–uniform scalar integrability is slightly less general than the definition of $\mathfrak{S}$–uniform scalar integrability in [3].

Remark 2.8. ($\mathfrak{S}$–uniform scalar integrability and Pettis integrability) From Lemma in [32], if P is a Pettis integrable law on E, then $\{P\}$ is $\mathfrak{S}$–uniformly scalarly integrable. The converse implication is true if E is a separable Fréchet space (see Proposition 2.9 below, and [45, 33] for the case of Banach spaces), but not in general: for example, if E is a separable Banach space, $(E, \sigma(E, E'))$ is Suslin, with the same Borel tribe and the same Pettis integrable laws as E. But the $\mathfrak{S}$–uniformly scalarly integrable laws on $(E, \sigma(E, E'))$ are simply the scalarly integrable laws on E, which are not always Pettis integrable (see Example 4.1 of [33]).

Proposition 2.9. *Suppose that E is a Fréchet space and let P be a law on E. If $\{P\}$ is $\mathfrak{S}$–uniformly scalarly integrable, then P is Pettis integrable.*

Proof. Let $B \in \mathcal{B}_E$. Let y be the linear form on E' defined by $y(x') = \int_B x'\,dP$. Let $E'_\sigma = (E', \sigma(E', E))$. The dual of E'_σ is E. Thus, in order to prove that $y \in E$, we only need to show that y is continuous on E'_σ. But, from a theorem of Banach ([8], Corollaire 1 page III.21), as E is complete, this amounts to the same as to show that, for any $A \in \mathfrak{S}$, the restriction of y to A is continuous for the topology induced by $\sigma(E', E)$. Let $A \in \mathfrak{S}$. If $(x'_n)_n$ is a sequence of elements of A which converges to an element x of A, it is straightforward from Vitali Theorem that $y(x'_n)$ converges to $y(x')$. As A is metrizable for $\sigma(E', E)$, this proves that y is continuous on A for $\sigma(E', E)$. □

The set of $\mathfrak{S}$–uniformly scalarly integrable random vectors defined on $(\Omega, \mathcal{F}, \mu)$ is endowed with the family of semi–norms

$$X \mapsto \mathfrak{N}_A(X) := \sup_{x' \in A} \int |\langle x',\, X\rangle|\,d\mu \quad (A \in \mathfrak{S})$$

(we have $\mathfrak{N}_A(X) < +\infty$ because P_X is $\mathfrak{S}$–uniformly scalarly integrable).

Here is an easy lemma.

Lemma 2.10. *If X and Y are Pettis integrable random vectors of E defined on $(\Omega, \mathcal{F}, \mu)$, then, for every $A \in \mathfrak{S}$,*

$$N_A(\mathbb{E}X - \mathbb{E}Y) \leq \mathfrak{N}_A(X - Y).$$

Proof. . Indeed we have, for any $A \in \mathfrak{S}$,

$$\begin{aligned} N_A(\mathbb{E}X - \mathbb{E}Y) &= \sup_{x' \in A} |\langle x',\, \int X\,d\mu - \int Y\,d\mu\rangle| \\ &\leq \sup_{x' \in A} \int |\langle x',\, X - Y\rangle|\,d\mu = \mathfrak{N}_A(X - Y). \end{aligned}$$

□

We now prove that, if E is quasi–complete, our definition of Pettis integrability of random vectors is equivalent to the usual one.

Proposition 2.11. *Assume that E is quasi–complete. Let X be a scalarly integrable random element of E defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Then X is Pettis integrable if and only if, for any $B \in \mathcal{F}$,* *there exists an element of E, that we shall denote by $\int_B X\,d\mu$, such that, for every $x' \in E'$,*

$$\langle x',\, \int_B X\,d\mu\rangle = \int_B \langle x',\, X\rangle\,d\mu.$$

Proof. The sufficiency is obvious.

For the necessity, let us assume that X is Pettis integrable. Let $\mathcal{F}_X$ be the subtribe of $\mathcal{F}$ generated by X. For every $\mathcal{F}_X$–measurable simple function $h : \Omega \to \mathbb{R}$, hX is Pettis integrable.

Now, if h is a bounded countably valued $\mathcal{F}_X$–measurable function defined on Ω, it follows from quasi–completeness of E and countable additivity of the mapping $\mathcal{F}_X \to E$, $B \to \int_B X\,d\mu$ (see [43], this is a consequence of the Orlicz–Pettis Theorem) that hX is still Pettis integrable.

Let h be a bounded $\mathcal{F}_X$–measurable function defined on Ω, and let $(h_n)_n$ be a sequence of $\mathcal{F}_X$–measurable countably valued functions which uniformly converges to h. It is easy to see that the sequence $(\int h_n X\,d\mu)_n$ is scalarly bounded, thus bounded. Furthermore, let $A \in \mathfrak{S}$, $\epsilon > 0$ and $n \in \mathbb{N}$ such that, for any $p \in \mathbb{N}$, $\|h_{n+p} - h_n\|_\infty \le \epsilon/\mathfrak{N}_A(X)$. We have, for any $p \in \mathbb{N}$, using Lemma 2.10,

$$\begin{aligned}
N_A(\mathbb{E}h_{n+p} - \mathbb{E}h_n X) &\le \mathfrak{N}_A((h_{n+p} - h_n)X)\\
&= \sup_{x'\in A} \int |h_{n+p} - h_n|\,|\langle x',\, X\rangle|\,d\mu\\
&\le \epsilon\,\mathfrak{N}_A(X),
\end{aligned}$$

which proves that $(\int h_n X\,d\mu)_n$ is Cauchy. From quasi–completeness of E, it has a limit $x \in E$ which satisfies $\langle x',\, x\rangle = \int \langle x',\, hX\rangle\,d\mu$.

Finally, let $B \in \mathcal{F}$. Let $h = \mathbb{E}(\mathbb{1}_B/\mathcal{F}_X)$ be the conditional expectation of $\mathbb{1}_B$ given $\mathcal{F}_X$. The mapping hX is Pettis integrable and we have, for any $x' \in E'$,

$$\langle x',\, \int hX\,d\mu\rangle = \int_B \langle x',\, X\rangle\,d\mu,$$

which proves our claim with $\int_B X\,d\mu = \int \mathbb{E}(\mathbb{1}_B/\mathcal{F}_X)X\,d\mu$. $\square$

In the sequel all weakly integrable laws will be Radon. The set of scalarly integrable (respectively Pettis integrable) Radon laws on E will be denoted by MW^1_E (respectively MP^1_E). These sets obviously remain the same if we replace $\tau_{\mathfrak{S}}$ by a topology consistent with $\langle E,\, E'\rangle$ which has the same Borel sets.

The set of Radon $\mathfrak{S}$–uniformly scalarly integrable laws on E will be denoted by MU^1_E. We have thus $\mathrm{MP}^1_E \subset \mathrm{MU}^1_E \subset \mathrm{MW}^1_E$.

From a result of the first author (given in [46], § 0.13, pages 17–19), if E is weakly sequentially complete, then $\mathrm{MP}^1_E = \mathrm{MW}^1_E$.

3. Narrow convergence and uniform integrability

The following general lemma is essential in this section. It is the second and last main step in our proof of the SLLN.

Lemma 3.1 **(Narrow convergence and uniform integrability).**

1. Let T be a separable topological space. Let $(P_\gamma)_{\gamma\in\Gamma}$ be a net of laws on T, narrowly converging to a law $P \in \mathcal{M}_1(T)$. Let $\mathcal{H}$ be a set of continuous functions on T with values in $\mathbb{R}$ and assume that $\mathcal{H}$ is uniformly integrable w.r.t. $(P_\gamma)_\gamma$. Then $\mathcal{H}$ is uniformly integrable w.r.t. P and, for every $f \in \mathcal{H}$,

$$P_\gamma(f) \to P(f). \tag{3.1}$$

2. If furthermore $\mathcal{H}$ is equicontinuous, then the convergence in (3.1) is uniform w.r.t. $\mathcal{H}$.

Proof.
1. *Step 1.* We first prove the Lemma in the case when $\mathcal{H}$ has a single element f. We assume also that $f \geq 0$, the extension to general f being standard. For every $a > 0$, let us define a continuous function $u_a : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$u_a(x) \leq x$ for every $x \geq 0$, $u_a(x) = x$ if $x \leq a$ and $u_a(x) = 0$ if $x > a + 1$.

We have, by Fatou's Lemma, $P(f) = \lim_{a\to+\infty} P(u_a(f))$. As, for any $a > 0$,

$$P(u_a(f)) = \lim_\gamma P_\gamma(u_a(f)) \leq \sup_\gamma P_\gamma(f) < +\infty,$$

this proves that f is P–integrable.

Let $\epsilon > 0$. There exists $a > 0$ such that $P(f\,\mathbb{1}_{f>a}) < \epsilon$ and $\sup_\gamma P_\gamma(f\,\mathbb{1}_{f>a}) < \epsilon$. Thus

$$\begin{aligned}\liminf P_\gamma(f) - \epsilon \leq \lim_\gamma P_\gamma(u_a(f)) \leq P(f)\\ \leq P(u_a(f)) + \epsilon \leq \limsup P_\gamma(f) + \epsilon,\end{aligned}$$

which proves Step 1.

Step 2. We now turn to the general case. Let $\epsilon > 0$. Let $a > 1$ such that

$$\sup_{f,\gamma} \int_{|f|>a-1} |f| \, dP_\gamma < \epsilon.$$

Let $v: x \mapsto x - u_{a-1}(x)$. Using Step 1, we get, for each $f \in \mathcal{H}$,

$$\int_{|f|>a} |f|\,dP \leq \int v(|f|)\,dP$$
$$= \lim_{\gamma\in\Gamma} \int v(|f|)\,dP_\gamma \leq \liminf_{\gamma\in\Gamma} \int_{|f|>a-1} |f|\,dP_\gamma < \epsilon,$$

thus

$$\sup_{f\in\mathcal{H}} \int_{|f|>a} |f|\,dP < \epsilon.$$

2. *Step 1.* Let us consider the semi–distance δ on T defined by $\delta(x,y) = \sup_{f\in\mathcal{H}} |f(x) - f(y)|$. From the equicontinuity of $\mathcal{H}$, δ is continuous. Let T/δ be the separable metric space, quotient of T by the equivalence relation $\delta(x,y) = 0$, and let S be its completion, endowed with the quotient distance $\widehat{\delta}$. The canonical projection $\mathrm{pr} : T \to S$ is continuous. Let us denote by $\widehat{Q}$ the image $\mathrm{pr}(Q)$ on S of a probability Q on T. The net $(\widehat{P}_\gamma)_{\gamma\in\Gamma}$ narrowly converges to $\widehat{P}$. On the other hand, each $f \in \mathcal{H}$ may be written $f = \widehat{f} \circ \mathrm{pr}$, for a function $\widehat{f} : T/\delta \to \mathbb{R}$. We have, for all $x, y \in T/\delta$,

$$\widehat{\delta}(x,y) = \sup_{f\in\mathcal{H}} |\widehat{f}(x) - \widehat{f}(y)|, \tag{3.2}$$

so the set $\{\widehat{f};\ f \in \mathcal{H}\}$ is uniformly equicontinuous, and its elements may be extended over S. Let us use the same notation for any function $\widehat{f}$ ($f \in \mathcal{H}$) and its extension, and let $\widehat{\mathcal{H}}$ be the set of these extensions. Formula (3.2) still holds true, and $\widehat{\mathcal{H}}$ is uniformly equicontinuous. Furthermore we have, for each $f \in \mathcal{H}$ and each probability Q on T,

$$\int_T f dQ = \int_S \widehat{f} d\widehat{Q}.$$

Therefore, we only need to show the lemma for S, $(\widehat{P}_\gamma)_{\gamma\in\Gamma}$ and $\widehat{\mathcal{H}}$.

