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MEASURE AND INTEGRATION

Compactness criteria for the stable topology

by

Paul RAYNAUD de FITTE

Summary. Adapting a general result of Topsøe [32], we prove a compactness criterion for the stable topology on the set of measures on the product of a measurable space and a Suslin (nonnecessarily regular) topological space. We also extend a compactness criterion of Jacod and Mémin [17] and we apply these results to the case of Young measures.

1. Introduction and preliminaries. Stable convergence has been discovered several times and is used under different names in the Calculus of Variations, in Control Theory, in Probability Theory, and in other fields. The expression "stable convergence" stems from Rényi [27] and is employed by probabilists, but this convergence was invented much earlier by L. C. Young [40].

Let us first present stable convergence in a particular case. Let $(f_n)_n$ be a sequence of measurable functions defined on a finite measure space $(\Omega, \mathcal{S}, \lambda)$, with values in $\mathbb{R}^d$ (the frame we will consider in this paper is much more general). The sequence $(f_n)_n$ is stably convergent if, for every $A \in \mathcal{S}$ and every bounded continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$, the sequence $(\int_A \varphi \circ f_n \, d\lambda)_n$ is convergent. The limit of (f_n) is the mapping $(A, \varphi) \mapsto \lim_n \int_A \varphi \circ f_n \, d\lambda$. It appears that this mapping can be identified with a measure μ on $\Omega \times \mathbb{R}^d$, defined by

$$\int_{\Omega\times\mathbb{R}^d} \mathbb{1}_A \otimes \varphi \, d\mu = \lim_n \int_A \varphi \circ f_n \, d\lambda$$

for every A and every φ as above (we denote by $\mathbb{1}_A$ the indicator function of A and $g \otimes h$ denotes the mapping $(\omega, x) \mapsto g(\omega)h(x)$). Note that the margin of μ on Ω is λ. In this convergence, each f_n can also be identified with a measure $\underline{\delta}_{f_n}$ on $\Omega \times \mathbb{R}^d$, namely $\underline{\delta}_{f_n} = \int_\Omega \delta_\omega \otimes \delta_{f_n(\omega)} \, d\lambda(\omega)$, where δ_x denotes the Dirac mass on x. In other

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words, $\underline{\delta}_{f_n}$ is supported by the graph of f_n and its margin on Ω is λ. We then have, for every A and every φ,

$$\int_{\Omega\times\mathbb{R}^d} \mathbb{1}_A \otimes \varphi \, d\underline{\delta}_{f_n} = \int_A \varphi \circ f_n \, d\lambda.$$

So, stable convergence of $(f_n)_n$ to μ can be expressed by

$$\lim_n \int_{\Omega\times\mathbb{R}^d} \mathbb{1}_A \otimes \varphi \, d\underline{\delta}_{f_n} = \int_{\Omega\times\mathbb{R}^d} \mathbb{1}_A \otimes \varphi \, d\mu$$

for every A and every φ. In this example, stable convergence can be seen as a convergence in the space of measures on $\Omega \times \mathbb{R}^d$ which have common margin λ on Ω. These measures are sometimes called Young measures, or relaxed controls, see *e.g.* [38] or [4] for an introduction to Young measures and their applications. So, stable convergence gives "relaxed" limits to sequences of functions, that is, limits which are not necessarily functions, but can be measures. For example, stable convergence is used to provide generalized solutions to variational problems or to control problems, see *e.g.* [28], or to provide weak solutions to stochastic differential equations [24]. Stable convergence is also used in limit theorems of probability, see [16, 21]. This convergence is topologisable, and compactness criteria for the stable topology are used to prove, for example, convergence results, or the existence of (relaxed) optimal controls. Assume that $(f_n)_n$ is a weakly convergent sequence in $\mathrm{L}^1(\Omega, \mathcal{S}, \lambda; \mathbb{R}^d)$. Then $(f_n)_n$ is sequentially relatively compact in the stable topology; if furthermore (f_n) is not strongly convergent, there exists a stably convergent subsequence $(f'_n)_n$, the limit of which is not a function from Ω to $\mathbb{R}^d$, but a Young measure which describes the oscillations of $(f'_n)_n$ around its weak limit [38, 39].

In this paper, we study stable convergence in the space of finite nonnegative measures on the product $\Omega \times \mathbb{T}$, where $(\Omega, \mathcal{S})$ is a measurable space and $\mathbb{T}$ is a Suslin topological space. Thus, the measures we consider do not necessarily have the same margin on Ω. This is the point of view chosen by Schäl [29] in the case when $\mathbb{T}$ is separable metrizable and Ω is topological, by Jacod and Mémin [17] and Galdéano [14] in the case when $\mathbb{T}$ is Polish, and by Balder [5] in the case when $\mathbb{T}$ is Suslin regular. The definition we choose yields a finer topology than the definition chosen by Schäl, by Jacod and Mémin and by Balder, but both definitions coincide in the metric case, as will be made clear in the sequel. With our definition, the stable topology is a particular case of Topsøe's w–topology [32]. However, Topsøe's criteria are not directly applicable in our situation, some work is necessary to adapt them. This is done in Section 2. As a consequence, we obtain a generalized version of Jacod and Mémin's compactness criterion [17, 18]. We also obtain a Prokhorov–type sufficient condition of compactness, similar to that obtained by Balder [5, Theorem 5.2], but with the advantage that is stated for a finer topology.

In Section 3, we investigate metrizability properties of the stable convergence. First, a version for stable convergence of the celebrated Portmanteau Theorem is given (for Young measures, this kind of result is sometimes called a semicontinuity theorem). This allows to give metrizability and sequential compactness criteria in the stable topology.

Finally, in Section 4, we apply our results to the particular case of Young measures (that is, measures on the product $\Omega \times \mathbb{T}$ with a prescribed margin on Ω). In particular, we obtain a slightly generalized version of Balder's sequential compactness criterion [2] and a generalization of a result on the equivalence of two tightness notions for Young measures, which was known in the case when $\mathbb{T}$ is Polish [3, 19].

Preliminary definitions and results. Throughout, $\mathbb{T}$ is a Hausdorff topological space. We denote respectively by $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{K}$ the sets of open, closed and compact subsets of $\mathbb{T}$. The σ–algebra of Borel subsets of $\mathbb{T}$ is denoted by $\mathcal{B}_{\mathbb{T}}$.

The space $\mathbb{T}$ is *Suslin* if it is the image of a Polish space by some continuous mapping. Most usual spaces of analysis are Suslin; we refer to [30] about Suslin spaces.

The space $\mathbb{T}$ is *submetrizable* if there exists a metrizable topology on $\mathbb{T}$ which is coarser than the original topology of $\mathbb{T}$. Every regular Suslin space is submetrizable.

In the sequel, we are also given a measure space $(\Omega, \mathcal{S})$. We call *random set* every element of $\mathcal{S} \otimes \mathcal{B}_{\mathbb{T}}$. If G is a random set and $\omega \in \Omega$, we denote by $G(\omega)$ the section $\{t \in \mathbb{T};\ (\omega, t) \in G\}$. The set of random sets G satisfying $G(\omega) \in \mathcal{G}$ for every $\omega \in \Omega$ is denoted by $\underline{\mathcal{G}}$, and its elements are called *open random sets*. We define similarly the set $\underline{\mathcal{F}}$ of *closed random sets* and the set $\underline{\mathcal{K}}$ of *compact random sets*.

By "measure", we mean nonnegative finite measure. The set of measures on a measurable space $(\mathbb{X}, \mathcal{X})$ is denoted by $\mathcal{M}^+(\mathbb{X}, \mathcal{X})$, or by $\mathcal{M}^+(\mathbb{X})$ when no ambiguity is to fear. In particular, $\mathcal{M}^+(\Omega)$ denotes $\mathcal{M}^+(\Omega, \mathcal{S})$, $\mathcal{M}^+(\mathbb{T})$ denotes $\mathcal{M}^+(\mathbb{T}, \mathcal{B}_{\mathbb{T}})$ and $\mathcal{M}^+(\Omega \times \mathbb{T})$ denotes $\mathcal{M}^+(\Omega \times \mathbb{T}, \mathcal{S} \otimes \mathcal{B}_{\mathbb{T}})$.

We refer to Topsøe's book [33] or to Bogachev's nice survey [6] for complements in Measure Theory.

We denote respectively by π_Ω and $\pi_{\mathbb{T}}$ the canonical projections from $\Omega \times \mathbb{T}$ onto Ω and $\mathbb{T}$. If $\mu \in \mathcal{M}^+(\Omega \times \mathbb{T})$, we denote by $(\pi_\Omega)_\sharp(\mu)$ the measure $\mu(. \times \mathbb{T})$ image of μ by π_Ω. Similarly, $(\pi_{\mathbb{T}})_\sharp(\mu)$ denotes the measure $\mu(\Omega \times .) \in \mathcal{M}^+(\mathbb{T})$.

