Sard’s Theorem for Hyper-Gevrey Functionals on the Wiener Space

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Let \((X, H, \mu)\) be the Wiener space; we define “twisted Sobolev spaces” associated to certain compact operators on \(H\). Wiener functionals belonging to one of these spaces are termed “hypersmooth”; we prove a measure-theoretic analog of Sard’s theorem for them. Our result applies in particular to solutions of Itô’s SDE’s with Gevrey class coefficients.

CONTENTS

1. Itô-Taylor formula for Gevrey functionals.
2. Classes of hyper-Gevrey functionals.
5. A covering lemma.


INTRODUCTION, NOTATIONS AND PREREQUISITES

Let \(X\) be the Wiener space, \(\mu\) the Wiener measure on \(X\), \(f: X \to \mathbb{R}\) a smooth map in Malliavin’s sense [10] and \(\mathcal{C}_0\) the set of critical points of \(f\); a classical result [10, p. 385] allows us to state that the measure \(f_\#(1 - 1_{\mathcal{C}_0})\mu\) is absolutely continuous relatively to the Lebesgue measure \(m\) on \(\mathbb{R}\). It is now rather natural to study the residual part \(\rho_f = f_\#(1_{\mathcal{C}_0})\mu\) of the image measure. Should \(X\) be replaced by a finite-dimensional Gaussian space, Sard’s well-known theorem [14] would imply that \(\rho_f\) would be singular with respect to \(m\); such is not the case here. Y. Katznelson and P. Malliavin [8] have even produced an example of an
f such that $\rho_f$ has a nonzero $C^\prime$ density with respect to $m$. The classical proof of Sard’s theorem is related to covering properties. It is well known that for $C^\prime$-functions defined on $\mathbb{R}^n$, the theorem fails to be true if $n$ is large enough. On the Wiener space there are essentially two distinct manifestations of the “lack of compactness”:

(i) The unit ball of linear forms on $L^2([0,1])$ is not compact, and the analogous fact is true for all the components of the Wiener chaos (linear forms being canonically identified to the elements of the first-order Wiener chaos).

(ii) Most interesting Wiener functionals are not differentiable in the Banach space sense.

In finite dimension, Ascoli’s theorem ensures that the unit ball of the space of Lipschitz functions is compact in the space of continuous functions. No equivalent exists on $X$ according to (i). Recently Da Prato, Malliavin, and Nualart [4] have given a full characterization of the compact subsets of the set of smooth functionals. We shall prove that solutions of certain stochastic differentials equations are “hyper-Gevrey functionals” in a sense to be made precise below. For such functionals, we shall prove Sard’s theorem by a covering argument based on an estimation of the oscillation of the function in a fiber containing a critical point. We shall generally follow the terminology and notation of [10]: $(X, H, \mu)$ will denote the usual Wiener space, $\nabla$, the gradient in Malliavin’s sense, $\mathcal{S}$ the Ornstein–Uhlenbeck operator, $\mathcal{C}_n$ the $n$th Wiener chaos; it is well known that $L^2(X)$ is the orthogonal direct sum of the $\mathcal{C}_n$. We shall denote by $j$ the canonical isomorphism

$$j: H \to L^2([0,1])$$

given by

$$j(x)(t) = x'(t).$$

Sard’s theorem in infinite dimension has recently been investigated in [13]; the setting, however, is rather different there.

A sketch of this paper appeared as part 5 of my Ph.D. thesis (Université Paris VI, 1993) written under the guidance of Professor Paul Malliavin; elements of part 4 of the thesis also appear in Section 3. The bulk of the writing-up was done while I was visiting the Institut für Angewandte Mathematik in Bonn (October 1993—February 1994); I take the opportunity to thank Michael Röckner and Hans Föllmer for inviting me and for helping in many ways to make my stay in Germany pleasant and fruitful. Sergio Albeverio and Niels Jacob greatly honored me by their invitations to lecture upon this material in Bielefeld and Erlangen, respectively; I then benefitted a lot from comments by members of the audience, notably Z. Brzeziak.
1. Itô–Taylor Formula for Gevrey Functionals