Step 2. So, we can (and shall) make without loss of generality the assumption that T is a Polish space endowed with a distance d, and that $\mathcal{H}$ satisfies, for all $x, y \in T$,

$$\sup_{f\in\mathcal{H}} |f(x) - f(y)| \leq \mathrm{d}(x,y). \tag{3.3}$$

From Van der Vaart and Wellner's representation theorem ([49], Theorem 1.10.3), there exist a net $(X_\gamma)_{\gamma\in\Gamma}$ of random elements of the Polish space T, defined on a probability space (S, Σ, λ), and a random element X of T defined on (S, Σ, λ), such that $P_{X_\gamma} = P_\gamma$ ($\gamma \in \Gamma$), $P_X = P$ and $(X_\gamma)_{\gamma\in\Gamma}$ converges almost uniformly to X (recall that Egorov's theorem is no more valid for nets). Now, the line of proof is similar to that of Proposition 2.1 in [3]: there exists for each $\epsilon > 0$ a set $B_\epsilon \in \Sigma$ such that $\lambda(B_\epsilon) \geq 1 - \epsilon$ and $(X_\gamma)_{\gamma\in\Gamma}$ converges uniformly to X on B_ϵ. Let $\epsilon > 0$. The family $(f \circ X_\gamma)_{f,\gamma}$ is uniformly integrable, thus there exists $\eta > 0$ such that, for any $A \in \Sigma$,

$$\lambda(A) \leq \eta \Rightarrow \sup_{f\in\mathcal{H},n\in\mathbb{N}} \int_A |f \circ X_\gamma| \, d\lambda < \frac{\epsilon}{3}.$$

Let $B = B_\eta$. Let $\gamma_0 \in \mathbb{N}$ such that

$$\gamma \geq \gamma_0 \Rightarrow \sup_{\omega\in B} \mathrm{d}(X_\gamma(\omega), X(\omega) < \frac{\epsilon}{3}.$$

We have, for $\gamma \geq \gamma_0$,

$$\begin{aligned}
&\sup_{f\in\mathcal{H}} |P_\gamma(f) - P(f)| \\
&\quad\leq \sup_{f\in\mathcal{H}} \left(\int_B |f \circ X_\gamma - f \circ X| \, d\lambda + |\int_{B^c} f \circ X_\gamma \, d\lambda| + |\int_{B^c} f \circ X \, d\lambda| \right) \\
&\quad\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

□

Here is a first application of Lemma 3.1, with the help of a Vitali theorem of K. Musiał [32] and of a Skorokhod representation theorem of A. Jakubowski [26]. This result will not be needed in the proof of the SLLN. Note that Musiał's result was first given by R. Geitz [23] for Banach spaces and perfect probability spaces. An alternative proof of Musiał's result is also given (for Banach spaces, but the arguments remain unchanged in locally convex spaces) in [9] and [2].

Theorem 3.2 (**Generalized Vitali convergence theorem**). *Let us assume that E is quasi–complete. Let $(P_n)_n$ be a $\mathfrak{S}$–uniformly scalarly integrable sequence in MP^1_E, narrowly converging to a law P on E. Let us also assume that $(P_n)_n$ is tight (we recall that, if E is not Prokhorov, narrow convergence does not imply tightness). Then $P \in MP^1_E$ and*

$$\lim_{n\to+\infty} \int x \, dP_n(x) = \int x \, dP(x).$$

Proof. From Jakubowski's result ([26], Theorem 2), as $(P_n)_n$ is tight and as the points of E are separated by a countable set of bounded continuous functions on E, there exist a subsequence $(P_{n_k})_k$ of $(P_n)_n$ and random elements X and X_k ($k \in \mathbb{N}$) of E, defined on $([0,1], \mathcal{B}_{[0,1]}, \lambda)$ (where λ is the Lebesgue measure on $[0,1]$), such that $P_X = P$, $P_{X_k} = P_{n_k}$ ($k \in \mathbb{N}$) and $(X_k)_k$ converges a.e. to X. From $\mathfrak{S}$–uniform scalar integrability of $(P_n)_n$ and Proposition 2.11, each X_n is Pettis integrable in the usual sense. Furthermore, for each $x' \in E'$, from the usual Vitali Theorem, $(\langle x', X_k\rangle)_k$ converges to $\langle x', X\rangle$ in $\mathrm{L}^1_{\mathbb{R}}$, because the sequence $(\langle x', X_k\rangle)_k$ is uniformly integrable. Thus, for any $B \in \mathcal{B}_{[0,1]}$ and any $x' \in E'$, we have

$$\begin{aligned}&\lim_{k\to+\infty} \Big|\int_B \langle x', X_k\rangle \, d\lambda - \int_B \langle x', X\rangle \, d\lambda\Big| \\ &\quad\le \lim_{k\to+\infty} \int_B |\langle x', X_k - X\rangle| \, d\lambda \le \lim_{k\to+\infty} \int_{[0,1]} |\langle x', X_k - X\rangle| \, d\lambda = 0.\end{aligned}$$

Furthermore, from Lemma 3.1, P is $\mathfrak{S}$–uniformly scalarly integrable, *i.e.* for every $A \in \mathfrak{S}$, $(\langle x', X\rangle)_{x'\in A}$ is uniformly integrable.

From Musiał's version of Vitali Theorem ([32], Theorem 1), this implies that $P \in \mathrm{MP}^1_E$.

But, from Lemma 3.1,

$$\lim_{n\to+\infty} P_n(x') = P(x')$$

for every $x' \in E'$, uniformly on any element of $\mathfrak{S}$, *i.e.*

$$\lim_{n\to+\infty} \langle x', \int x \, dP_n(x)\rangle = \langle x', \int x \, dP(x)\rangle$$

uniformly on any element of $\mathfrak{S}$. □

Remark 3.3. Even if (P_n) is $\mathfrak{S}$–uniformly scalarly integrable, (P_n) may be narrowly convergent but not tight. This can be illustrated as follows, by a slight adaptation of an example of X. Fernique ([20], Exemple 1.6.4). Let H be a separable Hilbert space and $(e_k)_{k\in\mathbb{N}}$ an orthogonal sequence of unit vectors of H. Assume that $E = (H, \sigma(H, H'))$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of elements of $]0,1[$ such that $(n^3 \log \alpha_n)_n$ is bounded by a number β. Set, for every $n \in \mathbb{N}$,

$$P_n = (1 - \alpha_n) \sum_{k\in\mathbb{N}} \alpha_n^k \delta_{ne_k}$$

(where δ_x denotes the Dirac measure concentrated on x). X. Fernique has shown that (P_n) is not tight but narrowly converges to δ_0.

It is easy to see that each P_n is Pettis integrable: for example, we have

$$P_n(\|.\|) = n(1 - \alpha_n) \sum_{k\in\mathbb{N}} \alpha_n^k = n < +\infty.$$

This proves that P_n is the law of a Bochner integrable random vector of H, thus ([43], Theorem 3 or [33], Proposition 5.1) P_n is a Pettis integrable law on H. As E and H have the same Pettis integrable laws, P_n is a Pettis integrable law on E.

Let us show that (P_n) is $\mathfrak{S}$–uniformly scalarly integrable. Let $x' = \sum_{k\in\mathbb{N}} \lambda_k e_k \in E$. For any $a > 0$ and $n \geq 1$, we have

$$P_n(|x'|\,\mathbb{1}_{\{|x'|>a\}}) \;=\; (1-\alpha_n)\sum_{|\lambda_k|>a/n} \alpha_n^k |n\lambda_k|.$$

But the number of indexes k such that $\lambda_k > a/n$ is at most $c_n = \|x'\|^2 n^2/a^2$. Let us denote by $[c_n]$ the integer part of c_n. As $(\alpha_n^k)_k$ is decreasing, and as $|\lambda_k| \leq \|x'\|$ for every $k \in \mathbb{N}$, we have thus

$$\begin{aligned}
P_n(|x'|\,\mathbb{1}_{\{|x'|>a\}}) &\leq (1-\alpha_n)\sum_{0\leq k\leq [c_n]-1} \|x'\| n \alpha_n^k \\
&= n\|x'\|(1-\alpha_n^{[c_n]}) \\
&\leq n\|x'\|\,(1-\exp(c_n \log \alpha_n)) \\
&\leq n\|x'\|(-c_n \log \alpha_n) \leq \frac{\beta\|x'\|^3}{a}.
\end{aligned}$$

Thus

$$\lim_{a\to+\infty} \sup_{n\in\mathbb{N}} P_n(|x'|\,\mathbb{1}_{\{|x'|>a\}}) = 0.$$

The following is an application of Lemma 3.1 and Theorem 3.2 providing a new Komlós type convergence result for Young measures.

We recall that a Young measure on E defined on a probability space $(\Omega, \mathcal{F}, \mu)$ is a measurable mapping from $(\Omega, \mathcal{F}, \mu)$ to $(\mathcal{M}_1(E), \flat(\tau_{\mathfrak{S}}))$. The set of Young measures on E defined on $(\Omega, \mathcal{F}, \mu)$ is denoted by $\mathcal{Y}(\Omega, \mathcal{F}, \mu, E)$.

Lectures on Young measures are [48] and [4].