As we mentionned in the Introduction, there are several ways to define a "stable topology" on $\mathcal{M}^+(\Omega \times \mathbb{T})$. One possible stable topology is described in the example in the Introduction: it is the topology of pointwise convergence on the test functions $\mathbb{1}_A \otimes \varphi$, where A runs over $\mathcal{S}$ and φ runs over the bounded continuous real valued functions on $\mathbb{T}$. This definition is chosen by Schäl [29], Jacod and Mémin [17] and Balder [5] (Schäl, followed by Balder, calls it the *ws–topology*). However, if $\mathbb{T}$ is not completely regular, the set of continuous functions can be restricted to the constants, see *e.g.* [31, Example 75], and then the ws–topology becomes very poor.

We choose a finer topology (Galdéano's definition [14] is intermediate between ours and that of [29, 17, 5]): we endow $\mathcal{M}^+(\Omega \times \mathbb{T})$ with the coarsest topology such that the mapping $\mu \mapsto \mu(\Omega \times \mathbb{T})$ is continuous and such that the mappings $\mu \mapsto \mu(G)$ are l.s.c. for every $G \in \underline{\mathcal{G}}$. In other words, a net $(\mu^\alpha)_{\alpha\in\mathbb{A}}$ in $\mathcal{M}^+(\Omega \times \mathbb{T})$ converges to $\mu \in \mathcal{M}^+(\Omega \times \mathbb{T})$ if and only if $\lim_\alpha \mu^\alpha(\Omega \times \mathbb{T}) = \mu(\Omega \times \mathbb{T})$ and, for each $G \in \underline{\mathcal{G}}$, $\liminf_\alpha \mu^\alpha(G) \geq \mu(G)$. We denote this topology by $\mathfrak{T}$ and we call it *stable topology*. This topology is a particular case of the w–topology of Topsøe [32]. From the Portmanteau Theorem 3.1 in this paper, when $\mathbb{T}$ is Suslin metrizable, the stable topology coincides with the ws–topology. Following Jacod and Mémin, we use the terminology "stable topology" because $\mathfrak{T}$ extends the so called "stable convergence" defined by Rényi [27, 26] for random variables.

We endow the space $\mathcal{M}^+(\mathbb{T})$ with the *narrow topology*, that is, the coarsest topology such that the mapping $\mu \mapsto \mu(\mathbb{T})$ is continuous and such that the mappings $\mu \mapsto \mu(G)$ are l.s.c. for every $G \in \mathcal{G}$. The space $\mathcal{M}^+(\Omega)$ is endowed with the *s–topology* (see [32]), that is, the coarsest topology such that the mapping $\mu \mapsto \mu(A)$ is continuous for every $A \in \mathcal{S}$. Thus $\mathfrak{T}$ generalizes both the narrow topology on $\mathcal{M}^+(\mathbb{T})$ and the s–topology on $\mathcal{M}^+(\Omega)$: if we take $\mathcal{S} = \{\emptyset, \Omega\}$, then $(\mathcal{M}^+(\Omega \times \mathbb{T}), \mathfrak{T})$ is homeomorphic to $\mathcal{M}^+(\mathbb{T})$ endowed with the narrow topology, and, similarly, if $\mathbb{T}$ has only one element, then $(\mathcal{M}^+(\Omega \times \mathbb{T}), \mathfrak{T})$ is homeomorphic to $\mathcal{M}^+(\Omega)$ endowed with the s–topology.

We call *universal completion* of $\mathcal{S}$ the σ–algebra $\mathcal{S}^* = \cap_{\mu\in\mathcal{M}^+(\Omega)}\mathcal{S}_\mu$, where $\mathcal{S}_\mu$ is the μ–completion of $\mathcal{S}$, that is, $\mathcal{S}_\mu$ is the union of $\mathcal{S}$ and the μ–negligible sets. Note that the sets $\mathcal{M}^+(\Omega, \mathcal{S})$ and $\mathcal{M}^+(\Omega, \mathcal{S}^*)$ can be identified via the canonical extension and restriction maps. Furthermore, the s–topologies on $\mathcal{M}^+(\Omega, \mathcal{S})$ and $\mathcal{M}^+(\Omega, \mathcal{S}^*)$ coincide. Indeed, let $(Q^\alpha)_{\alpha\in\mathbb{A}}$ be a net in $\mathcal{M}^+(\Omega)$ which converges to $Q \in \mathcal{M}^+(\Omega)$ in the s-topology relative to $\mathcal{S}$. Let $A \in \mathcal{S}^*$. We can find B and $\mathcal{N}$ in $\mathcal{S}$ such that $A \bigtriangleup B \subset \mathcal{N}$ and $Q(\mathcal{N}) = 0$. We then have $|Q^\alpha(A) - Q^\alpha(B)| \leq Q^\alpha(\mathcal{N}) \to 0$ and $Q^\alpha(B) \to Q(B)$, thus $Q^\alpha(A) \to Q(A)$, which proves that $(Q^\alpha)_\alpha$ converges to Q in the s-topology relative to $\mathcal{S}^*$. The converse inclusion of topologies is obvious.

If we replace $\mathcal{S}$ by $\mathcal{S}^*$ in the preceding definitions, we use notations such as $\underline{\mathcal{G}}^*$ or $\mathfrak{T}^*$. We say that $\mathcal{S}$ is *universally complete* if $\mathcal{S} = \mathcal{S}^*$.

Recall that the space $\mathbb{T}$ is said to be *second countable* if its topology has a countable base [13].

Proposition 1.1 *Assume that $\mathbb{T}$ is the union of a sequence $(\mathbb{T}_n)_n$ of second countable Suslin spaces which are Borel subsets of $\mathbb{T}$. Then $\mathfrak{T}^* = \mathfrak{T}$.*

Note that, if $\mathbb{T}$ is a countable union of Suslin spaces, it is Suslin. The hypothesis on $\mathbb{T}$ in Proposition 1.1 is of course satisfied if $\mathbb{T}$ is Polish and in some other interesting cases. This is the case if $\mathbb{T} = (\mathbb{E}', \sigma(\mathbb{E}', \mathbb{E}))$ for some separable Banach space $\mathbb{E}$ (consider the closed balls of $\mathbb{E}'$ with radius $n \in \mathbb{N}$). More generally, this is the case if $\mathbb{T} = (\mathbb{E}', \sigma(\mathbb{E}', \mathbb{E}))$ for some (locally convex) separable Fréchet space $\mathbb{E}$. Indeed, as $\mathbb{E}$ is barrelled, each bounded subset of $\mathbb{E}'_\sigma$ is relatively compact, and, from [7, Proposition 2 page IV.21], there exists a sequence $(K_n)_{n\geq 1}$ of closed bounded subsets of $\mathbb{E}'_\sigma$ such that each bounded subset of $\mathbb{E}'_\sigma$ is contained in some K_n. Furthermore, from the separability of $\mathbb{E}$, there exists a countable family of continuous functions on $\mathbb{E}'_\sigma$ which separates the points of $\mathbb{E}'_\sigma$, thus $\mathbb{E}'_\sigma$ is submetrizable, thus the compact sets K_n are metrizable, thus Polish. In this case, from Banach–Dieudonné Theorem (*e.g.* [7]), the topology τ of $\mathbb{E}'_\sigma$ is the topology τ_c of uniform convergence on compact subsets of $\mathbb{E}$. In particular, if $\mathbb{E}$ is a Fréchet–Montel space, then $\tau = \tau_c$ is also the topology τ_b of uniform convergence on bounded subsets of $\mathbb{E}$, that is, $\mathbb{T}$ is the strong dual $\mathbb{E}'_b$ of $\mathbb{E}$. This is the case if $\mathbb{T}$ is the space $\mathcal{S}'$ of tempered distributions.

The proof of Proposition 1.1 relies on the following lemma, which is an adaptation of results of Balder [3, Lemmas A4 and A6] (see also [8]). The first part of this lemma can also be deduced from results of Valadier ([35, 1.14], [36, Proposition 13]).

LEMMA 1.2 *Assume that $\mathbb{T}$ is the union of a sequence $(\mathbb{T}_n)_n$ of second countable Suslin spaces which are Borel subsets of $\mathbb{T}$. Let $H \in \underline{\mathcal{G}}^*$ and let $Q \in \mathcal{M}^+(\Omega)$. There exists $G \in \underline{\mathcal{G}}$ such that $H \subset G$ and $H(\omega) = G(\omega)$ for Q–almost every $\omega \in \Omega$.*

Proof. Assume first that $\mathbb{T}$ is a second countable Suslin space. Let $\mathcal{U}$ be a countable basis of $\mathbb{T}$. For each $\omega \in \Omega$, $H(\omega)$ is the union of all $U \in \mathcal{U}$ such that $U \subset H(\omega)$. For each $U \in \mathcal{U}$, let

$$E_U = \{\omega \in \Omega;\, U \subset H(\omega)\} = \left(\pi_\Omega\left((\Omega \times U) \cap H^c\right)\right)^c.$$

From the Projection Theorem (see [9, Theorem III.23]), since $\mathbb{T}$ is Suslin, each E_U is $\mathcal{S}^*$–measurable. Furthermore, we have

$$H = \bigcup_{U\in\mathcal{U}} (E_U \times U).$$

For each $U \in \mathcal{U}$, let $B_U \in \mathcal{S}$ be such that $E_U \subset B_U$ and $Q^*(E_U) = Q(B_U)$. We set

$$G = \bigcup_{U\in\mathcal{U}} (B_U \times U).$$

We then have $G \in \underline{\mathcal{G}}$ and $H \subset G$. For each $U \in \mathcal{U}$, let $\mathcal{N}_U \in \mathcal{S}^*$ be a Q–negligible set such that $E_U \cup \mathcal{N}_U = B_U$. Let $\mathcal{N} = \cup_{U\in\mathcal{U}}\mathcal{N}_U$. The set $\mathcal{N}$ is negligible and we have $G(\omega) = H(\omega)$ for every $\omega \in \Omega \setminus \mathcal{N}$, which proves the lemma in the case when $\mathbb{T}$ is second countable.