**Theorem 1.1.** Let \( g \in \mathcal{W}^{2,\gamma} (X) \) be
\[
\forall m \in \mathbb{N}, \| \nabla^m g \|_{L^2 L^{\gamma \gamma}} \leq C \gamma^m (m!)^\gamma \quad (\mathcal{W}^\gamma)
\]
where \( \alpha \geq 0 \) and \( \gamma \geq 1 \), then, denoting by \( \mathcal{C}_g \) the set of critical points of \( g \), i.e.,
\[
\mathcal{C}_g = \{ x \in X | \nabla g(x) = 0 \},
\]
one has, setting \( x' = \max(x, \frac{1}{2}) \):
\[
\delta(\| g(x, \cdot(t)) - g(x, (0)) \| \leq 1, \| g(x, (0)) \| \leq C_1 \exp(-C_2 t^{\frac{1}{4\gamma - 1}}))
\]
(\textit{where} \( x, \cdot(t) \) \textit{is the Ornstein–Uhlenbeck process on} \( X([9]) \)), \textit{with} \( C_1 \) \textit{and} \( C_2 \) \textit{depending only upon} \( x \) \textit{and} \( x' \).

A number of preliminary results will be useful.

**Lemma 1.2** (Stroock’s commutation formula [15]). \( \mathcal{L} \nabla = \nabla (\mathcal{L} + 1) \).

**Proof.** It is obvious via the chaos expansion. \qed

**Lemma 1.3.** Let \( f \in \mathcal{W}^{2,\gamma} (X) \), then
\[
\| \mathcal{L}^n \nabla f \|_{L^2 L^{\gamma \gamma}} \leq \| \mathcal{L}^{n+1} f \|_{L^2 L^{\gamma \gamma}}.
\]

**Proof.** Let \( f = \sum_{m \geq 0} f_m \) the decomposition of \( f \) on the Wiener chaos: one can write:
\[
\mathcal{L}^n \nabla f = \sum_{m \geq 0} \mathcal{L}^n \nabla f_m
\]
\[
= \sum_{m \geq 0} (1 - m)^n \nabla f_m,
\]
whence
\[
\| \mathcal{L}^n \nabla f \|_{L^2 L^{\gamma \gamma}} = \sum_{m \geq 0} (m - 1)^{2n} \| \nabla f_m \|_{L^2 L^{\gamma \gamma}}^2
\]
\[
= \sum_{m \geq 0} (m - 1)^{2n} m \| f_m \|_{L^2 L^{\gamma \gamma}}^2
\]
\[
\leq \sum_{m \geq 0} m^{2n + 2} \| f_m \|_{L^2 L^{\gamma \gamma}}^2
\]
\[
= \| \mathcal{L}^{n+1} f \|_{L^2 L^{\gamma \gamma}}. \qed
\]
By an elementary operator of type \((r, s)\) we shall mean the product, in an arbitrary order, of \(r\) factors \(L^p\) and \(s\) factors \(V\). The next two lemmas allow us to reduce the study of such operators to that of powers of \(L^p\).

**Lemma 1.4.** Let \(A\) be an elementary operator of type \((r, s)\); then

\[
\forall f \in W^{r, s}(X) \quad \|Af\|_{L^2(X)} \leqslant \|L^{r+s}f\|_{L^2(X)}.
\]

**Proof.** Let us proceed by induction over \(r + s\), as the result is obvious for \(r + s = 0\). In the other case, there are two possibilities:

(a) \(A\) is of the shape \(B L^p\), where \(B\) is of type \((r - 1, s)\). The induction hypothesis applied to \(B\) now gives us

\[
\|Af\|_{L^2(X)}^2 = \|BLf\|_{L^2(X)}^2
\]

\[
\leqslant \|L^{r+s-1}(Lf)\|_{L^2(X)}^2
\]

\[
= \|L^{r+s}f\|_{L^2(X)}^2,
\]

whence the desired conclusion.