Theorem 3.4. *Suppose that E is a separable Fréchet space and that $\mathcal{H}$ is a subset of $\mathcal{Y}(\Omega, \mathcal{F}, \mu, E)$ satisfying*

(i) *for any sequence $(\lambda^n)_n$ in $\mathcal{H}$, there exists a sequence $(\nu^n)_n$ in $\mathcal{Y}(\Omega, \mathcal{F}, \mu, E)$ such that, for every $n \in \mathbb{N}$, ν^n is in the convex hull $co\{\lambda^m;\, m \geq n\}$ of $\{\lambda^m;\, m \geq n\}$ and such that, for almost every $\omega \in \Omega$, $(\nu^n_\omega)_n$ narrowly converges in $\mathcal{M}_1(E)$;*

(ii) *for each $\omega \in \Omega$, the set $\mathcal{H}_\omega = \{\lambda_\omega;\, \lambda \in \mathcal{H}\}$ is $\mathfrak{S}$–uniformly scalarly integrable.*

Then, for any sequence $(\lambda^n)_n$ in $\mathcal{H}$, there exist a subsequence $(\lambda^{\alpha(n)})_n$ and $\lambda^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, \mu, E)$ such that, for each further subsequence $(\lambda^{\beta(n)})_n$, the following hold:

(a) *for almost every $\omega \in \Omega$, the sequence $\left(1/n\sum_{j=1}^n \lambda_\omega^{\beta(j)}\right)_n$ narrowly converges to λ^∞_ω,*

(b) $\lambda^\infty_\omega \in MP^1_E$ *a.e.*,
(c) *for almost every* $\omega \in \Omega$,

$$\lim_{n\to+\infty} \int_E x\, \frac{1}{n}\left(\sum_{j=1}^n \lambda_\omega^{\beta(j)}\right)(dx) = \int_E x\, \lambda^\infty_\omega(dx).$$

Proof. In view of (i) and Proposition 3.8 in [10], for any sequence $(\lambda^n)_n$ in $\mathcal{H}$, there exist a subsequence $(\lambda^{\alpha(n)})_n$ and $\lambda^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, \mu, E)$ such that, for each further subsequence $(\lambda^{\beta(n)})_n$ and for almost every $\omega \in \Omega$, the sequence $\left(1/n \sum_{j=1}^n \lambda_\omega^{\beta(j)}\right)_n$ $\flat(\tau_{\mathfrak{S}})$–converges to λ^∞_ω (the negligible subset depends on the subsequence $(\lambda^{\beta(n)})_n$). Let $(\lambda^{\beta(n)})_n$ be a subsequence of $(\lambda^{\alpha(n)})_n$ and N the corresponding negligible set. Fix $\omega \in \Omega \setminus N$. For simplicity set

$$P^n_\omega = \frac{1}{n}\sum_{j=1}^n \lambda_\omega^{\beta(j)}.$$

From Proposition 2.9, and Condition (ii), each P^n_ω is Pettis integrable. The conclusion follows from Theorem 3.2. □

The use of Lemma 3.1 in the proof of the SLLN will be made through the following theorem.

Theorem 3.5. *Let* (P_n) *and* (Q_n) *be two* $\flat(\tau_{\mathfrak{S}})$*–equivalent and* $\flat(\tau_{\mathfrak{S}})$*–relatively sequentially compact* $\mathfrak{S}$*–uniformly scalarly integrable sequences in* MP^1_E*. Then*

$$\lim_{n\to+\infty} (\int x\, dP_n - \int x\, dQ_n) = 0.$$

Proof. Let $A \in \mathfrak{S}$. Let (P'_n, Q'_n) be a subsequence of (P_n, Q_n). From the relative sequential compactness of (P_n) and the equivalence of (P_n) and (Q_n), there exists a subsequence (P''_n, Q''_n) of (P'_n, Q'_n) such that (P''_n) and (Q''_n) narrowly converge to a law P. From Lemma 3.1, P is $\mathfrak{S}$–uniformly scalarly integrable, and

$$\lim_{n\to+\infty} \sup_{x'\in A} |\int \langle x', x\rangle\, dP''_n - \int \langle x', x\rangle\, dP| = 0$$
$$= \lim_{n\to+\infty} \sup_{x'\in A} |\int \langle x', x\rangle\, dQ''_n - \int \langle x', x\rangle\, dP|,$$

hence

$$\lim_{n\to+\infty} N_A(\int x\, dP''_n - \int x\, dQ''_n) = 0.$$

Thus

$$\lim_{n\to+\infty} N_A\Big(\int x\,dP_n - \int x\,dQ_n\Big) = 0.$$

$\square$

4. Strong law of large numbers

Theorem 4.1 (SLLN for Pettis integrable random vectors). *Let $(X_n)_n$ be a sequence of pairwise independent Pettis integrable random elements of E, defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Assume that:*

(i) The sequence $(1/n\sum_{i=1}^n P_{X_i})_n$ is tight.
(ii) The sequence $(1/n\sum_{i=1}^n P_{X_i})_n$ is $\mathfrak{S}$–uniformly scalarly integrable.
(iii) For almost every $\omega\in\Omega$, the sequence $(1/n\sum_{i=1}^n \delta_{X_i(\omega)})_n$ is $\mathfrak{S}$–uniformly scalarly integrable.

Then $(X_n)_n$ satisfies the SLLN, that is

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^n (X_i - \mathbb{E}X_i) = 0 \ \textit{a.e.}$$

Proof. By (iii) and Theorem 2.4, there is an element Ω' of $\mathcal{F}$ with $\mu(\Omega')=1$, such that, for every $\omega\in\Omega'$, the sequence $(1/n\sum_{i=1}^n \delta_{X_i(\omega)})$ of empirical laws is tight and $\mathfrak{S}$–uniformly scalarly integrable and such that the sequences $(\frac{1}{n}\Sigma_{i=1}^n P_{X_i})_n$ and $(\frac{1}{n}\Sigma_{i=1}^n \delta_{X_i(\omega)})_n$ are equivalent.

Now let $\omega\in\Omega'$ be fixed. The sequences $(P_n)_n = (\frac{1}{n}\Sigma_{i=1}^n P_{X_i})_n$ and $(Q_n^\omega)_n = (\frac{1}{n}\Sigma_{i=1}^n \delta_{X_i(\omega)})_n$, are relatively sequentially compact for the topology $\flat(\tau_{\mathfrak{S}})$ of narrow convergence. Thus, from Theorem 3.5,

$$\lim_{n\to+\infty} \Big(\int x\,dP_n - \int x\,dQ_n^\omega\Big) = 0,$$

i.e.

$$\lim_{n\to+\infty} \Big(\frac{1}{n}\sum_{i=1}^n \mathbb{E}X_i - \frac{1}{n}\sum_{i=1}^n X_i(\omega)\Big) = 0.$$

$\square$

In the case when E is a separable Banach space, the following corollary is exactly Theorem 3 in [13]. A random element $X : (\Omega,\mathcal{F},\mu)\to E$ is said to be *absolutely summable* if, for each $A\in\mathfrak{S}$, $\int N_A(X)\,d\mu < +\infty$ (if E is a separable Banach space, we also say that X is *Bochner integrable*). If E is quasi–complete and X is absolutely summable and P_X is Radon, then X is Pettis integrable (see the proof of Theorem 3 in [43]).

Corollary 4.2 (SLLN of Cuesta and Matrán). *Let $(X_n)_n$ be a sequence of pairwise independent absolutely summable random elements of E, defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Assume that:*

- (i) *The sequence $(1/n \sum_{i=1}^n P_{X_i})_n$ is tight.*
- $(ii)'$ *For every $A \in \mathfrak{S}$, N_A is uniformly integrable w.r.t. the sequence $(1/n \sum_{i=1}^n P_{X_i})_n$.*
- $(iii)'$ *There exists $\Omega' \in \mathcal{F}$, with $\mu(\Omega') = 1$, such that, for any $\omega \in \Omega'$ and any $A \in \mathfrak{S}$,*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} \left(N_A(X_i(\omega)) - \mathbb{E}N_A(X_i)\right) = 0. \tag{4.1}$$

Then $(X_n)_n$ satisfies the SLLN.

Proof. Let us denote, as in the above proof, $P_n = 1/n \sum_{i=1}^n P_{X_i}$ and $Q_n^\omega = 1/n \sum_{i=1}^n \delta_{X_i(\omega)}$ $(n \in \mathbb{N}^*, \omega \in \Omega)$.

First, it is clear that Condition $(ii)'$ implies Condition (ii) of Theorem 4.1.

Moreover, from Theorem 2.4, $(Q_n^\omega)_n$ and $(P_n)_n$ are almost everywhere tight equivalent sequences, thus there exists $\Omega'' \in \mathcal{F}$ such that $\Omega'' \subset \Omega'$, $\mu(\Omega'') = 1$ and $(Q_n^\omega)_n$ and $(P_n)_n$ are equivalent for every $\omega \in \Omega''$.

Fix $\omega \in \Omega''$ and $A \in \mathfrak{S}$. Equation (4.1) may be written

$$\lim_{n\to+\infty} (Q_n^\omega(N_A) - P_n(N_A)) = 0. \tag{4.2}$$

Let us define, for every $a > 0$ a bounded continuous function $u_a : \mathbb{R}^+ \to \mathbb{R}^+$ such that $u_a(t) = t$ if $t \leq a-1$, $u_a(t) = 0$ if $t \geq a$ and $0 \leq u_a(t) \leq t$ for every $t \geq 0$. Set $v_a(t) = t - u_a(t)$ $(t \geq 0)$. From the tightness and the equivalence of $(Q_n^\omega)_n$ and $(P_n)_n$, we have, for every $a > 0$,

$$\lim_{n\to+\infty} (Q_n^\omega(u_a \circ N_A) - P_n(u_a \circ N_A)) = 0.$$

Using (4.2), this yields

$$\lim_{n\to+\infty} (Q_n^\omega(v_a \circ N_A) - P_n(v_a \circ N_A)) = 0.$$

But, from Condition $(ii)'$, we have

$$\lim_{n\to+\infty} P_n(v_a \circ N_A) = 0.$$

Thus,

$$\lim_{n\to+\infty} Q_n^\omega(v_a \circ N_A) = 0,$$

i.e. N_A is uniformly integrable w.r.t. $(Q_n^\omega)_n$.

Thus Condition (iii) of Theorem 4.1 is fulfilled, and $(X_n)_n$ satisfies the SLLN. □

Let E be a separable Banach space. Let E'_c be the dual space of E, endowed with the topology τ_c of uniform convergence on the compact subsets of E. We recall that E'_c is a Lusin space (because it is a countable union of metrizable compact sets), and that a random element of E'_c is Pettis integrable if and only if it is scalarly integrable. Indeed, from Mackey Theorem, the topology of E'_c is consistent with the duality $\langle E', E\rangle$, thus $\mathrm{MP}^1_{E'_c} = \mathrm{MP}^1_{E'_\sigma}$, where $E'_\sigma = (E', \sigma(E', E))$. Furthermore, as E'_σ is semi–reflexive, $\mathrm{MP}^1_{E'_\sigma} = \mathrm{MW}^1_{E'_\sigma}$ (see *e.g.* Corollary 4.1. of [43]).

If $(\Omega, \mathcal{F}, \mu)$ is a probability space, we shall denote by $\mathrm{L}^1_{E'}[E](\Omega, \mathcal{F}, \mu)$, or shortly by $\mathrm{L}^1_{E'}[E]$, the set of random vectors X defined on $(\Omega, \mathcal{F}, \mu)$ such that $P_X \in \mathrm{MP}^1_{E'_c}$ and $\int_\Omega \|X\|\, d\mu < +\infty$ (see [25] about this space). The elements of $\mathrm{L}^1_{E'}[E]$ are thus absolutely summable random vectors of E'_c. Note that X is not necessarily measurable for the norm topology on E', but the measurability of $\|X\|$ follows from the scalar measurability of X and the separability of E.