Assume now that $\mathbb{T}$ is the union of a sequence $(\mathbb{T}_n)_n$ of second countable Suslin spaces which are Borel subsets of $\mathbb{T}$. For each integer n, let $H_n = H \cap (\Omega \times \mathbb{T}_n)$. Applying the preceding result to each H_n, we construct a sequence $(G_n)_n$ such that, for each n, $H_n \subset G_n$, $G_n \in \mathcal{S} \otimes \mathcal{B}_{\mathbb{T}_n} \subset \mathcal{S} \otimes \mathcal{B}_{\mathbb{T}}$ and $H_n(\omega) = G_n(\omega)$ for Q–almost every ω. Let $\widetilde{G} = \cup_n G_n$. We have $H \subset \widetilde{G}$, $\widetilde{G} \in \mathcal{S} \otimes \mathcal{B}_{\mathbb{T}}$ and there exists a Q–negligible set $\mathcal{N} \in \mathcal{S}$ such that $H(\omega) = \widetilde{G}(\omega)$ for every $\omega \in \mathcal{N}^c$. We only need to take G such that

$$G(\omega) = \begin{cases} \widetilde{G}(\omega) = H(\omega) & \text{if } \omega \notin \mathcal{N}, \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

□

An extension of Lemma 1.2 to l.s.c. integrands instead of random sets can easily be obtained as in [3, 8], or by applying Lemma 1.2 to epigraphs, but we do not need it here.

Proof of Proposition 1.1. Clearly, we have $\mathfrak{T} \subset \mathfrak{T}^*$. Now, let $(\mu^\alpha)_{\alpha\in\mathbb{A}}$ be a net in $\mathcal{M}^+(\Omega \times \mathbb{T})$ which converges to some $\mu \in \mathcal{M}^+(\Omega \times \mathbb{T})$ for $\mathfrak{T}$. Let $Q = (\pi_\Omega)_\sharp(\mu)$ and, for each $\alpha \in \mathbb{A}$, let $Q^\alpha = (\pi_\Omega)_\sharp(\mu^\alpha)$. We thus have $Q^\alpha \to Q$ in the s–topology. Let $H \in \underline{\mathcal{G}}^*$ and let $G \in \underline{\mathcal{G}}$ be as in Lemma 1.2. Let $\mathcal{N}$ be the Q–negligible set $\{\omega \in \Omega;\ G(\omega) \neq H(\omega)\}$. We have

$$\mu^\alpha(G \setminus H) \leq Q^\alpha(\mathcal{N}) \to Q(\mathcal{N}) = 0,$$

thus

$$\mu(H) \leq \mu(G) \leq \liminf_\alpha \mu^\alpha(G) = \liminf_\alpha \mu^\alpha(H).$$

Thus $(\mu^\alpha)_{\alpha\in\mathbb{A}}$ converges to μ for $\mathfrak{T}^*$. □

2. Topsøe and Jacod–Mémin Criteria. We call *paving* on $\Omega \times \mathbb{T}$ any nonempty set of subsets of $\Omega \times \mathbb{T}$. We now list some properties of the pavings $\underline{\mathcal{G}}$ and $\underline{\mathcal{K}}$ that we shall need later. We present them in a similar way as in [32].

Lemma 2.1 (Properties of $\underline{\mathcal{G}}$ and $\underline{\mathcal{K}}$) *Assume that $\mathbb{T}$ is Suslin submetrizable and $(\Omega, \mathcal{S})$ is universally complete.*

I. $\underline{\mathcal{K}}$ contains $\emptyset$ and is closed under finite unions and countable intersections,

II. $\underline{\mathcal{G}}$ contains $\emptyset$ and is closed under finite unions and finite intersections (actually, $\underline{\mathcal{G}}$ is also closed under countable unions, but we shall not need it),

III. $K \setminus G \in \underline{\mathcal{K}}$ for all $K \in \underline{\mathcal{K}}$ and for all $G \in \underline{\mathcal{G}}$,

IV. $\underline{\mathcal{G}}$ separates the sets in $\underline{\mathcal{K}}$, that is, for any pair K_1, K_2 of disjoint elements of $\underline{\mathcal{K}}$, we can find a pair G_1, G_2 of disjoint elements of $\underline{\mathcal{G}}$ such that $K_1 \subset G_1$ and $K_2 \subset G_2$.

V'. Let $\mathfrak{M}$ be a subset of $\mathcal{M}^+(\Omega \times \mathbb{T})$ such that $(\pi_\Omega)_\sharp(\mathfrak{M})$ is relatively compact in the s-topology of $\mathcal{M}^+(\Omega)$. Then $\mathfrak{M}$ is uniformly σ–smooth on $\underline{\mathcal{G}}$ at $\emptyset$ w.r.t. $\underline{\mathcal{K}}$, *that is, for any countable family $(K_i)_{i\in I}$ of elements of $\underline{\mathcal{K}}$ which filters downwards to $\emptyset$, we have*

$$\inf_{i\in I} \sup_{\mu\in\mathfrak{M}} \inf_{G\in\underline{\mathcal{G}},\, G\supset K_i} \mu(G) = 0$$

(we say that $(K_i)_{i\in I}$ filters downwards to $\emptyset$ if $\cap_{i\in I}K_i=\emptyset$ and if, for any $i\in I$ and any $j\in I$, there exists $k\in I$ such that $K_k\subset K_i\cap K_j$).

REMARK 2.2 Note that we do not have Property V of [32], that is, semi–compactness of $\underline{\mathcal{K}}$ (a paving $\mathcal{C}$ is said to be *semi–compact* if, for any countable family of elements of $\mathcal{C}$ which has an empty intersection, there exists a finite subfamily which has an empty intersection). We shall see however that the weaker Property V' is sufficient to yield a result similar to [32, Theorem 4].

Proof of Lemma 2.1. Actually, only Properties IV and V' need a proof. We denote by d a continuous distance on $\mathbb{T}$ and by τ_0 the topology (coarser than the original topology τ of $\mathbb{T}$) generated by d.

Proof of IV. Let K_1, K_2 be disjoint elements of $\underline{\mathcal{K}}$.

Assume first that K_1 and K_2 have nonempty values. For each $\omega\in\Omega$, $K_1(\omega)$ and $K_2(\omega)$ are compact for τ_0, thus $d(K_1(\omega),K_2(\omega))>0$. Furthermore, as $(\Omega,\mathcal{S})$ is universally complete, the function $\phi:\ \omega\mapsto d(K_1(\omega),K_2(\omega))$ is $\mathcal{S}$-measurable. Indeed, let $\widetilde{\mathbb{T}}$ be the d–completion of $\mathbb{T}$. Then, for $i=1,2$, the set K_i is an element of $\mathcal{S}\otimes\mathcal{B}_{\widetilde{\mathbb{T}}}$, thus, using the Projection Theorem (see [9, Theorem III.23]), we have

$$\{\omega\in\Omega;\ K_i(\omega)\cap U\neq\emptyset\}=\pi_\Omega\left(K_i\cap(\Omega\times U)\right)\in\mathcal{S}$$

for any open subset U of $\widetilde{\mathbb{T}}$. But, from [9, Theorem III.9], this is equivalent to each of the following properties :

- for each $t\in\widetilde{\mathbb{T}}$, the function $\omega\mapsto d(t,K_i)$ is $\mathcal{S}$–measurable,
- there exists a sequence $(\varphi_n^i)_{n\in\mathbb{N}}$ of $\mathcal{S}$–measurable mappings $\Omega\to\widetilde{\mathbb{T}}$ such that, for every $\omega\in\Omega$, $\varphi_n^i(\omega)\in K_i(\omega)$ and $K_i(\omega)$ is the τ_0–closure of $\{(\varphi_n^i(\omega));\ n\in\mathbb{N}\}$.

Thus $\phi=\inf_{m,n\in\mathbb{N}}d(\varphi_m^1,\varphi_n^2)$ is $\mathcal{S}$–measurable.