(b) \(A\) is of the shape \(B V\), where \(B\) is of type \((r, s-1)\). We can then apply the induction hypothesis to \(B\) and to each of the components \(f_k = \langle \nabla f \mid e_k \rangle\); we get

\[
\|Bf_k\|_{L^2(X)}^2 \leqslant \|L^{r+s-1}f_k\|_{L^2(X)}^2.
\]

But one has

\[
\nabla f = \sum_{k=0}^{+\infty} f_k e_k,
\]

whence

\[
Af = \sum_{k=0}^{+\infty} Bf_k \otimes e_k
\]

\[
L^{r+s-1} \nabla f = \sum_{k=0}^{+\infty} (L^{r+s-1}f_k) e_k.
\]

Summing over \(k\) in the above inequalities we thus get

\[
\|Af\|_{L^2(X)}^2 \leqslant \|L^{r+s-1} \nabla f\|_{L^2(X)}^2
\]

\[
\leqslant \|L^{r+s}f\|_{L^2(X)}^2,
\]

where we have used Lemma 1.3. The result follows. \(\blacksquare\)
Lemma 1.5. Let \( f \in \mathcal{H}^{2,x}(X) \), then one has
\[
\|D^n f\|_{L^2(X)} \leq (3n)^{2n} \|f\|_{L^2(X)} + 3^{2n} \|\nabla^{2n} f\|_{L^2(X; H^{2n})}.
\]

Proof. Let us write the expansion of \( f \) using the “generalized Hermite polynomials”:
\[
f = \sum c_\phi H_\phi.
\]
We have
\[
\|D^n f\|_{L^2(X)} = \sum \|\phi\|^{2n} c_\phi^2
\]
and
\[
\|\nabla^{2n} f\|_{L^2(X; H^{2n})} = \sum (\|\phi\|-2n+1) c_\phi^2.
\]
For \( \|\phi\| \leq 3n \), one majorizes \( \|\phi\|^{2n} \) by \((3n)^{2n}\), for \( \|\phi\| > 3n \), one can write
\[
\prod_{j=1}^{2n} (\|\phi\| - j + 1) \geq (\|\phi\| - 2n)^{2n} \geq \left( \frac{\|\phi\|}{3} \right)^{2n},
\]
whence
\[
\|\phi\|^{2n} \leq 3^{2n} \prod_{j=1}^{2n} (\|\phi\| - j + 1).
\]
The result follows.

Corollary 1.6. Let \( g \in \mathcal{H}^{2,x}(X) \) satisfying the hypotheses of the theorem, then
\[
\forall n \in \mathbb{N}, \|D^n g\|_{L^2(X)} \leq 2c^2(9 \cdot \gamma^4 \cdot 2^4 \nu)^n n^{2\alpha'},
\]
where \( \alpha' = \max(\alpha, \frac{1}{2}) \).

Proof. By Lemma 1.5, we have
\[
\|D^n g\|_{L^2(X)} \leq (3n)^{2n} \|g\|_{L^2(X)} + 3^{2n} \|\nabla^{2n} g\|_{L^2(X; H^{2n})} \\
\leq (3n)^{2n} c^2 + 3^{2n} c^2 \gamma^{4n} (2n)^{2n} \\
\leq 3^{2n} c^2 \nu^{4n} \gamma^{4n} + 3^{2n} c^2 \gamma^{4n} (2n)^{4n} \\
\leq c^2 3^{2n} \nu^{4n} \gamma^{4n} + c^2 (2n)^{4n} \nu^{4n} \\
= 2c^2(9 \cdot \gamma^4 \cdot 2^4 \nu)^n n^{2\alpha'}.
\]
Lemma 1.7. Let $f \in \mathcal{H}^{2,\infty}(X)$, $A$ an elementary operator of type $(r, s)$, and $n$ an integer at least equal to $r + s$. Then one has, $\forall t \in [0, 1]$,

$$
\mathcal{E} \left( \| A f(\xi_n(t)) - A f(\xi_n(0)) \|_{\mathcal{H}^n}^2 \right) \leq \frac{4^n}{(n-r-s)!} \left( \frac{t^n}{r+s} \right) \| f \|_{L^n(X)}^2.
$$

Proof. $n$ being fixed, we shall use induction on $n - r - s$. For $r + s = n$ we simply write

$$
\mathcal{E} \left( \| A f(\xi_n(t)) - A f(\xi_n(0)) \|_{\mathcal{H}^n}^2 \right) \leq \mathcal{E} \left( \| A f(\xi_n(t)) - A f(\xi_n(0)) \|_{\mathcal{H}^n}^2 \right) \leq 2 \left( \mathcal{E} \left( \| A f(\xi_n(t)) \|_{L^n(X)}^2 \right) + \mathcal{E} \left( \| A f(\xi_n(0)) \|_{L^n(X)}^2 \right) \right) = 4 \left( \mathcal{E} \left( \| f \|_{L^n(X,H^n)}^2 \right) \right).
$$

because the Ornstein-Uhlenbeck process has the property that $(\xi_s)_s \mu = \mu$ for each $s \in \mathbb{R}$. It is now enough to apply Lemma 1.4.

