Master Thesis Proposal

Title: "Renormalized solution for quasilinear elliptic equations and finite volume approximation"

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This proposal has a part on the notion of renormalized solution and a part on the finite volume approximation.

For a data f belonging to L^1, if we consider the quasilinear equation

$$\begin{aligned} -\operatorname{div}(A(x,u)\nabla u) &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, with a matrix verifying appropriate conditions (Carathéodory function, ellipticity and bounded coefficients) it is well known since the work [5] of Boccardo-Gallouët that a solution in the sense of distributions exists. However even in the linear case, i.e. $A(x,s)=A(x)$, the obtained solution is not unique in general (see [16]). Roughly speaking the main obstacle is the fact that the solution u is not an admissible test function since we cannot expect to have u belonging to H^1_0 nor fu belonging to L^1.

Different notions have developed to extend the notion of weak solution so that we have existence, uniqueness and stability results for a large class of elliptic equations with L^1 data. Here we propose to use the notion of renormalized solutions, which was introduced by DiPerna and Lions in [8] for ordinary differential equations and which was extended to elliptic and parabolic equations (see among others [15, 14, 7, 3, 4]). It is well known that renormalized solution is convenient framework to deal with equation (1). There is a wide literature on the subject mainly for elliptic equations with Dirichlet boundary conditions, but a very few with Neumann boundary condition. The main difficulty in dealing with Neumann boundary conditions is the regularity of the solution which is not sufficient for the p-Laplace equation and p small to define the mean value.

Recently existence results for a class of nonlinear equations with L^1 data and Neumann boundary condition have been obtained in [1] (see also [2] for uniqueness results). Since in general the mean value of the solution is not well defined the authors use in [1] the median and prove the existence of renormalized solution having a null median.

For equation (1) with Neumann boundary conditions using the Boccardo-Gallouët estimates we can expect to define the mean value. The first aim of the present project is to adapt the method developed in [1] (see also [10] in a different framework) to prove the existence of a renormalized solution to

$$\begin{aligned} -\operatorname{div}(A(x,u)\nabla u) &= f \text{ in } \Omega \\ A(x,u)\nabla u\cdot\vec{n} &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2}$$

and having a null mean value.

As far as the approximation of elliptic or parabolic equations is concerned, Finite Element Methods and Finite Volume Methods are the most used. In this proposal we study the finite volume schemes and we refer to the book [11] in which the authors study finite volume approximation for linear or nonlinear elliptic, parabolic and hyperbolic equations. There is wide literature on the finite volume method and in general elliptic or parobolic equations with regular data (in the sense not belonging to L^1) are studied. In [9, 12] the convergence analysis of elliptic equation with L^1 data is studied and in both papers the authors prove, by adapting the method of Boccardo-Gallouët to the discrete case, that the finite volume approximation converges to a solution to the elliptic equation in the sense of distribution. Since the notion of solution in the sense of distribution is not the good one for L^1 data, a natural question is to know if the finite volume approximation converges to the renormalized solution. This question was solved recently in [13] for the non coercive elliptic equation

$$\begin{aligned} -\Delta u + \operatorname{div}(\mathbf{v}u) + bu = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3}$$

The second aim of the present project is, by mixing the methods developed in [6] and [13], to deal with elliptic equation with Neumann boundary conditions that is to prove that the finite volume scheme for the equation

$$\begin{aligned} -\operatorname{div}(\lambda(u)\nabla u) = f \text{ in } \Omega \\ \nabla u \cdot \vec{n} = 0 \text{ on } \partial\Omega \end{aligned} \tag{4}$$

converges to a renormalized solution (the existence of such a solution is the first aim). Here f belongs to L^1 and verifies the compatibility condition $\int_\Omega f = 0$ while λ is a continuous function verifying $0 < \lambda_0 \leq \lambda(r) \leq \lambda_1$, for any $r \in \mathbb{R}$.

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