Now set, for $i=1,2$,

$$G_i=\left\{(\omega,t)\in\Omega\times\mathbb{T};\ d(t,K_i(\omega))<\frac{\phi(\omega)}{3}\right\}.$$

It is clear that $G_1\cap G_2=\emptyset$ and that, for $i=1,2$, we have $K_i\subset G_i$ and $G_i(\omega)$ $(i=1,2)$ is τ–open for each $\omega\in\Omega$. Furthermore, for each $n\in\mathbb{N}$, the function

$$g_n^i:\ (\omega,t)\mapsto d\left(\varphi_n^i(\omega),t\right)-\frac{\phi(\omega)}{3}$$

is $\mathcal{S}\otimes\mathcal{B}_{\mathbb{T}}$–measurable, thus $G_i=\cup_{n\in\mathbb{N}}(g_n^i)^{-1}([-\infty,0[)$ belongs to $\mathcal{S}\otimes\mathcal{B}_{\mathbb{T}}$.

Let us now allow each K_i to have empty values. From the Projection Theorem, as $(\Omega, \mathcal{S})$ is universally complete, the sets $\{\omega \in \Omega;\ K_i(\omega) \neq \emptyset\}$ are $\mathcal{S}$-measurable. Consider the $\mathcal{S}$-measurable sets

$$\begin{aligned}
\Omega_0 &= \{\omega \in \Omega;\ K_1(\omega) = \emptyset \text{ and } K_2(\omega) = \emptyset\},\\
\Omega_1 &= \{\omega \in \Omega;\ K_1(\omega) = \emptyset \text{ and } K_2(\omega) \neq \emptyset\},\\
\Omega_2 &= \{\omega \in \Omega;\ K_1(\omega) \neq \emptyset \text{ and } K_2(\omega) = \emptyset\},\\
\Omega_3 &= \{\omega \in \Omega;\ K_1(\omega) \neq \emptyset \text{ and } K_2(\omega) \neq \emptyset\}.
\end{aligned}$$

The same arguments as above show the existence of two disjoint elements G'_1 and G'_2 of $\underline{\mathcal{G}}$, contained in $\Omega_3 \times \mathbb{T}$, such that $K_i \cap (\Omega_3 \times \mathbb{T}) \subset G'_i$ $(i = 1, 2)$. To prove Property IV, we only need to set

$$G_i(\omega) = \begin{cases} \emptyset & \text{if} \quad \omega \in \Omega_0,\\ \emptyset & \text{if} \quad \omega \in \Omega_i\\ \mathbb{T} & \text{if} \quad \omega \in \Omega_{3-i},\\ G'_i(\omega) & \text{if} \quad \omega \in \Omega_3. \end{cases}$$

Proof of V'. Let $(K_i)_{i \in I}$ be a countable family of elements of $\underline{\mathcal{K}}$ which filters downwards to $\emptyset$. For each $\omega \in \Omega$, $(K_i(\omega))_{i \in I}$ is a family of compact subsets of $\mathbb{T}$ which filters downwards to $\emptyset$, thus there exists an element i of I such that $K_i(\omega) = \emptyset$. We can enumerate the elements of I: $I = \{i_0, i_1, \dots\}$, and we can endow I with the ordering associated with this enumeration: $i_0 \leq i_1 \leq \dots$. For each $\omega \in \Omega$, let us denote by $\alpha(\omega)$ the smallest i such that $K_i(\omega) = \emptyset$. Using the Projection Theorem as in the proof of Property IV, we see that, for each $i \in I$, the set

$$A_i = \{\omega \in \Omega;\ \alpha(\omega) = i\} = (\pi_\Omega(K_i))^c \cap \bigcap_{j \leq i-1} \pi_\Omega(K_j)$$

is measurable. Thus the family $(A_i)_{i \in I}$ is a measurable partition of Ω. For each integer $n \in \mathbb{N}$, let $\Omega_n = A_{i_0} \cup \dots \cup A_{i_n}$. As $(K_i)_{i \in I}$ filters downwards to $\emptyset$, there exists an element j_1 of I such that $K_{j_1} \subset K_{i_0} \cap K_{i_1}$. We then have $K_{j_1}(\omega) = \emptyset$ on $A_{i_0} \cup A_{i_1} = \Omega_1$. By induction, we can construct a sequence $(K_{j_n})_{n \geq 1}$ such that $K_{j_n}(\omega) = \emptyset$ on Ω_n.

Now, we have $\cup_{n \in \mathbb{N}} \Omega_n = \Omega$. For each $n \geq 1$, the mapping $f_n : Q \mapsto Q(\Omega_n)$ is continuous on the closure of $(\pi_\Omega)_\sharp(\mathfrak{M})$ for the s–topology. By Dini Lemma, $(f_n)_n$ converges uniformly on the closure of $(\pi_\Omega)_\sharp(\mathfrak{M})$ to $f : Q \mapsto Q(\Omega)$. Let $\epsilon > 0$. We can thus find an $n \geq 1$ such that

$$\forall \mu \in \mathfrak{M} \quad (\pi_\Omega)_\sharp(\mu)(\Omega_n^c) \leq \epsilon.$$

Let $G_n = \Omega_n^c \times \mathbb{T}$. We have $K_{j_n} \subset G_n$ and

$$\forall \mu \in \mathfrak{M} \quad \mu(G_n) = (\pi_\Omega)_\sharp(\mu)(\Omega_n^c) \leq \epsilon.$$

This shows that

$$\inf_{i\in I}\sup_{\mu\in\mathfrak{M}}\inf_{G\in\underline{\mathcal{G}},\,G\supset K_i}\mu(G)\leq\epsilon.$$

As ϵ is arbitrary, this proves Property V'. □

We can now give an adaptation of Topsøe's compactness criterion ([32, Corollary 2], see also [34, 22]).

Recall that a subset $\mathfrak{K}$ of a topological space $\mathbb{T}$ is *net–compact* if every net of elements of $\mathfrak{K}$ admits a subnet which converges in $\mathbb{T}$, or equivalently, if every universal net of elements of $\mathfrak{K}$ is convergent in $\mathbb{T}$ (see [20] about subnets and universal nets). We say that $\mathfrak{K}$ is *relatively compact* if it is contained in a compact subset of $\mathbb{T}$. Thus every relatively compact subset of $\mathbb{T}$ is net–compact. The converse implication is true if $\mathbb{T}$ is regular (see the proof in [25] or [22]).

If $\mathcal{C}$ and $\mathcal{E}$ are two pavings on $\Omega\times\mathbb{T}$, we say that $\mathcal{E}$ *dominates* $\mathcal{C}$ if each element of $\mathcal{C}$ is contained in some element of $\mathcal{E}$.

Theorem 2.3 (Topsøe Criterion) *Assume that $\mathbb{T}$ is Suslin submetrizable. Let $\mathfrak{M}$ be a subset of $\mathcal{M}^+(\Omega\times\mathbb{T})$. Then $\mathfrak{M}$ is $\mathfrak{T}^*$–net–compact if and only if Conditions (i) and (ii) below are satisfied.*

- (i) *The set $(\pi_\Omega)_\sharp(\mathfrak{M})$ is net–compact for the s–topology.*
- (ii) *For any subfamily $\underline{\mathcal{G}}'$ of $\underline{\mathcal{G}}^*$ which dominates $\underline{\mathcal{K}}$ and for each $\epsilon>0$, there exists a finite subfamily $\underline{\mathcal{G}}''$ of $\underline{\mathcal{G}}'$ such that, for every $\mu\in\mathfrak{M}$, we can find $G\in\underline{\mathcal{G}}''$ such that $\mu(G^c)<\epsilon$.*

Conditions of net–compactness in the s–topology have been given in [32, 15, 1]. In particular, if a subset $\mathfrak{M}$ of $\mathcal{M}^+(\Omega)$ is relatively compact, then it is *equicontinuous*, that is, for each decreasing sequence $(A_n)_n$ in $\mathcal{S}$ such that $\cap_n A_n=\emptyset$, we have $\lim_n\sup_{\mu\in\mathfrak{M}}\mu(A_n)=0$ [15]. This result will be helpful in Corollary 3.4.

Proof. With the help of Lemma 2.1, this is a simple adaptation of Topsøe's proof [32, 34]. We sketch the proof for the convenience of the reader, and we detail some parts that will be helpful to prove Theorem 2.6.

To simplify notations, we assume that $(\Omega,\mathcal{S})$ is universally complete.