Let us now assume that $r + s < n$, and let $(w_r)_{r < \infty}$ be a Hilbertian basis of $H^{r,s}$. Let us set

$$
A f(x) = \sum_{j=0}^{\infty} f_j(x) w_j.
$$

Each $f_j$ belongs to $\mathcal{H}^{2,\infty}(X)$, and thus satisfies Itô's formula [3, p. 207],

$$
f_j(\xi_n(t)) - f_j(\xi_n(0)) = \int_0^t \mathcal{L} f_j(\xi_n(u)) \, du + M_j(\omega, t),
$$

where

$$
M_j(\omega, t) = \int_0^t \| \nabla f_j(\xi_n(u)) \|_{H^n}^2 \, du \quad \text{is a martingale.}
$$

It follows that

$$
\mathcal{E} \left( (f_j(\xi_n(t)) - f_j(\xi_n(0)))^2 \right) \leq 2 \mathcal{E} \left( \left( \int_0^t \mathcal{L} f_j(\xi_n(u)) \, du \right)^2 \| f_j(\xi_n(0)) \|_{H^n}^2 + M_j(\omega, t)^2 \right).
$$
But \( \mathcal{L}Af(x_{\omega}(0)) = 0 \) and \( \nabla Af(x_{\omega}(0)) = 0 \) for almost all \( \omega \) such that \( x_{\omega}(0) \in \mathcal{C}_{r} \) by [7, Théorème 4]. Schwarz’s inequality and the definition of \( M_{j} \) thus allow us to write

\[
\delta((f_{j}(x_{\omega}(t)) - f_{j}(x_{\omega}(0)))^{2} 1_{x_{\omega}(0)})
\leq 2t \int_{0}^{t} \delta((\mathcal{L}^{2}f_{j}(x_{\omega}(u)) - \mathcal{L}^{2}f_{j}(x_{\omega}(0)))^{2} 1_{x_{\omega}(0)}) du
\]

\[
+ 2t \int_{0}^{t} \delta(||\nabla f_{j}(x_{\omega}(u)) - \nabla f_{j}(x_{\omega}(0))||_{H_{r}}^{2} 1_{x_{\omega}(0)}) du.
\]

By the dominated convergence theorem one may, while summing on \( j \), exchange integration and summation; moreover, one has

\[
\mathcal{L}Af = \sum_{j \in \mathbb{N}} \mathcal{L}f_{j} w_{j}
\]

and

\[
\nabla Af = \sum_{j \in \mathbb{N}} \nabla f_{j} \otimes w_{j}.
\]

It appears that

\[
\delta(||Af(x_{\omega}(t)) - Af(x_{\omega}(0))||_{H_{r}}^{2} 1_{x_{\omega}(0)})
\leq 2t \int_{0}^{t} \delta(||\mathcal{L}Af(x_{\omega}(u)) - \mathcal{L}Af(x_{\omega}(0))||_{H_{r}}^{2} 1_{x_{\omega}(0)}) du
\]

\[
+ 2t \int_{0}^{t} \delta(||\nabla Af(x_{\omega}(u)) - \nabla Af(x_{\omega}(0))||_{H_{r}}^{2} 1_{x_{\omega}(0)}) du.
\]

The induction hypothesis now gives us

\[
\delta(||Af(x_{\omega}(t)) - Af(x_{\omega}(0))||_{H_{r}}^{2} 1_{x_{\omega}(0)})
\leq 2 \|L^{n}f\|_{L^{2}(X)} \frac{4^{n-r-s}}{(n-r-s-1)!} \left( \int_{0}^{t} u^{n-r-s-1} du + t \int_{0}^{t} u^{n-r-s-1} du \right)
\]

\[
= \frac{2 \cdot 4^{n-r-s}}{(n-r-s-1)!} \frac{t^{n-r-s}}{n-r-s} \cdot \left( 1 + t \right) \|L^{n}f\|_{L^{2}(X)}^{2}
\]