Corollary 4.3 (SLLN for elements of $\mathbf{L}^1_{E'}[E]$). *Let E be a separable Banach space. Let $(X_n)_n$ be a sequence of pairwise independent elements of $L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$. Assume that:*

- (i) *The sequence $(P_n)_n = (1/n \sum_{i=1}^n P_{X_i})_n$ is a tight sequence of elements of $\mathcal{M}(E'_c)$.*
- $(ii)'$ *The function $\|.\|$ is uniformly integrable w.r.t. $(1/n \sum_{i=1}^n P_{X_i})_n$.*
- $(iii)'$ *The sequence of real random variables $(\|X_n\|)_n$ satisfies the SLLN, that is,*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^n (\|X_i\| - \mathbb{E}\|X_i\|) = 0 \quad a.e.$$

Then $(X_n)_n$ satisfies the SLLN in E'_c, i.e. *there exists $\Omega' \in \mathcal{F}$, with $\mu(\Omega') = 1$, such that, for any $\omega \in \Omega'$ and any compact subset K of E,*

$$\lim_{n\to+\infty} \sup_{x\in K} \left| \frac{1}{n} \sum_{i=1}^n \langle x,\, X_i(\omega) - \mathbb{E}X_i\rangle \right| = 0.$$

Proof. Clearly, Conditions (i) and (ii) of Theorem 4.1 are satisfied. Let us prove that Condition (iii) of Theorem 4.1 is also satisfied.

Let us denote, as usual, $P_n = 1/n \sum_{i=1}^n P_{X_i}$ and $Q_n^\omega = 1/n \sum_{i=1}^n \delta_{X_i(\omega)}$ $(n \in \mathbb{N}^*, \omega \in \Omega)$. For every $a \geq 0$ and every $t \geq 0$, let us set $u_a(t) = t\, \mathbb{1}_{[0,a]}(t)$ and $v_a(t) = t - u_a(t)$. The function $\|.\| : E'_c \to \mathbb{R}^+$ is l.s.c. because it is the supremum of a family of continuous functions: $\|x'\| = \sup_{x\in E, \|x\|\leq 1} \langle x,\, x'\rangle$ for each $x' \in E'$. Thus $u_a \circ \|.\|$ is bounded and Borel, and we can apply the SLLN of Csörgo, Tandori and Totik ([12], Theorem 1), or of Etemadi ([19], Corollary 1): there exists $\Omega'' \in \mathcal{F}$, with $\Omega'' \subset \Omega'$ and $\mu(\Omega'') = 1$, such that, for each $\omega \in \Omega''$,

$$\lim_{n\to+\infty} (Q_n^\omega(u_a \circ \|.\|) - P_n(u_a \circ \|.\|)) = 0. \tag{4.3}$$

Let $\omega \in \Omega''$. Condition $(iii)'$, may be written

$$\lim_{n\to+\infty} (Q_n^\omega(\|.\|) - P_n(\|.\|)) = 0. \tag{4.4}$$

From (4.3) and (4.4), we have

$$\lim_{n\to+\infty} (Q_n^\omega(v_a \circ \|.\|) - P_n(v_a \circ \|.\|)) = 0.$$

But, from Condition $(ii)'$, we have

$$\lim_{n\to+\infty} P_n(v_a \circ \|.\|) = 0,$$

thus

$$\lim_{n\to+\infty} Q_n^\omega(v_a \circ \|.\|) = 0, \tag{4.5}$$

i.e. $\|.\|$ is uniformly integrable w.r.t. $(Q_n^\omega)_n$.

Let K be a compact subset of E. There exists $c > 0$ such that $N_K \le c\|.\|$. Thus, from (4.5), using de la Vallée Poussin criterion, N_K is uniformly integrable w.r.t. $(Q_n^\omega)_n$.

This implies that Condition (iii) of Theorem 4.1 is fulfilled, and $(X_n)_n$ satisfies the SLLN in E'_c. □

Remark 4.4. In particular, if E is a separable Banach space and $(X_n)_n$ is a pairwise independent identically distributed sequence of elements of $\mathrm{L}^1_{E'}[E]$, then $(X_n)_n$ satisfies the SLLN in E'_c. So, Corollary 4.3 provides a version of Etemadi's SLLN [18] for elements of $\mathrm{L}^1_{E'}[E]$.

In the case of a Banach space endowed with its weak topology, it is possible to weaken slightly the assumption of tightness:

Corollary 4.5 (SLLN in the $\sigma(F, F')$ topology of a Banach space F). *Let F be a separable Banach space, and let us assume that $E = (F, \sigma(F, F'))$ (in this case E and F have the same bounded subsets, the same Borel tribe and the same Pettis integrable laws). Let $(X_n)_n$ be a sequence of pairwise independent Pettis integrable random elements of E, defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Assume that:*

- (i) *The sequence $(P_n)_n = (1/n \sum_{i=1}^n P_{X_i})_n$ is* tight relatively to bounded sets, i.e. *for each $\epsilon > 0$, there exists a bounded subset M of E such that, for every $n \in \mathbb{N}^*$, $P_n(M) > 1 - \epsilon$.*

(*ii*) *Each element of* $E' = F'$ *is uniformly integrable w.r.t. the sequence* $(1/n \sum_{i=1}^{n} P_{X_i})_n$.

(*iii*) *For almost every* $\omega \in \Omega$, *each element of* $E' = F'$ *is uniformly integrable w.r.t. the sequence* $(1/n \sum_{i=1}^{n} \delta_{X_i(\omega)})_n$.

Then $(X_n)_n$ *satisfies the SLLN.*

Proof. We are going to show that Theorem 4.1 applies to the sequence (X_n) considered as a sequence of random elements of the space $G = (F'', \sigma(F'', F'))$.

For any normed space N and $r \geq 0$, let us denote by $B_N(0, r)$ the closed ball of center 0 and radius r in N. we have

$$F'' = \cup_{n\in\mathbb{N}} B_{F''}(0, n) = \cup_{n\in\mathbb{N}} \overline{B_F(0, n)}^{F''}.$$

Each $B_{F''}(0, n)$ is compact for the topology induced by $\sigma(F'', F')$. Let $\mathcal{D}$ be a countable subset of F' that separates the points of F (such a set exists because F has the Lindelöf property, see for example [7, 40]). The topology $\sigma(F'', F')$ coincides on $B_{F''}(0, n)$ with the coarser Hausdorff topology of pointwise convergence on the elements of $\mathcal{D}$, which is metrizable. Therefore, the topology induced by $\sigma(F'', F')$ on $B_{F''}(0, n)$ is compact and metrizable, thus Polish.

So, G is a countable union of Suslin spaces, thus it is Suslin ([7], Proposition 8 page IX.60).

Furthermore, E is a Lusin subspace of the regular space G, thus E is a Borel subset of G ([7] (Lemme 7 page IX.67). So, any random element of E may be viewed as a random element of G and any law on E as a law on G.

As the closure in G of any bounded subset of E is compact, Condition (*i*) implies that (P_n) is a tight sequence of MP^1_G. The conclusion follows from application of Theorem 4.1 in G. □

Remark 4.6. Suppose that, for almost every $\omega \in \Omega$, $(X_n(\omega))_n$ is a bounded sequence in E. Then Condition (iii) in Theorem 4.1 is satisfied. We use this fact in the example below, which shows that Theorem 4.1 also applies to sequences of non–Bochner integrable random vectors of a Banach space.

Example 4.7. [**SLLN for Pettis non-Bochner integrable random vectors**] Let E be a separable Banach space, and let us denote by $\overline{B}_{E'}$ the closed unit ball of E'. Let U be a Pettis integrable random element of E, defined on $(\Omega, \mathcal{F}, \mu)$ (it is more interesting if we choose U non–Bochner integrable). As the sequence

$$(u_n)_{n\in\mathbb{N}^*} = (\mu\{\|U\| > n\})_{n\in\mathbb{N}^*}$$

converges to 0, we can extract a subsequence $(u_{\alpha_n})_{n\in\mathbb{N}^*}$ such that $\sum_{n\in\mathbb{N}^*} u_{\alpha_n} < +\infty$ and $u_{\alpha_n} < 1$ for every $n \in \mathbb{N}^*$. Since E is Polish, U is tight ([6], Theorem 1.4. page 10 or [35], Theorem 3.2. page 29). For each $\epsilon > 0$, there exists a compact set $K_\epsilon \subset E$ such that $\mu\{U \in K_\epsilon\} \geq 1-\epsilon$. From a theorem of Mazur, the closed convex hull $\tilde{K}_\epsilon$ of the compact set $\{0\}\bigcup K_\epsilon$ is also compact, and we have

$$\forall t \in [0,1], \quad \mu\{tU \in \tilde{K}_\epsilon\} \geq 1-\epsilon. \tag{4.6}$$

We consider now an independent sequence of random elements of E, with same law as U, defined on a probability space (S, Σ, λ). For each $n \in \mathbb{N}^*$, set

$$X_n = \frac{1}{\alpha_n} U_n.$$

From Equation (4.6), as $1/\alpha_n \in [0,1]$ for each $n \in \mathbb{N}^*$, the sequence $(X_n)_{n\in\mathbb{N}^*}$ is tight. Thus Condition (i) in Theorem 4.1 is fulfilled.

Furthermore,

$$\begin{aligned}
&\lim_{a\to+\infty} \sup_{n\in\mathbb{N}^*} \sup_{x'\in\overline{B}_{E'}} \int_{|\langle x',\, X_n\rangle|>a} |\langle x',\, X_n\rangle|\, d\lambda \\
&= \lim_{a\to+\infty} \sup_{n\in\mathbb{N}^*} \sup_{x'\in\overline{B}_{E'}} \frac{1}{\alpha_n} \int_{|\langle x',\, U\rangle|>\alpha_n a} |\langle x',\, U\rangle|\, d\mu = 0
\end{aligned}$$

because $(1/\alpha_n)_{n\in\mathbb{N}^*}$ is bounded and because $\{U\}$ is $\mathfrak{S}$–uniformly scalarly integrable. Thus $(X_n)_{n\in\mathbb{N}^*}$ is $\mathfrak{S}$–uniformly scalarly integrable, and Condition (ii) in Theorem 4.1 is fulfilled.

Finally, we have

$$\begin{aligned}
\lambda\{\forall n \in \mathbb{N}^*, \|X_n\| \leq 1\} &= \prod_{n\in\mathbb{N}^*} \mu\{\|U\| \leq \alpha_n\} \\
&= \prod_{n\in\mathbb{N}^*} (1 - \mu\{\|U\| > \alpha_n\}) \\
&= \prod_{n\in\mathbb{N}^*} (1 - u_{\alpha_n}).
\end{aligned}$$

As $\sum_{n\in\mathbb{N}^*} u_{\alpha_n} < +\infty$ and $u_{\alpha_n} \neq 1$ for every $n \in \mathbb{N}^*$, it follows from a classical result on infinite products that $\prod_{n\in\mathbb{N}^*}(1-u_{\alpha_n})$ converges to a limit other than 0. Thus $\lambda\{\forall n \in \mathbb{N}^*, \|X_n\| \leq 1\} > 0$. Let $A = \{\omega \in$

S; $(X_n(\omega))_{n\in\mathbb{N}^*}$ is bounded$\}$. We have

$$\begin{aligned}\lambda(A) &= \lambda\left(\bigcup_{m\in\mathbb{N}}\{\omega\in S;\, \forall n\in\mathbb{N}^*, \|X_n(\omega)\|\le m\}\right)\\ &\ge \lambda\{\omega\in S;\, \forall n\in\mathbb{N}^*, \|X_n(\omega)\|\le 1\}\\ &> 0.\end{aligned}$$

But the event A does not depend on the values of the first terms of $(X_n)_{n\in\mathbb{N}^*}$. We deduce from Kolmogorov's zero-one law that $\lambda(A)=1$. Thus $(X_n)_{n\in\mathbb{N}^*}$ is almost everywhere bounded (but not necessarily by a constant !). This implies Condition (*iii*) in Theorem 4.1.