The "only if" part of the proof is exactly as in [32] or [34], and does not rely on Properties I to V'. The shortest way ([34, Theorem 3.1]) goes as follows. Assume that $\mathfrak{M}$ is $\mathfrak{T}$–net–compact. Then (i) obviously holds true. If (ii) is not satisfied, there exists a family $\underline{\mathcal{G}}'=(G_K)_{K\in\underline{\mathcal{K}}}$, with $G_K\supset K$ for each $K\in\underline{\mathcal{K}}$, such that, for each finite subfamilily $\underline{\mathcal{G}}''$ of $\underline{\mathcal{G}}'$, there exists $\mu\in\mathfrak{M}$ satisfying $\min\{\mu(G^c);\,G\in\underline{\mathcal{G}}''\}>\epsilon$. Now, the family

$$(\mathcal{O}_{G_K})_{K\in\underline{\mathcal{K}}}:=\big(\{\mu\in\mathcal{M}^+(\Omega\times\mathbb{T});\ \mu(G_K^c)<\epsilon\}\big)_{K\in\underline{\mathcal{K}}}$$

is an open cover of $\mathcal{M}^+(\Omega \times \mathbb{T})$ (indeed, as $\mathbb{T}$ is Suslin, each measure on $\mathbb{T}$ is Radon, thus, for each $\mu \in \mathcal{M}^+(\Omega \times \mathbb{T})$ there exists $H \in \mathcal{K}$ such that $\mu(\Omega \times H^c) < \epsilon$; if we take $K = \Omega \times H$, we thus have $\mu \in \mathcal{O}_{G_K}$). By net–compactness of $\mathfrak{M}$, we can extract a finite subfamily $\underline{\mathcal{G}}''$ of $\underline{\mathcal{G}}'$ such that $(\mathcal{O}_G)_{G\in\underline{\mathcal{G}}''}$ is an open cover of $\mathfrak{M}$ (see [22, Proposition 1]), which leads to a contradiction.

Assume now that (i) and (ii) are satisfied. Let $(\mu^\alpha)_{\alpha\in\mathbb{A}}$ be a universal net in $\mathfrak{M}$. Thanks to (i), we have $\sup_{\mu\in\mathfrak{M}} \mu(\Omega \times \mathbb{T}) < +\infty$. We can thus define a bounded set function $\nu : \underline{\mathcal{G}} \to [0, +\infty[$ by

$$\forall G \in \underline{\mathcal{G}} \quad \nu(G) = \lim_\alpha \mu^\alpha(G).$$

The mapping ν is monotone (that is, $G \subset G' \Rightarrow \nu(G) \le \nu(G')$), additive (that is, $G \cap G' = \emptyset \Rightarrow \nu(G \cup G') = \nu(G) + \nu(G')$) and subadditive (that is, $\nu(G \cup G') \le \nu(G) + \nu(G')$). Let $(K_i)_{i\in I}$ be a countable family of elements of $\underline{\mathcal{K}}$ which filters downwards to $\emptyset$. By (i) and Property V' of Lemma 2.1, we have

$$\inf_{i\in I} \inf_{G\in\underline{\mathcal{G}},\, G\supset K_i} \nu(G) \le \inf_{i\in I} \sup_{\mu\in\mathfrak{M}} \inf_{G\in\underline{\mathcal{G}},\, G\supset K_i} \mu(G) = 0 = \nu(\emptyset), \tag{2.1}$$

that is, ν is σ–smooth at $\emptyset$ w.r.t. $\underline{\mathcal{K}}$. Let us define a set function μ on $\mathcal{S} \otimes \mathcal{B}_\mathbb{T}$ by

$$\forall B \in \mathcal{S} \otimes \mathcal{B}_\mathbb{T} \quad \mu(B) = \sup_{K\subset B,\, K\in\underline{\mathcal{K}}} \inf_{G\in\underline{\mathcal{G}},\, G\supset K} \nu(G). \tag{2.2}$$

From Properties I to IV of Lemma 2.1 and from (2.1) and [32, Theorem 2], the set function μ is a measure on $\mathcal{S} \otimes \mathcal{B}_\mathbb{T}$. Moreover, we obviously have

$$\forall G \in \underline{\mathcal{G}} \quad \mu(G) \le \liminf_\alpha \mu^\alpha(G). \tag{2.3}$$

To conclude that $(\mu^\alpha)_\alpha$ converges to μ, we thus only need to prove that

$$\mu(\Omega \times \mathbb{T}) = \lim_\alpha \mu^\alpha(\Omega \times \mathbb{T}). \tag{2.4}$$

Assume that (2.4) is not satisfied. There exist $\epsilon > 0$ and, for each $K \in \underline{\mathcal{K}}$, an element G_K of $\underline{\mathcal{G}}$ such that $G_K \supset K$ and $\lim_\alpha \mu^\alpha(G_K) + 2\epsilon \le \lim_\alpha \mu^\alpha(\Omega \times \mathbb{T})$. We then obtain a contradiction with (ii) by taking $\underline{\mathcal{G}}' = \{G_K;\, K \in \underline{\mathcal{K}}\}$. This proves that $\mathfrak{M}$ is net–compact. □

From Topsøe's criterion we get generalizations of Prokhorov's compactness criterion. We first need to define some notions of tightness in $\mathcal{M}^+(\Omega \times \mathbb{T})$.

Let us say that a subset $\mathfrak{M}$ of $\mathcal{M}^+(\Omega \times \mathbb{T})$ is *flexibly tight* if, for each $\epsilon > 0$, there exists $K \in \underline{\mathcal{K}}$ such that $\sup_{\mu\in\mathfrak{M}} \mu(K^c) < \epsilon$. Let us say that $\mathfrak{M}$ is *strictly tight* if, for each $\epsilon > 0$, there exists $K \in \mathcal{K}$ such that $\sup_{\mu\in\mathfrak{M}} \mu(\Omega \times K^c) < \epsilon$. Thus $\mathfrak{M}$ is strictly tight if and only if $(\pi_\mathbb{T})_\sharp(\mathfrak{M})$ is tight in the usual sense.

Applying Theorem 2.3, we immediately have the following result, which extends [5, Theorem 5.2] in that $\mathbb{T}$ is not necessarily regular and that we obtain compactness for a topology which is finer than the ws–topology.

Proposition 2.4 *Assume that $\mathbb{T}$ is Suslin submetrizable and let $\mathfrak{M}$ be a flexibly tight subset of $\mathcal{M}^+(\Omega\times\mathbb{T})$ such that $(\pi_\Omega)_\sharp(\mathfrak{M})$ is net–compact for the s–topology. Then $\mathfrak{M}$ is net–compact for $\mathfrak{T}^*$ (and thus also for $\mathfrak{T}$).*

We can weaken the hypothesis on $\mathbb{T}$ at the cost of replacing flexible tightness by strict tightness.

Proposition 2.5 *Assume that the compact subsets of $\mathbb{T}$ are metrizable. Let $\mathfrak{M}$ be a stricly tight subset of $\mathcal{M}^+(\Omega\times\mathbb{T})$ such that $(\pi_\Omega)_\sharp(\mathfrak{M})$ is net–compact for the s–topology. Then $\mathfrak{M}$ is net–compact for $\mathfrak{T}^*$ (and thus also for $\mathfrak{T}$).*

Proof. Let $(K_n)_{n\geq 1}$ be an increasing sequence in $\mathcal{K}$ such that, for each $n\geq 1$, $\sup_{\mu\in\mathfrak{M}}\mu(\Omega\times K_n^c)<1/n$. Let $\mathbb{T}_0=\cup_{n\geq 1}K_n$. We have $\mu(\Omega\times\mathbb{T}_0^c)=0$ for each $\mu\in\mathfrak{M}$. Furthermore, if $(\mu^\alpha)_{\alpha\in\mathbb{A}}$ is a net in $\mathfrak{M}$ which converges to some $\mu\in\mathcal{M}^+(\Omega\times\mathbb{T})$, we have, for each $n\geq 1$, $\mu(K_n^c)\leq\liminf_\alpha\mu^\alpha(K_n^c)\leq 1/n$, Thus the closure $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ for $\mathfrak{T}$ can be identified with a subset of $\mathcal{M}^+(\Omega\times\mathbb{T}_0)$ and the topology $\mathfrak{T}$ coincides on $\overline{\mathfrak{M}}$ with the stable topology on $\mathcal{M}^+(\Omega\times\mathbb{T}_0)$. Moreover, $\mathbb{T}_0$ is a countable union of Lusin spaces, thus it is Lusin. We can thus apply Proposition 2.4 in $\mathcal{M}^+(\Omega\times\mathbb{T}_0)$ to conclude that $\mathfrak{M}$ is net–compact. □

Recall that a Hausdorff topological space $\mathbb{T}$ is Prokhorov if and only if every compact subset of τ–regular measures on $\mathbb{T}$ is tight (see *e.g.* [6]). We can now give a generalization of a criterion of Jacod and Mémin ([17, Théorème 2.8], see also [29, Theorem 3.10] and [5, Theorem 5.2]), which was given for the Polish case.

Corollary 2.6 (Jacod and Mémin's criterion) *Assume that $\mathbb{T}$ is Suslin submetrizable and Prokhorov. Let $\mathfrak{M}\subset\mathcal{M}^+(\Omega\times\mathbb{T})$. Then $\mathfrak{M}$ is $\mathfrak{T}$–net–compact if and only if Conditions (a) and (b) below are satisfied.*

- (a) *$(\pi_\Omega)_\sharp(\mathfrak{M})$ is net–compact in the s–topology on $\mathcal{M}^+(\Omega)$.*
- (b) *$(\pi_\mathbb{T})_\sharp(\mathfrak{M})$ is net–compact in the narrow topology on $\mathcal{M}^+(\mathbb{T})$.*

Proof. The necessary condition is obvious.