\[
\leq \frac{4^{n-r-s+1}}{(n-r-s)!} \cdot t^{n-r-s} \cdot \|L^{n}f\|_{L^{2}(X)}^{2}.
\]

\[ \blacksquare \]
Proof of Theorem 1.1. Let us apply Lemma 1.7 to \( f = g \) and \( A = \text{Id} \); we get

\[
\forall t \in [0, 1], \forall n \in \mathbb{N}, \quad \delta(|g(x_n(t)) - g(x_n(0))|^2 \mathbf{1}_{\mathbb{R}^n}(x_n(0))) \leq 4^{n+1} \frac{t^n}{n!} \|f\|^2_{L^2(X)}.
\]

By Corollary 1.6, this is less than

\[
4^{n+1} \frac{t^n}{n!} 2e^{2(9 \cdot \gamma^4 \cdot 2^{4s})n} n^{4s}.
\]

But \( n! \geq (n/e)^n \), whence \( \forall t \in [0, 1], \forall n \in \mathbb{N}, \)

\[
\delta(|g(x_n(t)) - g(x_n(0))|^2 \mathbf{1}_{\mathbb{R}^n}(x_n(0))) \leq 8e^{2(36 \cdot e \cdot \gamma^4 \cdot 2^{4s} \cdot t)n} n^{4s}.
\]

Choosing \( n = E((1/e)(36 \cdot e \cdot \gamma^4 \cdot 2^{4s} \cdot t)^{1/(4s+1)}) \), we see that the right-hand side is less than

\[
8e^{2(36 \cdot e \cdot \gamma^4 \cdot 2^{4s} \cdot t)n} e^{-(4s+1)n(36 \cdot e \cdot \gamma^4 \cdot 2^{4s} \cdot t)^{-n}}
\]

\[
= 8e^{2e^{4s+1} - (4s+1)(36 \cdot e \cdot \gamma^4 \cdot 2^{4s} \cdot t)}
\]

\[
\leq 8e^{2e^{4s+1} - 4s+1} e^{-(4s+1)(36 \cdot e \cdot \gamma^4 \cdot 2^{4s} \cdot t)}
\]

Whence the result with \( c_1 = 8e^{2e^{4s+1}} \) and

\[c_2 = \frac{4s+1}{e} (36 \cdot e \cdot 16^s)^{-1/(4s+1)}.
\]

Remark. All infinite dimensional separable Gaussian probability spaces are canonically isomorphic; the results of this section are therefore valid for any of them. We shall use that remark in Section 4.

2. Classes of Hyper-Gevrey Functionals

The definitions in this paragraph can be considered as the axiomatic basis for the sequel of the paper. Let \((X, H, \mu)\) be an abstract Wiener space; a positive definite self-adjoint compact operator \(A\) on \(H\) will be called of order \(a\) if

\[
\lambda_1 \geq \cdots \geq \lambda_n \geq \cdots
\]

being its eigenvalues in decreasing order, there are \(c > 0\) and \(c' > 0\) such that:

\[
\forall n > 0, \quad c'n^{-a} \leq \lambda_n \leq cn^{-a}.
\]
Now, such an $\mathcal{A}$ being given, we will define the Sobolev spaces $\mathcal{S}^p_{r,\mathcal{A}}$ by their norm:

$$\|\phi\|_{\mathcal{S}^p_{r,\mathcal{A}}} = \sum_{j=0}^{r} \|((\mathcal{A}^{\otimes j})^{-1} \nabla^j \phi\|_{L^p(X, H^{\otimes j})}.$$  

Then we shall the class $\mathcal{S}^p_{r,\mathcal{A}}(X)$ of hypersmooth functionals of order $(p, a, \alpha)$ by the condition that there exists a compact operator $\mathcal{A}$ of order $a$ and real numbers $c_1(p)$ and $c_2(p)$ such that $\forall r \in \mathbb{N}$:

$$\|g\|_{\mathcal{S}^p_{r,\mathcal{A}}} \leq c_1(p) c_2(p)^r (r!)^\alpha.$$  

It is a consequence of Appendix A1 that Itô functionals provided by solutions of SDE's with coefficients of Gevrey class $\alpha$ are elements of $\mathcal{S}^p_{r,\mathcal{A}}(X)$ for each $p \geq 1$ and each $a < \frac{1}{2}$; the operator $\mathcal{A}$ can be taken to be $\mathcal{L}$, and $c_1(p)$ and $c_2(p)$ are explicitly computable from $p$, $a$, and the derivatives of $\mathcal{A}$. We define

$$\mathcal{S}^p_{r,\mathcal{A}}(X) = \bigcap_{p \geq 1} \mathcal{S}^p_{r,\mathcal{A}}(X).$$