So, the sequence $(X_n)_{n\in\mathbb{N}^*}$ satisfies the hypothesis of Theorem 4.1. If we assume that U is not Bochner integrable, then the X_n are not Bochner integrable either.

5. Generalized Kantorovich functionals on the space of $\mathfrak{S}$–uniformly scalarly integrable laws

We recall that MU^1_E is the set of $\mathfrak{S}$–uniformly scalarly integrable Radon laws on E.

Definition 5.1. If P and Q are elements of MU^1_E, we shall denote by $D(P,Q)$ the set of probabilities π on $E\times E$ with marginals P and Q, *i.e.* such that $\pi(.\times E)=P$ and $\pi(E\times .)=Q$. For each $A\in\mathfrak{S}$, we define a mapping $d_A:\ \mathrm{MU}^1_E\times\mathrm{MU}^1_E\to[0,+\infty[$ by

$$d_A(P,Q)=\inf_{\pi\in D(P,Q)}\sup_{x'\in A}\int_{E\times E}|\langle x',\,x-y\rangle|\,d\pi(x,y).$$

This definition of d_A extends in an obvious way on $\mathrm{MW}^1_E\times\mathrm{MW}^1_E$ or on $\mathcal{M}_1(E)\times\mathcal{M}_1(E)$ (with possibly infinite values in those cases).

The mapping d_A is a particular case of *generalized Kantorovich functional* in the sense of [38], page 137. We do not know if there exists a dual expression of d_A, as in the Kantorovich–Rubinštein Theorem. Before we prove that d_A is a semi–distance on MU^1_E, we need some preliminaries.

Definition 5.2. Let $A\in\mathfrak{S}$ and $P,Q\in\mathrm{MU}^1_E$. We shall say that a probability $\pi\in D(P,Q)$ is a *d_A–optimal coupling* of (P,Q) if

$$d_A(P,Q)=\sup_{x'\in A}\int_{E\times E}|\langle x',\,x-y\rangle|\,d\pi(x,y).$$

Proposition 5.3. *Let $A\in\mathfrak{S}$. Every $(P,Q)\in MU^1_E\times MU^1_E$ has a d_A–optimal coupling.*

Proof. Let P and Q be elements of MU^1_E. Then $D(P,Q)$ is a tight subset of the space of Borel measures on $E \times E$. Indeed, let $\epsilon > 0$. As P and Q are Radon, there exist two compact subsets K_1 and K_2 of E such that $P(K_1^c) < \epsilon$ and $Q(K_2^c) < \epsilon$. Let $\pi \in D(P,Q)$. We have

$$\pi(K_1 \times K_2)^c = \pi((K_1 \times E)^c \cup (E \times K_2)^c) \leq \pi(K_1 \times E)^c + \pi(E \times K_2)^c < 2\epsilon.$$

From the generalized Prokhorov's theorem ([44], Theorem 9.1 (iii) or [40], Theorem 3 page 379), $D(P,Q)$ is relatively compact, thus relatively sequentially compact, for the narrow topology on $\mathcal{M}(E \times E)$.

Furthermore, A is uniformly integrable w.r.t. $\{P,Q\}$ because P and Q are in MU^1_E. By de la Vallée Poussin's criterion, there exists an N-function φ such that

$$\sup_{x' \in A} \int_E \varphi(\langle x', x\rangle) dP(x) < +\infty \text{ and } \sup_{x' \in A} \int_E \varphi(\langle x', x\rangle) dQ(x) < +\infty.$$

Thus, for every $\pi \in D(P,Q)$, we have

$$\sup_{x' \in A} \int_{E \times E} \varphi(\langle x', x\rangle) d\pi(x,y) < +\infty$$

$$\text{and } \sup_{x' \in A} \int_{E \times E} \varphi(\langle x', y\rangle) d\pi(x,y) < +\infty.$$

Using the monotonicity of φ and the triangular inequality, this yields

$$\sup_{\pi \in D(P,Q)} \sup_{x' \in A} \int_{E \times E} \varphi(\langle x', x - y\rangle) d\pi(x,y) < +\infty,$$

which proves that the family $((x,y) \to \langle x', x-y\rangle)_{x' \in A}$ is uniformly integrable w.r.t. $D(P,Q)$.

Let (π_n) a sequence of elements of $D(P,Q)$ such that

$$\lim_{n \to +\infty} \sup_{x' \in A} \int_{E \times E} |\langle x', x - y\rangle| \, d\pi_n(x,y) = d_A(P,Q).$$

By relative sequential compactness of $D(P,Q)$, there exists a subsequence (π'_n) of (π_n) narrowly converging to a law $\pi \in D(P,Q)$. As $\mathcal{H} = \{(x,y) \mapsto \langle x', x - y\rangle;\, x' \in A\}$ is uniformly integrable w.r.t. (π'_n), we have, from Lemma 3.1,

$$\sup_{x' \in A} \int_{E \times E} |\langle x', x - y\rangle| \, d\pi(x,y)$$

$$= \lim_{n \to +\infty} \sup_{x' \in A} \int_{E \times E} |\langle x', x - y\rangle| \, d\pi'_n(x,y) = d_A(P,Q).$$

□

The following lemma will allow us to represent sequences that converge for d_A by sequences of random elements of E.

Lemma 5.4. *Let T be a topological space. Let I be a non empty set and $(\pi_i)_{i\in I}$ be a family of Radon laws on $T \times T$. Let p_1 and p_2 be the projections defined by $p_1(x, y) = x$ and $p_2(x, y) = y$ for all $(x, y) \in T \times T$. We assume that $p_2(\pi_i) = P$ for some fixed law P on T and for each $i \in I$. Then there exist a probability space (S, Σ, λ), a family $(X_i)_{i\in I}$ of random elements of T defined on (S, Σ, λ) and a random element X of T defined on (S, Σ, λ), such that $P_{(X_i,X)} = \pi_i$ for each $i \in I$.*

Proof. Let X be the identity mapping on $(T, \mathcal{B}_T, P)$. Clearly $P_X = P$. From Theorem 1 in [47] there exists for each $i \in I$ a regular conditional probability distribution of π_i given X, *i.e.* a family $(\pi_i^x)_{x\in T}$ of laws on T such that, for each $A \in \mathcal{B}_T$, the mapping $x \mapsto \pi_i^x(A)$ is measurable, and, for each $A, B \in \mathcal{B}_T$, $\pi_i(A \times B) = \int_B \pi_i^x(A)\, dP(x)$. If T is Suslin, this is also a consequence of the well known Theorem of Jirina ([27], Theorem 3.3), because the Borel tribe of T is countably generated ([40], Corollary page 108). For each $x \in T$, let π^x be the probability $\otimes_{i\in I}\pi_i^x$ on $(T^I, \otimes_{i\in I}\mathcal{B}_T)$ and let π be the probability on $(T^I \times T, (\otimes_{i\in I}\mathcal{B}_T) \otimes \mathcal{B}_T)$ defined by

$$\pi(A \times B) = \int_B \pi^x(A)\, dP(x)$$

for all $A \in \otimes_{i\in I}\mathcal{B}_T$ and $B \in \mathcal{B}_T$. Let $((X_i)_{i\in I}, X)$ be the identity mapping on $(T^I \times T, (\otimes_{i\in I}\mathcal{B}_T) \otimes \mathcal{B}_T, \pi)$. Then $P_{(X_i,X)} = \pi_i$ for each $i \in I$. □

Let $A \in \mathfrak{S}$. Let I be a non empty set, and let $(P_i)_i \in (\mathrm{MU}^1_E)^I$ and $P \in \mathrm{MU}^1_E$. For each $i \in I$, let π_i be a d_A–optimal coupling of (P_i, P). From Lemma 5.4, there exist a probability space (S, Σ, λ), a family (X_i) of random elements of E defined on (S, Σ) and a random element X of E defined on (S, Σ), such that $P_{(X_i,X)} = \pi_i$ for every $i \in I$.

Definition 5.5. With the above notations, $((S, \Sigma, \lambda), (X_i), X)$ is called a *d_A-representation of* $((P_i), P)$.

Proposition 5.6. *Let $A \in \mathfrak{S}$. The functional d_A is a semi–distance on MU^1_E. Its restriction to MP^1_E satisfies the following Lipschitz property: for any elements P and Q of MP^1_E,*

$$N_A\left(\int x\, dP(x) - \int x\, dQ(x)\right) \le d_A(P, Q). \tag{5.1}$$

If E is Lindelöf, the semi–distance d_A is a distance if and only if A separates the points of E.

Proof. Let P, Q, R be elements of MU^1_E. It is clear that $d_A(P,P)=0$ and $d_A(P,Q)=d_A(Q,P)$. Let $((S,\Sigma,\lambda),(X,Y),Z)$ be a d_A-representation of $((P,Q),R)$. We have

$$\begin{aligned} d_A(P,R)+d_A(R,Q) &= \sup_{x'\in A}\int_S |\langle x',\, X-Z\rangle|\,d\lambda \\ &\quad +\sup_{x'\in A}\int_S |\langle x',\, Y-Z\rangle|\,d\lambda \\ &\geq \sup_{x'\in A}\int_S |\langle x',\, X-Z\rangle|+|\langle x',\, Y-Z\rangle|d\lambda \\ &\geq \sup_{x'\in A}\int_S |\langle x',\, X-Y\rangle|\,d\lambda \\ &\geq d_A(P,Q). \end{aligned}$$

Now, assume that P and Q are in MP^1_E. Let π be a d_A–optimal coupling of (P,Q) and let (U,V) be a random element of $E\times E$ with law π, defined on a probability space (S,Σ,λ). Using Lemma 2.10, we have

$$\begin{aligned} N_A\left(\int x\,dP(x)-\int x\,dQ(x)\right) &= N_A(\mathbb{E}U-\mathbb{E}V) \\ &\leq \mathfrak{N}_A(U-V)=d_A(P,Q). \end{aligned}$$

If A does not separate the points of E, then d_A is not a distance, because of the embedding $E\to\mathrm{MP}^1_E\subset\mathrm{MU}^1_E$, $x\mapsto\delta_x$. Assume now that A separates the points of E and that E is Lindelöf. Let P and Q be elements of MU^1_E. If $d_A(P,Q)=0$, then, for every x' in A, $\int|\langle x',\, x-y\rangle|d\pi(x,y)=0$. Thus, for every $x'\in A$ and any d_A–optimal coupling π of (P,Q), $\langle x',\, x-y\rangle=0$ π-almost everywhere. Using a countable subset of A which separates the points of E (thanks to the Lindelöf property), we deduce that, π-almost everywhere, $\langle x', x-y\rangle=0$ for every $x'\in A$, and therefore $P=Q$. □

We investigate now, for sequences in MU^1_E, the relations between convergence for the semi–distances d_A $(A\in\mathfrak{S})$, $\mathfrak{S}$–uniform scalar integrability and narrow convergence.