Assume now that (a) and (b) are satisfied. As $\mathbb{T}$ is Suslin, every element of $\mathcal{M}^+(\mathbb{T})$ is τ–regular. From (b) and the Prokhorov property, $\mathfrak{M}$ is stricly tight. The result thus follows from Proposition 2.5. □

3. Metrizability, sequential compactness. We start with a result which is very similar to a classical one for narrow convergence [33, Theorem 8.1]. Similar results (but for sequences) are given in [17, 5]. Let us fix some definitions and notations. A (bounded) measurable mapping $f:\Omega\times\mathbb{T}\to\mathbb{R}$ is called a *(bounded) integrand*. We say that f is a *(bounded) continuous integrand* (resp. a *(bounded) l.s.c. integrand*) if furthermore $f(\omega,.)$ is continuous (resp. l.s.c.) for every $\omega\in\Omega$. If $f:\Omega\to\mathbb{R}$ and $g:\mathbb{T}\to\mathbb{R}$ are measurable mappings, we denote by $f\otimes g$ the integrand defined by $(f\otimes g)(\omega,t)=f(\omega)g(t)$ for every $(\omega,t)\in\Omega\times\mathbb{T}$.

Theorem 3.1 (Portmanteau Theorem) *Assume that $\mathbb{T}$ is metrizable Suslin. Let d be a distance on $\mathbb{T}$ which is compatible with the topology of $\mathbb{T}$. Let $(\mu^\alpha)_{\alpha\in\mathbb{A}}$ be a net in $\mathcal{M}^+(\Omega\times\mathbb{T})$ and let $\mu\in\mathcal{M}^+(\Omega\times\mathbb{T})$. The following conditions are equivalent.*

1. *$(\mu^\alpha)_\alpha$ converges to μ in the stable topology.*
2. *$\liminf_\alpha \mu^\alpha(f)\geq\mu(f)$ for every bounded l.s.c. integrand.*
3. *$\lim_\alpha \mu^\alpha(f)=\mu(f)$ for every bounded continuous integrand.*
4. *$\lim_\alpha \mu^\alpha(\mathbb{1}_A\otimes g)=\mu(\mathbb{1}_A\otimes g)$ for every $A\in\mathcal{S}$ and every bounded mapping $g:\mathbb{T}\to\mathbb{R}$ which is Lipschitz for d.*

Proof. In Conditions 1, 2, 3, we can restrict without loss of generality the range of the mappings f to the open interval $]0,1[$.

$1\Rightarrow 2$. For any integrand $f:\Omega\times\mathbb{T}\to\,]0,1[$, and for any $\mu\in\mathcal{M}^+(\Omega\times\mathbb{T})$, we have

$$\frac{1}{n}\left(1+\sum_{k=1}^{n-1}\mu\{f>k/n\}\right)\geq\mu(f)\geq\frac{1}{n}\sum_{k=1}^{n-1}\mu\{f>k/n\}.$$

Assume that f is l.s.c. For every real number a, the set $\{f>a\}$ is in $\underline{\mathcal{G}}$. Using Condition 1, we thus have, for every integer $n\geq 1$,

$$\begin{aligned}\liminf_\alpha\mu^\alpha(f)\geq\liminf_\alpha\left\{\frac{1}{n}\sum_{k=1}^{n-1}\mu^\alpha\{f>k/n\}\right\}&\geq\frac{1}{n}\sum_{k=1}^{n-1}\liminf_\alpha\mu^\alpha\{f>k/n\}\\ &\geq\frac{1}{n}\sum_{k=1}^{n-1}\mu\{f>k/n\}\geq\mu(f)-1/n,\end{aligned}$$

which proves 2.

$2\Rightarrow 3$ and $3\Rightarrow 4$ are obvious.

$4\Rightarrow 1$.

First step (this part of the proof is inspired from [18]). Let $f:\Omega\times\mathbb{T}\to\mathbb{R}$ be a bounded integrand such that $f(\omega,.)$ is Lipschitz for d. We shall prove that $\lim_\alpha\mu^\alpha(f)=\mu(f)$.

From the classical Portmanteau Theorem for narrow convergence on separable metric spaces (see *e.g.* [12]), Condition 4 means that, for every $A\in\mathcal{S}$, the net $(\mu^\alpha(\mathbb{1}_A\otimes .))_\alpha$ of elements of $\mathcal{M}^+(\mathbb{T})$ narrowly converges to the measure $\mu(\mathbb{1}_A\otimes .)$. But this is independent from the distance d. We can thus choose a distance d such that $(\mathbb{T},d)$ be totally bounded (see [13, 23]). Then the space $\mathcal{C}_u(\mathbb{T},d)$ of d–uniformly continuous functions on $\mathbb{T}$ is separable for the supremum norm $\|.\|_\infty$. In particular, the set $\mathrm{BL}_d(\mathbb{T})$ of bounded Lipschitz functions is separable. Let $\mathcal{D} = \{h_0, h_1, \dots\}$ be a countable dense subset of $\mathrm{BL}_d(\mathbb{T})$. Let $\epsilon > 0$. For each $\omega \in \Omega$, let

$$N(\omega) = \inf\{n \in \mathbb{N};\ \|f(\omega, .) - h_n\|_\infty < \epsilon\}.$$

For each integer $n \geq 1$, let

$$A_n = \{\omega \in \Omega;\ N(\omega) = n\}.$$

Let $h = \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} \otimes h_n$. Condition 4 implies in particular that $\lim_\alpha \mu^\alpha(\Omega \times \mathbb{T}) = \mu(\Omega \times \mathbb{T})$. We thus have

$$\limsup_\alpha |\mu^\alpha(f - h)| \leq \epsilon\, \mu(\Omega \times \mathbb{T}) \ \text{ and } \ |\mu(f - h)| \leq \epsilon\, \mu(\Omega \times \mathbb{T}). \tag{3.1}$$

Now, there exists an integer n_0 such that $\mu(\cup_{n>n_0}(A_n \times \mathbb{T})) < \epsilon\, \mu(\Omega \times \mathbb{T})$. By Condition 4, we have $\lim_\alpha \mu^\alpha(\cup_{n>n_0}(A_n \times \mathbb{T})) = \mu(\cup_{n>n_0}(A_n \times \mathbb{T}))$. Let $\widetilde{h} = \sum_{n\leq n_0} \mathbb{1}_{A_n} \otimes h_n$. We thus have

$$\limsup_\alpha \left|\mu^\alpha(h - \widetilde{h})\right| \leq \epsilon\, \mu(\Omega \times \mathbb{T}) \ \text{ and } \ \left|\mu(h - \widetilde{h})\right| \leq \epsilon\, \mu(\Omega \times \mathbb{T}). \tag{3.2}$$

Morever, we also have

$$\lim_\alpha \mu^\alpha(\widetilde{h}) = \mu(\widetilde{h}). \tag{3.3}$$

Gluing together (3.1), (3.2) and (3.3) for all $\epsilon > 0$, we obtain that $\lim_\alpha \mu^\alpha(f)$ exists and satisfies

$$\forall \epsilon > 0 \quad \left|\lim_\alpha \mu^\alpha(f) - \mu(f)\right| \leq 4\epsilon\, \mu(\Omega \times \mathbb{T}).$$

We have thus proved that $(\mu^\alpha(f))_\alpha$ converges to $\mu(f)$.

Second step. From Proposition 1.1, we can assume without loss of generality that $\mathcal{S}$ is universally complete. Let $F \in \underline{\mathcal{F}}$. We only need to prove that $\limsup_\alpha \mu^\alpha(F) \leq \mu(F)$. From the Projection Theorem, the set

$$\Omega' = \{\omega \in \Omega;\ F(\omega) \neq \emptyset\}$$

is in $\mathcal{S}$. Furthermore, F admits a Castaing representation, that is, there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of $\mathcal{S}$–measurable mappings defined on Ω', with values in $\mathbb{T}$, such that, for every $\omega \in \Omega$, $F(\omega)$ is the closure of $\{\sigma_n(\omega);\ n \in \mathbb{N}\}$ (see [9, Theorem III.22]). We define a continuous integrand g_d on $\Omega' \times \mathbb{T}$, with values in $[0, +\infty]$, by

$$g_d(\omega, t) = \begin{cases} d(t, F(\omega)) = \inf_{n\in\mathbb{N}} d(t, \sigma_n(\omega)) & \text{if } \omega \in \Omega', \\ +\infty & \text{otherwise.} \end{cases}$$