3. Heat Propagation for the Ornstein-Uhlenbeck Operator

**Lemma 3.1.** Given $x \in ]0, 1[\ and \ r \in ]0, 1[\ there exists $c_{r,\alpha}$ such that, whenever $A \subset X$ is a Borelian set with $\mu(A) \geq \alpha$ then one has

$$\forall t \geq r, \ E((P, 1_A)^{-t}) \leq c_{r,\alpha}.$$  

**Proof.** We shall use the symmetrization procedure in the sense of Ehrard [5] and Borrell [1]. Let us denote by $A^\prime$ the symmetrization of $A$ in the sense of [1, p. 3] (with $c = g = 0$, $n = 1$, $f = 1_A$). Then for any $p: \mathbb{R}^+ \to \mathbb{R}$ convex and increasing, we have

$$E(p(\mu A)) \leq E(p(\mu A^\prime)).$$  

For $\varepsilon > 0$ let us denote

$$p_\varepsilon(\xi) = (\varepsilon + \xi)^{-r} + r e^{-r-1} \xi.$$  

Then $p_\varepsilon$ is convex and increasing; thus

$$E(p_\varepsilon(\mu A)) \leq E(p_\varepsilon(\mu A^\prime)).$$

580 129 1-14
Furthermore,
\[ E(\mathcal{P}, \mathbf{1}_A) = \mu(A) \]
\[ E(\mathcal{P}, \mathbf{1}_{A^c}) = \mu(A^c) = \mu(A). \]

Therefore (i) implies that
\[ E((\varepsilon + \mathcal{P}, \mathbf{1}_A)^{-1}) \leq E((\varepsilon + \mathcal{P}, \mathbf{1}_{A^c})^{-1}). \]

Letting \( \varepsilon \to 0 \) we get
\[ E((\mathcal{P}, \mathbf{1}_A)^{-1}) \leq E((\mathcal{P}, \mathbf{1}_{A^c})^{-1}) \]
(this is obvious when the right-hand side is infinite; in the other case, Lebesgue's dominated convergence theorem can be applied.)

The computation of the right-hand side for a half-space reduces the problem to a problem in one dimension. On \( \mathbb{R} \) we shall use the Mehler formula,
\[ p_t(\xi_0, \eta) = (2\pi)^{-1/2} \beta_t^{1/2} e^{-\frac{1}{2} x^2 \xi_0 - \frac{1}{2} \eta^2}. \]

where
\[ \beta_t = (1 - e^{-2t})^{-1} \]
and \( p_t \) is defined by
\[ P_t f(\xi_0) = \int_{\mathbb{R}} f(\eta) p_t(\xi_0, \eta) \, d\eta. \]

We determine \( \lambda_\epsilon \) by the condition
\[ \int_{-\lambda_\epsilon}^{\lambda_\epsilon} d\mu_\mu(\eta) = x. \]

where
\[ d\mu_\mu(\eta) = (2\pi)^{-1/2} e^{-\frac{1}{2} \eta^2} \, d\eta. \]

One has trivially:
\[ -\frac{1}{2} (e^{-\epsilon \xi_0 - \eta})^2 \geq -e^{-2\epsilon \xi_0^2} - \eta^2. \]

Therefore, by an elementary computation, we get the result whenever
\[ t > \frac{1}{2} \log(1 + 2r). \]

