Navier-Stokes/Biot coupling for poroelastic nonlinear shells

Phd subject proposed by A. Blouza
LMRS, Université de Rouen

Biot models describe the linked interaction between fluids and deformation in porous media. The presence of a moving fluid in a porous medium affects its mechanical response. In the same time, the change in the mechanical state of the porous skeleton influences the behavior of the fluid inside the pores. These two coupled deformation-diffusion phenomena lie at the heart of the theory of poroelasticity. More precisely, the two key phenomena can be summarized as follows:

- fluid-to-solid coupling: occurs when a change in the fluid pressure or fluid mass induces a deformation of the porous skeleton.
- solid-to-fluid coupling: occurs when modifications in the stress of the porous skeleton induce change in fluid pressure or fluid mass.

Biot model

The fluid domain is bounded by a deformable porous matrix consisting of a skeleton and connecting pores filled with fluid, whose dynamics is described by the Biot model. The classical quasi-static Biot poroelasticity system describes coupled elastic deformations and diffusive flow in porous medium. The material comprising the porous medium is supposed to be incompressible, which means that deformations in the medium occur due to the deformations of the porous skeleton. The Biot model is:

$$-\nabla\cdot\sigma(\mathbf{U}) + \nabla p_p = f_p, \qquad \text{in } \Omega_p$$

$$\frac{\partial}{\partial t}(\phi\beta\, p_p + \nabla\cdot\mathbf{U}) + \nabla\cdot\mathbf{V} = g_p(x,t), \qquad \text{in } \Omega_p,$$

where p_p is the fluid pressure, $\mathbf{U}$ is the displacement vector of the porous medium,

$$\sigma(U) = \mu\frac{\nabla\mathbf{U} + \nabla\mathbf{U}^t}{2} + \lambda\nabla\cdot\mathbf{U}\,\mathbf{Id}$$

is a second order symmetric stress tensor (expressing the Hooke's law) and

$$\mathbf{V} = \frac{-\kappa}{\eta}\nabla p_p,$$

is the fluid velocity vector (expressing the Darcy's law), λ (dilation moduli) and μ (shear moduli) are Lamé coefficients of the porous medium, φ is the porosity, β is the compressibility of the fluid, κ is the permeability of the porous medium, η is the viscosity of the fluid, $\mathbf{Id}$ is the unit tensor, and $g_p(x,t)$ is a source term, which describes e.g., injection process or extraction process.

The coupling first order terms in the system have the following meaning: the term ∇p_p in the first equation results from the additional stress in the medium coming from the fluid pressure, the term $\nabla\cdot\mathbf{u}$ in the second equation represents the additional fluid content due to local volume change.

The time-dependent Darcy problem can be regarded as a limit of the Biot system when the rigidity of the solid phase becomes infinity, i.e. the poroelastic structure is in fact a rigid body.

Shell model

In 3d elasticity, the undeformed body occupies a region Ω. Under loading, a point $x \in \Omega$ moves to $x + U(x)$. Let $\Gamma_0 \cup \Gamma_1$ be a partition of $\partial\Omega$ with meas $(\Gamma_0) > 0$. Then the equilibrium equations of 3d elasticity are:

$$\begin{aligned} -\text{div}\,\sigma(U) &= f && \text{in } \Omega \text{ (force balance)} \\ \sigma(U) &= H : e(U) && \text{in } \Omega \text{ (constitutive)} \\ U &= 0 && \text{on } \Gamma_0 \text{ (clamping)} \\ \sigma \cdot n &= h && \text{in } \Gamma_1 \text{ (boundary traction)} \end{aligned}$$

The corresponding weak form is given by : Find $U \in W_0 = \{V \in H^1(\Omega,^3), V = 0 \text{ on } \Gamma_0\}$ such that

$$\int_\Omega \Big(H(x) : e(U)(x)\Big) : e(V)\, dx = \int_\Omega f(x) \cdot V(x)\, dx + \int_{\Gamma_1} h(x) \cdot V(x)\, d\Gamma \;\forall V \in W_0,$$

where $e(U) = (\nabla U(x) + \nabla U(x)^t)/2$ is the stain tensor and the : operator represents tensor contraction.