For each $\epsilon > 0$, the set

$$G_{d,\epsilon} = \{g_d < \epsilon\}$$

is in $\underline{\mathcal{G}}$, and we have

$$F = \cap_{n\geq 1} G_{d,1/n}.$$

For each $\epsilon > 0$, let us define a bounded integrand $f_{d,\epsilon}$ on $\Omega \times \mathbb{T}$ by

$$f_{d,\epsilon}(\omega, t) = \begin{cases} 1 & \text{if} \quad (\omega, t) \in F \\ 1/\epsilon(\epsilon - g_d(\omega, t)) & \text{if} \quad 0 \leq g_d(\omega, t) \leq \epsilon \\ 0 & \text{if} \quad g_d(\omega, t) > \epsilon. \end{cases}$$

For every $\omega \in \Omega$, $f(\omega, .)$ is Lipschitz for d. From the first step, we thus have

$$\lim_\alpha \mu^\alpha(f_{d,\epsilon}) = \mu(f_{d,\epsilon}).$$

Furthermore, we have

$$\mathbb{1}_F \leq f_{d,\epsilon} \leq \mathbb{1}_{G_{d,\epsilon}}.$$

This yields

$$\mu(F) = \inf_{n\geq 1} \mu(f_{d,1/n}) = \inf_{n\geq 1} \lim_\alpha \mu^\alpha(f_{d,1/n}) \geq \limsup_\alpha \inf_{n\geq 1} \mu^\alpha(f_{d,1/n}) = \limsup_\alpha \mu^\alpha(F).$$

$\square$

In the preceding theorem, if $\{(\pi_\Omega)_\sharp(\mu^\alpha)\,;\, \alpha \in \mathbb{A}\}$ is equicontinuous (see the definition in the comments after Theorem 2.3), we can relax Condition 4 by replacing the σ–algebra $\mathcal{S}$ by a subset which generates $\mathcal{S}$, which is stable under finite intersection, and which contains Ω. This is shown slightly more generally in the following result.

THEOREM 3.2 *With the same notations and hypothesis as in Theorem 3.1, assume furthermore that $\{(\pi_\Omega)_\sharp(\mu^\alpha)\,;\, \alpha \in \mathbb{A}\}$ is equicontinuous. Let $\mathcal{C}$ be a set of non-negative $\mathcal{S}$–measurable bounded functions which generates $\mathcal{S}$, which is stable under multiplication of two elements, and which contains the constant function 1. Then Conditions 1,2,3,4 of Theorem 3.1 are equivalent to*

5. *$\lim_\alpha \mu^\alpha(f \otimes g) = \mu(f \otimes g)$ for every $f \in \mathcal{C}$ and every bounded mapping $g : \mathbb{T} \to \mathbb{R}$ which is Lipschitz for d.*

Proof. The implication $3 \Rightarrow 5$ is clear, thus we only need to prove $5 \Rightarrow 4$.

Without loss of generality, we restrict in Condition 5 the range of the mappings g to the interval $[0, 1]$. Let $\mathcal{A}$ be the set of bounded measurable mappings $f : \Omega \to \mathbb{R}$ such that $\lim_\alpha \mu^\alpha(f \otimes g) = \mu(f \otimes g)$ for every bounded mapping $g : \mathbb{T} \to \mathbb{R}$ which is Lipschitz for d. The set $\mathcal{A}$ is a vector space over $\mathbb{R}$ and contains the constant functions. Let us check that $\mathcal{A}$ is stable under monotone limits of uniformly bounded sequences. Let $(f_n)_n$ be an increasing uniformly bounded sequence of elements of $\mathcal{A}$ and let $f = \sup_n f_n$. Let $\epsilon > 0$. From equicontinuity of the margins of elements of $\mathfrak{M}$ on Ω, there exists an integer n_0 such that

$$\sup_\alpha \mu^\alpha((f - f_{n_0}) \otimes \mathbb{1}_\mathbb{T}) < \epsilon \text{ and } \mu((f - f_{n_0}) \otimes \mathbb{1}_\mathbb{T}) < \epsilon.$$

We thus have, for any mapping $g : \mathbb{T} \to [0, 1]$ which is Lipschitz for d,

$$\begin{aligned}\limsup_\alpha \mu^\alpha(f \otimes g) &\le \lim_\alpha \mu^\alpha(f_{n_0} \otimes g) + \epsilon \le \mu(f \otimes g) + \epsilon = \sup_n \mu(f_n \otimes g) + \epsilon \\ &= \sup_n \lim_\alpha \mu^\alpha(f_n \otimes g) + \epsilon \le \liminf_\alpha \sup_n \mu^\alpha(f_n \otimes g) + \epsilon \\ &= \liminf_\alpha \mu^\alpha(f \otimes g) + \epsilon.\end{aligned}$$

Thus, ϵ being arbitrary,

$$\lim_\alpha \mu^\alpha(f \otimes g) = \mu(f \otimes g).$$

From the Functional Monotone Class Theorem (see [10, Théorème 21, page 20] and [11, page 231]), $\mathcal{A}$ contains all bounded measurable functions $f : \Omega \to \mathbb{R}$. □

Corollary 3.3 (metrizability) *Assume that $\mathbb{T}$ is Suslin metrizable and that $\mathcal{S}$ is essentially countably generated (that is, there exists a countable subset $\mathcal{C}$ of $\mathcal{S}$ such that $\mathcal{S}$ is contained in the universal completion of the σ–algebra generated by $\mathcal{C}$). Let $\mathfrak{M}$ be a subset of $\mathcal{M}^+(\Omega \times \mathbb{T})$ such that $(\pi_\Omega)_\sharp(\mathfrak{M})$ is equicontinuous. Then $\mathfrak{M}$ is metrizable.*

Note that the metrizability condition on $\mathbb{T}$ cannot be removed. Indeed, if Q is any measure on Ω such that $Q(\Omega) \ne 0$, then $t \mapsto Q \otimes \delta_t$ is a continuous embedding from $\mathbb{T}$ into $\mathcal{M}^+(\Omega \times \mathbb{T})$ (we denote by δ_t the Dirac measure concentrated on t).

Proof. From Proposition 1.1, we can assume without loss of generality that $\mathcal{S}$ is countably generated. Let $\mathcal{C}$ be a countable algebra which generates $\mathcal{S}$. Let $\mathcal{A}$ be the set of indicator functions of elements of $\mathcal{C}$.

On the other hand, we can find a distance d on $\mathbb{T}$ which is compatible with the topology of $\mathbb{T}$ and such that $(\mathbb{T}, d)$ is totally bounded (see [13, 23]). The space $\mathrm{BL}_d(\mathbb{T})$ of (necessarily bounded) Lipschitz mappings from $\mathbb{T}$ to $\mathbb{R}$ is separable for the norm $\|.\|_\infty$. Let $\mathcal{E}$ be a countable dense subset of $\mathrm{BL}_d(\mathbb{T})$. We can assume without loss of generality that $\mathbb{1}_\mathbb{T} \in \mathcal{E}$.

Let $(\mu^\alpha)_{\alpha\in\mathbb{A}}$ be a net in $\mathfrak{M}$ and let $\mu \in \mathcal{M}^+(\Omega \times \mathbb{T})$. Assume that

$$\forall (f, g) \in \mathcal{C} \times \mathcal{E} \quad \lim_\alpha \mu^\alpha(f \otimes g) = \mu(f \otimes g).$$

We have, in particular, $\lim_\alpha \mu^\alpha(\Omega\times\mathbb{T}) = \mu(\Omega\times\mathbb{T})$. Let $g \in \mathrm{BL}_d(\mathbb{T})$. For each $\epsilon > 0$, we can find $g_\epsilon \in \mathcal{E}$ such that $\|g - g_\epsilon\|_\infty \le \epsilon$. We thus have, for each $f \in \mathcal{C}$,

$$\limsup_\alpha |\mu^\alpha(f\otimes g) - \mu^\alpha(f\otimes g_\epsilon)| \le \|f\|_\infty\, \epsilon \limsup_\alpha \mu^\alpha(\mathbb{1}_{\Omega\times\mathbb{T}}) = \|f\|_\infty\, \epsilon\, \mu(\Omega\times\mathbb{T}),$$

thus, ϵ being arbitrary, $(\mu^\alpha(f\otimes g))_\alpha$ converges and we have

$$\lim_\alpha \mu^\alpha(f\otimes g) = \mu(f\otimes g)$$

From Theorem 3.2, this proves that $(\mu^\alpha)_\alpha$ converges to μ. Thus the topology $\mathfrak{T}$ is the coarsest topology such that, for each $(f,g) \in \mathcal{C}\times\mathcal{E}$, the mapping $\mu \mapsto \mu(f\otimes g)$ is continuous. This proves that the topology $\mathfrak{T}_{\mathfrak{M}}$ induced by $\mathfrak{T}$ on $\mathfrak{M}$ is metrizable. Indeed, if we denote $\mathcal{C}\times\mathcal{E} = \{(f_n, g_n);\ n \in \mathbb{N}\}$, then a distance Δ which is compatible with $\mathfrak{T}_{\mathfrak{M}}$ is given by

$$\Delta(\mu,\nu) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{|\mu(f_n\otimes g_n) - \nu(f_n\otimes g_n)|}{1 + |\mu(f_n\otimes g_n) - \nu(f_n\otimes g_n)|}.$$

□

We say that a subset $\mathfrak{K}$ of a topological space $\mathbb{T}$ is *relatively sequentially compact* if every sequence of elements of $\mathfrak{K}$ admits a convergent subsequence. This terminology is not fully consistent with that for net–compactness, but we follow established use.