First, the topology on MU^1_E associated with the semi–distances d_A $(A\in\mathfrak{S})$ and the narrow topology on MU^1_E are not comparable.

Example 5.7. (Narrow convergence does not imply convergence for the semi–distances d_A $(A\in\mathfrak{S})$) Let $e\in E$, $e\neq 0$, and set

$$P_n = \frac{1}{n}\delta_{ne} + \frac{n-1}{n}\delta_0.$$

Then (P_n) converges narrowly to $P = \delta_0$, whereas $\int x\, dP_n(x) = e$ does not converge to $\int x\, dP(x) = 0$, thus, from (5.1), (P_n) does not converge to P for the semi–distances d_A $(A \in \mathfrak{S})$.

Example 5.8. (Convergence for the semi–distances d_A $(A \in \mathfrak{S})$ does not imply narrow convergence) This example is borrowed from [17]. Assume that E is a Hilbert space, and that $(e_k)_{k\in\mathbb{N}}$ is an orthogonal sequence of unit vectors of E. We denote by $\overline{B}_{E'}$ the closed unit ball of $E' = E$. The uniform structure on $\mathrm{MU}^1_E = \mathrm{MW}^1_E = \mathrm{MP}^1_E$ generated by the semi–distances d_A $(A \in \mathfrak{S})$ is also generated by the only distance $\mathrm{d} = d_{\overline{B}_{E'}}$. Let $P_n = 2^{-n}\sum_{k=1}^{2^n}\delta_{e_k}$ $(n \in \mathbb{N}^*)$. The set $D(P_n, \delta_0)$ has a single element $P_n \otimes \delta_0$ and we have

$$\begin{aligned}
\mathrm{d}(P_n, \delta_0) &= \sup_{x'\in\overline{B}_{E'}} \iint |\langle x',\, x-y\rangle|\, dP_n(x)\, d\delta_0(y)\\
&= \sup_{x'\in\overline{B}_{E'}} \int |\langle x',\, x\rangle|\, dP_n(x)\\
&= \sup_{x'\in\overline{B}_{E'}} 2^{-n}\sum_{k=1}^{2^n} |\langle x',\, e_k\rangle|\\
&\leq 2^{-n/2}
\end{aligned}$$

thus $\lim_{n\to\infty}\mathrm{d}(P_n, \delta_0) = 0$. But (P_n) does not converge narrowly to δ_0: let $\varphi(x) = \min\{\|x\|, 1\}$. The function φ is bounded and continuous, $P_n(\varphi) = 1$ for every $n \in \mathbb{N}$, but $\delta_0(\varphi) = 0$. A generalization of this example in arbitrary infinite dimensional Banach spaces is provided by Example 1 in [17]: with the same notations as in [17], just take $P_n = 2^{-n}\sum_{k=1}^{2^n}\delta_{e^n_k}$.

Theorem 5.9. *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence in MU^1_E and let P be a law on E.*
1. Let $A \in \mathfrak{S}$.

- a. *If $\lim_n d_A(P_n, P) = 0$, then A is uniformly integrable w.r.t. $\{P_n;\, n \in \mathbb{N}\} \cup \{P\}$.*
- b. *If $(P_n)_n$ narrowly converges to P and A is uniformly integrable w.r.t. (P_n), then $\lim_n d_A(P_n, P) = 0$.*

2. Assume now that E is quasi–complete and that Condition 1.b is fulfilled for every $A \in \mathfrak{S}$. If $(P_n)_{n\in\mathbb{N}}$ is tight and if each P_n is in MP^1_E, then $P \in MP^1_E$.

Proof.
1. a. Let us first prove that A is uniformly integrable w.r.t. P. Let $\epsilon > 0$. Let $m \in \mathbb{N}$ such that $d_A(P_m, P) \leq \epsilon$. Let π be a d_A–optimal coupling of (P_m, P). Let X_m and X be random elements of E defined on a probability space (S, Σ, λ), such that $P_{(X_m,X)} = \pi$. We have

$$\begin{aligned}\sup_{x'\in A}\int_S |\langle x',\, X\rangle|\,d\lambda &\leq \sup_{x'\in A}\int_S |\langle x',\, X-X_m\rangle|\,d\lambda + \sup_{x'\in A}\int_S |\langle x',\, X_m\rangle|\,d\lambda\\ &= d_A(P_m,P) + \sup_{x'\in A}\int_S |\langle x',\, X_m\rangle|\,d\lambda < +\infty,\end{aligned}$$

because X_m is $\mathfrak{S}$–uniformly scalarly integrable. Hence,

$$\lim_{a\to+\infty}\sup_{x'\in A}\lambda\{|\langle x',\, X\rangle| > a\} \leq \lim_{a\to+\infty}\frac{1}{a}\sup_{x'\in A}\int_S |\langle x',\, X\rangle|\,d\lambda = 0. \tag{5.2}$$

On the other hand,

$$\begin{aligned}&\lim_{a\to+\infty}\sup_{x'\in A}\int_{|\langle x',\, X\rangle|>a} |\langle x',\, X\rangle|\,d\lambda\\ &\quad\leq \lim_{a\to+\infty}\sup_{x'\in A}\int_{|\langle x',\, X\rangle|>a} |\langle x',\, X-X_m\rangle|\,d\lambda + \sup_{x'\in A}\int_{|\langle x',\, X\rangle|>a} |\langle x',\, X_m\rangle|\,d\lambda\\ &\quad\leq \epsilon + \lim_{a\to+\infty}\sup_{x'\in A}\int_{|\langle x',\, X\rangle|>a} |\langle x',\, X_m\rangle|\,d\lambda.\end{aligned}$$

Now, from (5.2), as X_m is $\mathfrak{S}$–uniformly scalarly integrable,

$$\lim_{a\to+\infty}\sup_{x'\in A}\int_{|\langle x',\, X\rangle|>a} |\langle x',\, X_m\rangle|\,d\lambda = 0.$$

So, ϵ being arbitrary, the previous inequality shows that A is uniformly integrable w.r.t. P.

We now prove that A is uniformly integrable w.r.t. (P_n). Take a d_A-representation $((S, \Sigma, \lambda), (X_n), X)$ of $((P_n), P)$. We have

$$\begin{aligned}&\lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_{\{|\langle x',\, .\rangle|>a\}} |\langle x',\, x\rangle|\,dP_n(x)\\ &\quad= \lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_{\{|\langle x',\, X_n\rangle|>a\}} |\langle x',\, X_n\rangle|\,d\lambda\\ &\quad\leq \lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_{\{|\langle x',\, X_n\rangle|>a\}} |\langle x',\, X_n - X\rangle|\,d\lambda\\ &\qquad+ \lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_{\{|\langle x',\, X_n\rangle|>a\}} |\langle x',\, X\rangle|\,d\lambda.\end{aligned}$$

Observe next that

$$\begin{aligned}
&\lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\lambda\{|\langle x',\,X_n\rangle|>a\}\\
\leq\;& \lim_{a\to+\infty}\frac{1}{a}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_S |\langle x',\,X_n\rangle|\,d\lambda\\
\leq\;& \lim_{a\to+\infty}\frac{1}{a}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_S |\langle x',\,X_n-X\rangle|\,d\lambda\\
&+\lim_{a\to+\infty}\frac{1}{a}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_S |\langle x',\,X\rangle|\,d\lambda\\
\leq\;& \lim_{a\to+\infty}\frac{1}{a}\sup_{n\in\mathbb{N}} d_A(P_n,P)+\lim_{a\to+\infty}\frac{1}{a}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_S |\langle x',\,X\rangle|\,d\lambda\\
=\;& 0.
\end{aligned}$$

As A is uniformly integrable w.r.t. λ (because X is $\mathfrak{S}$–uniformly scalarly integrable), we have thus

$$\lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_{\{|\langle x',\,X_n\rangle|>a\}}|\langle x',\,X\rangle|\,d\lambda=0.$$

For every $\epsilon>0$, let n_ϵ be an integer such that

$$\sup_{n>n_\epsilon}\sup_{x'\in A}\int_S|\langle x',\,X_n-X\rangle|\,d\lambda\leq\epsilon.$$

The family $(x\mapsto|\langle x',\,X_n-X\rangle|)_{x'\in A,n\leq n_\epsilon}$ is uniformly integrable w.r.t. λ, thus

$$\lim_{a\to+\infty}\sup_{n\leq n_\epsilon}\sup_{x'\in A}\int_{\{|\langle x',\,X_n\rangle|>a\}}|\langle x',\,X_n-X\rangle|\,d\lambda=0$$

for every $\epsilon>0$. Thus

$$\lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_{\{|\langle x',\,.\rangle|>a\}}|\langle x',\,x\rangle|\,dP_n(x)=0.$$

b. Note that $(P_n)_n$ is not necessarily tight. Otherwise, we could use A. Jakubowski's version of Skorokhod's representation theorem ([26], Theorem 2), and conclude with Lemma 3.1. In order to use a Skorokhod representation, we are going to project the laws P_n and P on a quotient space which is Polish.

Let B be the vector subspace of E' generated by A and let $B^\circ\subset E$ be the orthogonal of B. Let pr_A be the canonical projection of E onto the quotient space E/B° and let $\|.\|_A$ be the norm on E/B° defined by $\|\mathrm{pr}_A(x)\|_A=N_A(x)=\sup_{x'\in A}|\langle x',\,x\rangle|$. We denote by E_A the linear space E/B° endowed with $\|.\|_A$, and by $\widehat{E}_A$ its completion. Thus $\widehat{E}_A$ is a separable Banach space.

The adjoint mapping $(\mathrm{pr}_A)'$ of pr_A is an algebraic isomorphism of the dual $\widehat{E}'_A$ of E/B° onto the closure $\overline{B}$ of B for $\sigma(E', E)$. Indeed, let $x' \in \overline{B}$. Let us define $y' \in \widehat{E}'_A$ by $\langle y', y\rangle = \langle x', x\rangle$ for $x \in E$ and $y = \mathrm{pr}_A(x) = x + B^\circ = x + (\overline{B})^\circ$. Then $x' = (\mathrm{pr}_A)'(y')$. Thus $\overline{B} \subset (\mathrm{pr}_A)'(\widehat{E}'_A)$. Conversely, let $x' = (\mathrm{pr}_A)'(y')$ for some $y' \in \widehat{E}'_A$. We have, for every $x \in B^\circ$, $\langle x', x\rangle = \langle y', \mathrm{pr}_A(x)\rangle = 0$ (because $\mathrm{pr}_A(x) = 0$), hence, by the bipolar theorem, $x' \in (B^\circ)^\circ = \overline{B}$. Thus $(\mathrm{pr}_A)'(\widehat{E}'_A) = \overline{B}$. The injectivity of $(\mathrm{pr}_A)'$ is immediate.