A shell is physical body defined by a chart

$$\Phi(x_1, x_2, x_3) = \varphi(x_1, x_2) + x_3 a_3(x_1, x_2)$$

of the reference coordinates $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega \text{ and } |x_3| < t(x_1, x_2)/2\}$, where $t : \omega \to \mathbb{R}^+$ is the shell thickness, and $\varphi : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi \in \mathcal{C}^1(\bar{\omega}; \mathbb{R}^3)$ is the shell midsurface chart such that $a_\alpha = \partial_\alpha \varphi$, $\alpha = 1, 2$, is a basis.

In curvilinear coordinates, the weak form for $3d$ elasticity (without boundary traction) appears

$$\int_\Omega H^{ijkl} e_{ij}(U) e_{kl}(V) \sqrt{g}\, dx = \int_\Omega f(x) \cdot V(x) \sqrt{g}\, dx.$$

Under the **Naghdi's model**, the displacement U is determined by a midsurface displacement $u : \omega \to \mathbb{R}^3$ and the rotation r, $r_\alpha : \omega \to \mathbb{R}$, with $r_\alpha = r \cdot a_\alpha$

$$U(x_1, x_2, x_3) = \underbrace{u(x_1, x_2)}_{displacement} + x_3 \underbrace{r(x_1, x_2)}_{rotation}$$

Then, one can integrate over x_3 and by using kinematics assumptions and $\sigma_{33} = 0$ we obtain the 2d model: Find $(u, r) \in \mathcal{W}$

$$\int_\omega \Big\{ t a^{\alpha\beta\rho\sigma} \Big[\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) + \frac{t^2}{12} \chi_{\alpha\beta}(u, r)\chi_{\rho\sigma}(v, s) \Big] + t \frac{E}{1+\nu} a^{\alpha\beta} \delta_\alpha(u, r)\delta_\beta(v, s) \Big\} \sqrt{a}\, dx, = \int_\omega f \cdot v \sqrt{a}\, dx.$$

where

$$\mathcal{W} = \big\{(v, s) \in H^1(\omega; \mathbb{R}^3)^2;\; v = s = 0 \text{ on } \gamma_0\big\}.$$

and the tensors γ, χ, δ describe respectively membrane stretching, bending and transverse shear.

Navier-Stokes equations

The fluid that occupies the domain Ω_f may be described by the Navier-Stokes equations:

$$\begin{aligned} \nu \Delta \mathbf{u_f} + (\mathbf{u_f} \cdot \nabla)\mathbf{u_f} + \nabla p_f &= \mathbf{f_f}, \\ \nabla \cdot \mathbf{u_f} &= 0, \end{aligned}$$

with $\mathbf{u_f}$ the velocity of the fluid, p_f the pressure of the fluid and ν the viscosity of the fluid and $\mathbf{f_f}$ the body force.

Fluid-Poroelastic interaction

In the presence of a fluid-structure interaction coupling conditions must be applied at the interface between the structure and the fluid. The coupling conditions may be continuity of the velocity at the interface and/or continuity of the displacement. In the case of a blood flow (without porous media layer) it is common to consider the following model:

$$\nu\Delta\mathbf{u_f} + (\mathbf{u_f}\cdot\nabla)\mathbf{u_f} + \nabla p_f = \nabla\cdot\sigma_{\mathbf{f}}(u_f, p),$$
$$\nabla\cdot\mathbf{u_f} = 0,$$

where the fluid stress tensor is $\sigma_{\mathbf{f}}(u_f, p) = -p\mathbf{I} + 2\mathbf{D}(\mathbf{u_f})$ with $\mathbf{D}(\mathbf{u_f}) = \frac{1}{2}(\nabla\mathbf{u_f} + (\nabla\mathbf{u_f})^T)$.

At the fluid structure interface the coupling condition is:

$$\sigma_{\mathbf{f}}(u_f, p)n = -pn,$$

with n the (outward) normal to interface Γ.

Main focus of recent researches on the discretization of this problem consists in establishing an iterative scheme for enforcing the coupling condition.

Aim of the Phd

1. Derive the model by:
   - Modifying the equations above for taking into account in each one the presence of the shell described by a nonlinear Naghdi type, the porous media and the fluid.
   - Writing the coupling conditions at the interfaces taking into account the porosity of the shell.
2. Mathematical analysis of the model: establish existence and uniqueness of the model.
3. Write a numerical scheme for the model: a finite element scheme and an iterative algorithm that deals with the coupling conditions.
4. Derive error indicators that allow to balance between the discretization error and the error due to the use of an iterative algorithm for ensuring the coupling conditions.
5. Develop a code in Freefem++ for this interaction problem. Application to a blood flow in an artery.