Corollary 3.4 (sequential compactness from compactness) *Assume that $\mathbb{T}$ is Suslin submetrizable and that $\mathcal{S}$ is essentially countably generated. Let $\mathfrak{M}$ be a relatively compact subset of $\mathcal{M}^+(\Omega\times\mathbb{T})$. Then the closure $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ for $\mathfrak{T}$ is metrizable, and thus $\mathfrak{M}$ is relatively sequentially compact.*

Proof. Let us denote by τ the topology of $\mathbb{T}$ and by τ_0 a metrizable topology which is coarser than τ. The topology τ_0 is also Suslin and has the same Borel sets as τ (see [30]). Therefore the set $\mathcal{M}^+(\Omega\times\mathbb{T})$ remains unchanged if we replace τ by τ_0. Let us denote by $\mathfrak{T}(\tau_0)$ the stable topology on $\mathcal{M}^+(\Omega\times\mathbb{T})$ associated with τ_0. Obviously $\mathfrak{T}(\tau_0)$ is coarser than $\mathfrak{T}$. Now, the topologies $\mathfrak{T}$ and $\mathfrak{T}(\tau_0)$ coincide on $\overline{\mathfrak{M}}$ because $\overline{\mathfrak{M}}$ is compact, Moreover, as $(\pi_\Omega)_\sharp\left(\overline{\mathfrak{M}}\right)$ is compact, it is equicontinuous. From Corollary 3.3, the set $\mathfrak{M}$ is thus metrizable for $\mathfrak{T}(\tau_0)$. Thus $\overline{\mathfrak{M}}$ is metrizable compact for $\mathfrak{T}$. □

4. Application to Young measures. Let P be a fixed probability measure on $(\Omega,\mathcal{S})$. We denote by $\mathcal{Y}(P)$ the set of elements μ of $\mathcal{M}^+(\Omega\times\mathbb{T})$ such that $(\pi_\Omega)_\sharp(\mu) = P$ and we endow $\mathcal{Y}(P)$ with the topology induced by $\mathfrak{T}$. The elements of $\mathcal{Y}(P)$ are called *Young measures*, see *e.g.* [38, 4]. All preceding results have obvious versions in $\mathcal{Y}(P)$ (in particular, the compactness criteria are quite simplified!).

Theorem 4.1 (Compactness criteria for Young measures) *Assume that $\mathbb{T}$ is Suslin submetrizable.*

1. *Every flexibly tight subset of $\mathcal{Y}(P)$ is net–compact.*
2. *If $\mathbb{T}$ is Prokhorov, a subset $\mathfrak{M}$ of $\mathcal{Y}(P)$ is net–compact if and only if $(\pi_{\mathbb{T}})_\sharp(\mathfrak{M})$ is net–compact in the narrow topology.*
3. *Every relatively compact subset of $\mathcal{Y}(P)$ is relatively sequentially compact.*

Remark 4.2 In particular, every flexibly tight subset of $\mathcal{Y}(P)$ is relatively sequentially compact: this extends slightly a result of relative sequential compactness by Balder [2], given for regular Suslin spaces.

Proof of Theorem 4.1. Part 1 is a particular case of Proposition 2.4.

Let us prove Part 2. We have $(\pi_\Omega)_\sharp(\mathfrak{M}) = \{P\}$ for every subset $\mathfrak{M}$ of $\mathcal{Y}(P)$, thus the condition of net–compactness of $(\pi_\Omega)_\sharp(\mathfrak{M})$ in the s–topology is always trivially satisfied, in particular, the equicontinuity condition in Theorem 3.2 and Corollary 3.3 is also automatically satisfied. Thus, Part 2 follows from Corollary 2.6.

Now, for Part 3, if we assume that $\mathcal{S}$ is essentially countably generated, the relative sequential compactness follows from Corollary 3.4. So, let us show how to drop this assumption on $\mathcal{S}$. Let $\mathfrak{M}$ be a relatively compact subset of $\mathcal{Y}(P)$. By reasoning on the closure of $\mathfrak{M}$, we can assume without loss of generality that $\mathfrak{M}$ is compact.

Firstly, let τ_W be the coarsest topology on $\mathcal{Y}(P)$ such that the mappings $\mu \mapsto \mu(\mathbb{1}_A \otimes g)$ are continuous for each $A \in \mathcal{S}$ and for each bounded continuous function $g : \mathbb{T} \to \mathbb{R}$. This topology is coarser than the stable topology, but, as $\mathbb{T}$ is submetrizable, τ_W is Hausdorff. As $\mathfrak{M}$ is compact for the stable topology, both topologies coincide on $\mathfrak{M}$, thus we only need to prove that $\mathfrak{M}$ is relatively sequentially compact for τ_W.

Secondly, as $\mathbb{T}$ is Suslin, it is a Radon space (see [30]), thus every element μ of $\mathcal{Y}(P)$ is disintegrable (see [37]), that is, there exists a Borel mapping $\mu_. : \omega \mapsto \mu_\omega$, $\Omega \to \mathcal{M}^+(\mathbb{T})$ such that μ_ω is a probability for each $\omega \in \Omega$ and such that

$$\mu(A \times B) = \int_A \mu_\omega(B)\, dP(\omega) \tag{4.1}$$

for all $A \in \mathcal{S}$ and all $B \in \mathcal{B}_{\mathbb{T}}$. Now, it is well known that the space $\mathcal{M}^+(\mathbb{T})$ is Suslin (see [30, Theorem 7 page 385]). Thus the Borel σ–algebra $\mathcal{B}_{\mathcal{M}^+(\mathbb{T})}$ is countably generated [30, Corollary page 108]). Let $(\mu^n)_n$ be a sequence in $\mathfrak{M}$ and let $\mathcal{S}_0$ be the σ–algebra generated by the mappings $\omega \mapsto \mu^n_\omega$. From Corollary 3.4, $(\mu^n)_n$ is a relatively sequentially compact sequence in $\mathcal{M}^+(\Omega \times \mathbb{T}, \mathcal{S}_0 \otimes \mathcal{B}_{\mathbb{T}})$. Let $(\nu^n)_n$ be a subsequence of $(\mu^n)_n$. There exist a further subsequence $(\lambda^n)_n$ of $(\nu^n)_n$ and an element μ of $\mathcal{M}^+(\Omega \times \mathbb{T}, \mathcal{S}_0 \otimes \mathcal{B}_{\mathbb{T}})$ such that $(\lambda^n)_n$ stably converges to μ in

$\mathcal{M}^+(\Omega \times \mathbb{T}, \mathcal{S}_0 \otimes \mathcal{B}_{\mathbb{T}})$. The measure μ can easily be extended to a measure (also denoted by μ) on $\mathcal{S} \otimes \mathcal{B}_{\mathbb{T}}$, with the help of formula (4.1), and we have $\mu \in \mathcal{Y}(P)$. Let $A \in \mathcal{S}$ and let $g : \mathbb{T} \to \mathbb{R}$ be a bounded continuous function. We have

$$\begin{aligned}
\lim_n \lambda^n(\mathbb{1}_A \otimes g) &= \lim_n \int_A \lambda^n_\omega(g)\, dP(\omega) \\
&= \lim_n E\left(\mathbb{1}_A \lambda^n_\cdot(g) | \mathcal{S}_0\right) \\
&= \lim_n E\left(E\left(\mathbb{1}_A | \mathcal{S}_0\right) \lambda^n_\cdot(g)\right) \\
&= \lim_n \lambda^n\left(E\left(\mathbb{1}_A | \mathcal{S}_0\right) \otimes g\right) \\
&= \mu\left(E\left(\mathbb{1}_A | \mathcal{S}_0\right) \otimes g\right) \\
&= \mu(\mathbb{1}_A \otimes g).
\end{aligned}$$

Thus $(\lambda^n)_n$ converges to μ for τ_W. □

An immediate consequence of Theorem 4.1 is the following corollary, which was known in the case when $\mathbb{T}$ is Polish (Balder [3, page 573], Valadier with another proof [19, Theorem 2.4]).

Corollary 4.3 (Equivalence between tightness notions) *Assume that* $\mathbb{T}$ *is Suslin submetrizable and Prokhorov. Then every flexibly tight subset of* $\mathcal{Y}(P)$ *is strictly tight.*

Proof. As $\mathbb{T}$ is Suslin, every element of $\mathcal{M}^+(\mathbb{T})$ is τ–regular. The result is thus an immediate consequence of Parts 1 and 2 of Theorem 4.1. □

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Laboratoire de Mathématiques Raphaël Salem, UMR CNRS 6085, Site Colbert, Faculté des Sciences, Université de Rouen, 76 821 Mont–Saint–Aignan Cedex, France

e-mail: prf@univ-rouen.fr

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