We shall denote by $\widehat{x'}$ the inverse image of an element $x' \in \overline{B}$ by $(\mathrm{pr}_A)'$ and by $\widehat{A}$ the inverse image of A.

Let us denote by $\widehat{Q}$ the image $\mathrm{pr}_A(Q)$ on $\widehat{E}_A$ of a probability Q on E. Similarly, if π is a law on $E \times E$, we shall denote by $\widehat{\pi}$ the image on $\widehat{E}_A \times \widehat{E}_A$ of π by $\mathrm{pr}_A \otimes \mathrm{pr}_A$. Let Q and R be laws on E. Then $\pi \in D(Q, R)$ if and only if $\widehat{\pi} \in D(\widehat{Q}, \widehat{R})$ and we have

$$\sup_{x' \in A} \int_{E\times E} |\langle x', x - y\rangle| \, d\pi(x, y) = \sup_{\widehat{x}' \in \widehat{A}} \int_{\widehat{E}_A \times \widehat{E}_A} |\langle \widehat{x}', x - y\rangle| \, d\widehat{\pi}(x, y).$$

Furthermore, if Q and R are tight, there exists a countable union T of compact subsets of E such that $Q(T) = R(T) = 1$. Thus $\pi(T \times T) = 1$ for any $\pi \in D(Q, R)$, and also, for any $\pi \in D(\widehat{Q}, \widehat{R})$, $\pi(\mathrm{pr}_A(T) \times \mathrm{pr}_A(T)) = 1$. From [40], Theorem 12 page 126, since $T \times T$ is a $K_{\sigma\delta}$, the mapping $\pi \mapsto \widehat{\pi}$, from $\mathcal{M}_1(T \times T)$ to $\mathcal{M}_1(\mathrm{pr}_A(T) \times \mathrm{pr}_A(T))$, is surjective. Thus, if Q and R are tight, a law $\pi \in D(Q, R)$ is a d_A–optimal coupling of (Q, R) if and only if $\widehat{\pi}$ is a $d_{\widehat{A}}$–optimal coupling of $(\widehat{Q}, \widehat{R})$, and we have $d_A(Q, R) = d_{\widehat{A}}(\widehat{Q}, \widehat{R})$.

Now, the sequence $(\widehat{P}_n)_n$ narrowly converges to $\widehat{P}$ and $\widehat{E}_A$ is Polish thus, from Skorokhod's representation theorem ([41], page 281), there exist a sequence $(X_n)_n$ of random elements of $\widehat{E}_A$ defined on a probability space (S, Σ, λ) and a random element X of $\widehat{E}_A$ defined on (S, Σ, λ), such that $P_{X_n} = \widehat{P}_n$ ($n \in \mathbb{N}$), $P_X = \widehat{P}$ and $(X_n)_n$ converges a.e. to X.

For each $n \in \mathbb{N}$, let π_n be a d_A–optimal coupling of (P_n, P) and $\widehat{\pi}_n$ its image by pr_A. We have

$$\begin{aligned} d_A(P_n, P) &= \sup_{x' \in A} \int_{E\times E} |\langle x', x - y\rangle| \, d\pi_n(x, y) \\ &= \sup_{\widehat{x}' \in \widehat{A}} \int_{\widehat{E}_A \times \widehat{E}_A} |\langle \widehat{x}', x - y\rangle| \, d\widehat{\pi}_n(x, y) \\ &\le \sup_{\widehat{x}' \in \widehat{A}} \int_{\Omega} |\langle \widehat{x}', X_n - X\rangle| \, d\lambda \end{aligned}$$

because $\widehat{\pi}_n$ is a $d_{\widehat{A}}$–optimal coupling of $(\widehat{P}_n, \widehat{P})$. Set $Q_n = P_{X_n - X}$. The sequence $(Q_n)_n$ narrowly converges to δ_0 and $\widehat{A}$ is uniformly integrable w.r.t. $(Q_n)_n$. Using the preceding inequality and Lemma 3.1, we get

$$\begin{aligned} \lim_{n\to+\infty} d_A(P_n, P) &\leq \lim_{n\to+\infty} \sup_{\widehat{x}'\in\widehat{A}} \int_{\widehat{E}_A} |\langle \widehat{x}', x\rangle| \, dQ_n \\ &= 0. \end{aligned}$$

A longer but more elementary proof of 1.b can also be obtained by following the same steps as in [5] (Lemma 8.3) for the Lévy–Wasserstein distance in a separable Banach space. The projection onto the quotient space $\widehat{E}_A$ cannot be avoided, in order to use Prokhorov's compactness theorem.

2. From 1.a, (P_n) is $\mathfrak{S}$–uniformly scalarly integrable, and Theorem 3.2 yields the conclusion. □

Remark 5.10. Part 1.b of Theorem 5.9 also holds true for nets. The only substantial modification in the proof consists in using X. Fernique's version of Skorokhod's representation theorem [21] (for nets or filters) instead of Skorokhod's original result. Thus, on $\mathfrak{S}$–uniformly scalarly integrable subsets of MU^1_E, the topology induced by the semi–distances d_A $(A \in \mathfrak{S})$ is coarser than that induced by $\flat(\tau_{\mathfrak{S}})$.

Corollary 5.11. *With the same hypothesis and notations as in Theorem 3.4, we have, for almost every $\omega \in \Omega$,*

$$\forall A \in \mathfrak{S}, \ \lim_{n\to+\infty} d_A\left(\frac{1}{n}\sum_{j=1}^{n} \lambda_\omega^{\beta(j)},\ \lambda_\omega^\infty\right) = 0.$$

Proof. Replace Theorem 3.2 by Theorem 5.9 in the proof of Theorem 3.4. □

Corollary 5.12. *Let (P_n) be a $\flat(\tau_{\mathfrak{S}})$–relatively sequentially compact sequence in MU^1_E, and let P be a law on E. The following are equivalent:*

(*i*) *For every $A \in \mathfrak{S}$, $\lim_{n\to\infty} d_A(P_n, P) = 0$,*
(*ii*) *$(P_n)_n$ converges narrowly to P and $(P_n)_n$ is $\mathfrak{S}$–uniformly scalarly integrable.*

If (i) and (ii) are true, then $P \in MU^1_E$.

Proof. From Theorem 5.9, we have immediately $(ii) \Rightarrow (i)$, and (i) implies $\mathfrak{S}$–uniform scalar integrability of $(P_n)_n$. Assume (i). Let (P'_n) be a subsequence of (P_n). By relative sequential compactness of (P_n), there exists a subsequence (P''_n) of (P'_n), narrowly converging to an element Q of MP^1_E. But, (P''_n) being $\mathfrak{S}$–uniformly scalarly integrable, we have also from Theorem 5.9 that, for every $A \in \mathfrak{S}$, $\lim_{n\to\infty} d_A(P''_n, Q) = 0$. Thus $Q = P$. We have proved that each subsequence (P'_n) of (P_n) has a subsequence (P''_n) which converges to P for $\flat(\tau_{\mathfrak{S}})$. Thus $(P_n)_n$ narrowly converges to P.

Finally, if If (i) and (ii) are true, then $P \in \mathrm{MU}^1_E$ from Lemma 3.1 or from Theorem 5.9. □

The following theorem is a generalization of Corollary 5.12, the limit P being replaced by a sequence (P_n). It improves Theorem 3.5.

Theorem 5.13. *Let (P_n) and (Q_n) be two sequences in MU^1_E such that (P_n) is $\flat(\tau_{\mathfrak{S}})$–relatively sequentially compact and $\mathfrak{S}$–uniformly scalarly integrable. The following are equivalent:*

(i) (Q_n) is $\flat(\tau_{\mathfrak{S}})$–relatively sequentially compact and, for every $A \in \mathfrak{S}$,
$\lim_{n\to\infty} d_A(P_n, Q_n) = 0$,

(ii) (P_n) and (Q_n) are $\flat(\tau_{\mathfrak{S}})$–equivalent and (Q_n) is $\mathfrak{S}$–uniformly scalarly integrable.

Proof. . $(i) \Rightarrow (ii)$. Let $A \in \mathfrak{S}$. For each integer n, let π_n be a d_A-optimal coupling of (Q_n, P_n). Let $(X_n, Y_n)_{n\in\mathbb{N}}$ be a sequence of random elements of $E \times E$ defined on a probability space (S, Σ, λ), such that $P_{(X_n,Y_n)} = \pi_n$ for each $n \in \mathbb{N}$. We have, for every $a > 0$,

$$\begin{aligned}
\sup_{x'\in A} \int_{|\langle x', .\rangle|>a} |\langle x', x\rangle|\, dQ_n(x) &= \sup_{x'\in A} \int_{|\langle x', Y_n\rangle|>a} |\langle x', Y_n\rangle|\, d\lambda \\
&\leq \sup_{x'\in A} \int_{|\langle x', Y_n\rangle|>a} |\langle x', Y_n - X_n\rangle|\, d\lambda \\
&\quad + \sup_{x'\in A} \int_{|\langle x', Y_n\rangle|>a} |\langle x', X_n\rangle|\, d\lambda \\
&\leq d_A(P_n, Q_n) \\
&\quad + \sup_{x'\in A} \int_{|\langle x', Y_n\rangle|>a} |\langle x', X_n\rangle|\, d\lambda.
\end{aligned}$$

Now,

$$\lambda\{|\langle x', Y_n\rangle| > a\} \leq \lambda\left(\left\{|\langle x', X_n\rangle| > \frac{a}{2}\right\} \cup \left\{|\langle x', X_n - Y_n\rangle| > \frac{a}{2}\right\}\right).$$

Furthermore,

$$\lim_{a\to+\infty} \sup_{n\in\mathbb{N}} \sup_{x'\in A} \lambda\left\{|\langle x', X_n\rangle| > \frac{a}{2}\right\} = 0$$

(because $(\langle x', X_n\rangle)_{x'\in A, n\in\mathbb{N}}$ is uniformly integrable w.r.t. λ), and

$$\lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\lambda\left\{|\langle x',\, X_n - Y_n\rangle| > \frac{a}{2}\right\}$$
$$\leq \lim_{a\to+\infty}\frac{2}{a}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int|\langle x',\, X_n - Y_n\rangle| d\lambda = 0.$$

Thus

$$\lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\lambda\{|\langle x',\, Y_n\rangle| > a\} = 0.$$

As $(\langle x', X_n\rangle)_{x'\in A, n\in\mathbb{N}}$ is uniformly integrable w.r.t. λ, this yields

$$\lim_{a\to+\infty}\sup_{n\in\mathbb{N}}\sup_{x'\in A}\int_{|\langle x',\, Y_n\rangle|>a}|\langle x',\, X_n\rangle| d\lambda = 0.$$

Thus, for every $A\in\mathfrak{S}$, $(\langle x', Y_n\rangle)_{x'\in A, n\in\mathbb{N}}$ is uniformly integrable w.r.t. λ, *i.e.* (Q_n) is $\mathfrak{S}$–uniformly scalarly integrable.

If (P'_n) is a subsequence of (P_n) which narrowly converges to a law P, then, for every $A\in\mathfrak{S}$, $d_A(P'_n, P)\to 0$ from Theorem 5.9, because (P_n) is $\flat(\tau_{\mathfrak{S}})$–relatively sequentially compact and $\mathfrak{S}$–uniformly scalarly integrable. But we have $d_A(P'_n, Q'_n)\to 0$ (where (Q'_n) is the subsequence of (Q_n) having the same subscripts as (P'_n)), thus $\lim_{n\to+\infty} d_A(Q'_n, P) = 0$, and, from (i) and $\mathfrak{S}$–uniform scalar integrability of (Q_n), Corollary 5.12 implies that (Q'_n) narrowly converges to P.

On the other hand, if (Q'_n) is a subsequence of (Q_n) which narrowly converges to a probability Q, then, for every $A\in\mathfrak{S}$, $\lim_{n\to+\infty} d_A(Q'_n, Q) = 0$ from Theorem 5.9, because (Q_n) is $\mathfrak{S}$–uniformly scalarly integrable. Let (P'_n) be the subsequence of (P_n) with same subscripts as (Q'_n). As $\lim_{n\to+\infty} d_A(P'_n, Q'_n) = 0$, we have also $\lim_{n\to+\infty} d_A(P'_n, Q) = 0$, and (P'_n) converges narrowly to Q from Corollary 5.12, because (P'_n) is $\flat(\tau_{\mathfrak{S}})$–relatively sequentially compact.

$(ii)\Rightarrow(i)$. From the equivalence of (P_n) and (Q_n), the relative sequential compactness of (P_n) implies that of (Q_n).

For the other part of (i), it is sufficient to prove that, for each $A\in\mathfrak{S}$ and for each subsequence $(P'_n, Q'_n)_n$ of $(P_n, Q_n)_n$, there exists a subsequence $(P''_n, Q''_n)_n$ of $(P'_n, Q'_n)_n$ such that $\lim_{n\to+\infty} d_A(P''_n, Q''_n) = 0$. Choose $(P''_n, Q''_n)_n$ such that (P''_n) narrowly converges to a measure P. Then $\lim_{n\to+\infty} d_A(P''_n, P) = 0$ from Theorem 5.9, because (P_n) is $\flat(\tau_{\mathfrak{S}})$–relatively sequentially compact and $\mathfrak{S}$–uniformly scalarly integrable. But (Q''_n) also converges narrowly to P, because (Q_n) and (P_n) are equivalent, and thus, in the same way, $\lim_{n\to+\infty} d_A(Q''_n, P) = 0$. Therefore $\lim_{n\to+\infty} d_A(P''_n, Q''_n) = 0$. □

Theorem 5.13 yields a small improvement of the SLLN given in Theorem 4.1. The line of proof is similar to that of Theorem 4 in [14], where the semi–distances d_A are replaced by the Lévy–Wasserstein distance. The result obtained may be seen as a new Glivenko–Cantelli type theorem.

Corollary 5.14. *With the same hypothesis and notations as in Theorem 4.1, except that the laws P_{X_n} ($n \in \mathbb{N}$) are only supposed to be in MU^1_E and not necessarily in MP^1_E, we have, for almost every $\omega \in \Omega$,*

$$\forall A \in \mathfrak{S}, \lim_{n\to+\infty} d_A(1/n \sum_{i=1}^{n} \delta_{X_i(\omega)}, \frac{1}{n}\Sigma_{i=1}^n P_{X_i}) = 0.$$

Proof. Let us define Ω' as in the proof of Theorem 4.1, and let $\omega \in \Omega'$. The sequences $(P_n)_n = (\frac{1}{n}\Sigma_{i=1}^n P_{X_i})_n$ and $(Q_n^\omega)_n = (\frac{1}{n}\Sigma_{i=1}^n \delta_{X_i(\omega)})_n$ satisfy the hypothesis and condition (ii) of Theorem 5.13 because, from Theorem 2.4, they are tight and $\flat(\tau_{\mathfrak{S}})$–equivalent. Thus we have $\lim_{n\to+\infty} d_A(P_n, Q_n) = 0$ for every $A \in \mathfrak{S}$. □

If the random vectors X_n ($n \in \mathbb{N}$) are Pettis integrable, the SLLN of Theorem 4.1 follows from the previous result and from the Lipschitz property (5.1) of d_A given in Proposition 5.6.

6. The "sharp topology" on the space of $\mathfrak{S}$–uniformly scalarly integrable laws

Definition 6.1. The supremum of the narrow topology on MU^1_E, induced by $\flat(\tau_{\mathfrak{S}})$, and of the topology associated to the semi–distances d_A ($A \in \mathfrak{S}$), is called the *sharp topology associated with* $\tau_{\mathfrak{S}}$ and denoted $\sharp(\mathfrak{S})$.

Theorem 6.2. *A subset $\mathcal{D}$ of MU^1_E is relatively sequentially compact for $\sharp(\mathfrak{S})$ if and only if it is $\mathfrak{S}$–uniformly scalarly integrable and relatively sequentially compact for $\flat(\tau_{\mathfrak{S}})$.*

Proof. If $\mathcal{D}$ is relatively sequentially compact for $\sharp(\mathfrak{S})$, then it is relatively sequentially compact for $\flat(\tau_{\mathfrak{S}})$. Thus, for any sequence (P_n) of elements of $\mathcal{D}$, there exists a subsequence (P'_n) of (P_n) which converges to a law P on E for the semi–distances d_A ($A \in \mathfrak{S}$). From Theorem 5.9, (P'_n) is $\mathfrak{S}$–uniformly scalarly integrable. So, any sequence (P_n) of elements of $\mathcal{D}$ has a subsequence (P'_n) which is $\mathfrak{S}$–uniformly scalarly integrable. Suppose that $\mathcal{D}$ is not $\mathfrak{S}$–uniformly scalarly integrable, *i.e.* there exists an $A \in \mathfrak{S}$ which is not uniformly integrable w.r.t. $\mathcal{D}$. Then, there exists $\epsilon > 0$ such that

$$\lim_{a\to+\infty} \sup_{P\in\mathcal{D}} \sup_{x'\in A} \int_{|\langle x',\,.\rangle|>a} |\langle x',\, x\rangle|\, dP > \epsilon.$$

We define recursively a strictly increasing sequence (a_n) of non negative integers and a sequence (P_n) of elements of $\mathcal{D}$ in the following way:

(α) a_0 is chosen arbitrarily,

(β) for each a_n, P_n is such that

$$\sup_{x'\in A}\int_{|\langle x',\,.\rangle|>a_n}|\langle x',\,x\rangle|\,dP_n>\frac{\epsilon}{2},$$

(γ) once a_n and P_n are fixed, a_{n+1} is chosen such that $a_{n+1}\geq a_n+1$ (thus we shall have $\lim_{n\to+\infty}a_n=+\infty$) and

$$\sup_{x'\in A}\int_{|\langle x',\,.\rangle|>a_{n+1}}|\langle x',\,x\rangle|\,dP_n<\frac{\epsilon}{2}.$$

It is clear that no subsequence of (P_n) is $\mathfrak{S}$–uniformly scalarly integrable, we have thus a contradiction. So $\mathcal{D}$ is $\mathfrak{S}$–uniformly scalarly integrable.

Conversely, assume that $\mathcal{D}$ is relatively sequentially compact for $\flat(\tau_{\mathfrak{S}})$ and $\mathfrak{S}$–uniformly scalarly integrable. Then every sequence (P_n) in $\mathcal{D}$ has a subsequence (P'_n) which narrowly converges to a law P. From Corollary 5.12, (P'_n) converges to P for the semi–distances d_A ($A\in\mathfrak{S}$) and $P\in\mathrm{MU}^1_E$, thus (P'_n) also converges to P for $\sharp(\mathfrak{S})$. This proves that $\mathcal{D}$ is relatively sequentially compact for $\sharp(\mathfrak{S})$. □

Now, let $\mathcal{M}$ be a convex subset of MU^1_E, endowed with a topology τ. Let us say that $(\mathcal{M},\tau)$ *satisfies Property* $(\mathcal{H})$ if

(a) $\mathcal{M}$ contains the Dirac masses ($x\in E$),

(b) If $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ are sequences in $\mathcal{M}\cap\mathrm{MP}^1_E$ such that (P_n) and (Q_n) are $\flat(\tau_{\mathfrak{S}})$–equivalent and such that $\{P_n;\,n\in\mathbb{N}\}$ and $\{Q_n;\,n\in\mathbb{N}\}$ are relatively sequentially compact for τ, then $\lim_{n\to+\infty}\int x dP_n-\int x dQ_n=0$.

From Theorem 6.2 and Theorem 5.13, it is clear that $(\mathrm{MU}^1_E,\sharp(\mathfrak{S}))$ satisfies Property $(\mathcal{H})$. Thus, Theorem 4.1 is a particular case of the following SLLN.

Theorem 6.3. *Let $\mathcal{M}$ be a subset of MP^1_E, endowed with a topology τ, such that $(\mathcal{M},\tau)$ satisfies Property $(\mathcal{H})$ defined above. Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of pairwise independent Pettis integrable random elements of E defined on a probability space $(\Omega,\mathcal{F},\mu)$, and let us denote $P_n=1/n\sum_{i=1}^n P_{X_i}$ and $Q_n^\omega=1/n\sum_{i=1}^n\delta_{X_i}$ ($n\in\mathbb{N}^*$, $\omega\in\Omega$). Assume that*

(i) $(P_n)_n$ is tight,

(ii) the set $\{P_n;\,n\in\mathbb{N}^*\}$ is a relatively sequentially compact subset of $(\mathcal{M},\tau)$,

(iii) for almost every $\omega \in \Omega$, $\{Q_n^\omega; n \in \mathbb{N}^*\}$ is a relatively sequentially compact subset of $(\mathcal{M}, \tau)$.

Then (X_n) satisfies the SLLN.

Proof. From Theorem 2.4, there exists $\Omega' \in \mathcal{F}$, with $\mu(\Omega') = 1$, such that, for every $\omega \in \Omega'$, the sequence $(Q_n^\omega)_n$ is tight and $(P_n)_n$ and $(Q_n^\omega)_n$ are equivalent. The conclusion follows from Property $(\mathcal{H})$. $\square$

In the case when E is a Banach space, another example of topological space of laws with property $(\mathcal{H})$ is given by the set MB_E^1 of Bochner integrable laws on E, endowed with the topology τ_{LW} associated with the *Lévy–Wasserstein distance* (see [38, 14, 15]). This topology is finer than the narrow topology and, from Theorem 3.2.4. of [15], a subset $\mathcal{D} \subset \mathrm{MB}_E^1$ is relatively compact for τ_{LW} (or, equivalently, relatively sequentially compact, because τ_{LW} is metrizable) if and only if $\mathcal{D}$ is tight and the norm $\|.\|$ is uniformly integrable w.r.t. $\mathcal{D}$.

In the case $(\mathcal{D}, \tau) = (\mathrm{MB}_E^1, \tau_{LW})$, if E is a separable Banach space, Theorem 6.3 is equivalent to Theorem 3 in [13].

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