But \( \log(1 + 2r) < 2r. \)
Lemma 3.2. Let $G$ be a Hilbert space, for each $p \geq 1$, each $n \in \mathbb{N}$, and each $g \in \mathcal{H}^{\nu, \tau} (X; G)$, one has $\forall t \geq 0$

$$E(\| g - P_t g \|_{G}^{p} \mathbf{1}_{\mathcal{C}_t}) \leq \frac{2^{p}}{\prod_{j=0}^{p-1} (1 + jp)} t^{n} \| P^{n} g \|_{L^{p} (X; G)}^{p},$$

where

$$\mathcal{C}_t = \{ x \in X \mid \nabla g (x) = 0 \}.$$

Proof. We shall proceed by induction over $n$. For $n = 0$ we have

$$E(\| g - P_t g \|_{G}^{p} \mathbf{1}_{\mathcal{C}_t}) \leq E(\| g - P_t g \|_{G}^{p})$$

$$= \| g - P_t g \|_{L^{p} (X; G)}^{p}$$

$$\leq (\| g \|_{L^{p} (X; G)} + \| P_t g \|_{L^{p} (X; G)})^{p}$$

$$\leq (2 \| g \|_{L^{p} (X; G)})^{p}$$

$$= 2^{p} \| g \|_{L^{p} (X; G)}^{p},$$

because $P_t$ is a contraction on $L^{p} (X; G)$; the result follows.

Let us assume the result to be true at rank $n$ and let $g \in \mathcal{H}^{\nu, \tau} (X; G)$; we have

$$g - P_t g = \int_{0}^{t} \left( - P_u \mathcal{L} g \right) du$$

(it is enough to check the equality for $g$ a scalar function, and thus for $g \in C_{m}$; but then it reduces to the obvious:

$$1 - e^{-mt} = \int_{0}^{t} me^{-mu} du.$$

But $\mathcal{L} g = 0$ almost surely on $\mathcal{C}_t$ by Théorème 4 of [7, p. 86], whence

$$\mathbf{1}_{\mathcal{C}_t} (g - P_t g) = \int_{0}^{t} \mathbf{1}_{\mathcal{C}_t} (\mathcal{L} g - P_u \mathcal{L} g) du.$$

One thus has

$$\| (g - P_t g) \mathbf{1}_{\mathcal{C}_t} \|_{G} \leq \int_{0}^{t} \| \mathcal{L} g - P_u \mathcal{L} g \|_{G} \mathbf{1}_{\mathcal{C}_t} du.$$
Hölder’s inequality now implies that
\[
\|(g - P_1 g) \mathbf{1}_{\mathcal{K}}\|_{L^q} \leq \left( \int_0^t \|v\| du \right)^{1/q} \left( \int_0^t \|g - P_u \mathcal{L}g\|_{L^p} \mathbf{1}_{\mathcal{K}} \|du\right)^{1/p},
\]
where \(q\) is the conjugate exponent of \(p\). It now follows that
\[
\|(g - P_1 g) \mathbf{1}_{\mathcal{K}}\|_{L^q} \leq t^{p-1} \int_0^t \|g - P_u \mathcal{L}g\|_{L^p} \mathbf{1}_{\mathcal{K}} \|du\.
\]
But the induction hypothesis applied to \(\mathcal{L}g\) gives us
\[
\forall u \in [0, t], \quad E(\|\mathcal{L}g - P_u \mathcal{L}g\|_{L^p} \mathbf{1}_{\mathcal{K}}) \leq \frac{2^p}{\prod_{j=0}^{p-1} (1 + j \rho)} u^{\rho p} \|\mathcal{L}^{n+1}g\|_{L^p(\mathcal{K} \times \Omega)}.
\]
But \(Q_u \in \mathcal{C}_{\mathcal{K}}\) by Théorème 4 of [7], whence
\[
E(\|g - P_1 g\|_{L^p} \mathbf{1}_{\mathcal{K}})
\leq t^{p-1} \int_0^t E(\|\mathcal{L}g - P_u \mathcal{L}g\|_{L^p} \mathbf{1}_{\mathcal{K}}) \|du\)
\leq t^{p-1} \int_0^t \frac{2^p}{\prod_{j=0}^{p-1} (1 + j \rho)} u^{\rho p} \|\mathcal{L}^{n+1}g\|_{L^p(\mathcal{K} \times \Omega)} \|du\)
\leq t^{p-1} \frac{2^p}{\prod_{j=0}^{p-1} (1 + j \rho)} \int_0^t u^{\rho p} \|\mathcal{L}^{n+1}g\|_{L^p(\mathcal{K} \times \Omega)} \|du\)
\leq \frac{2^p \|\mathcal{L}^{n+1}g\|_{L^p(\mathcal{K} \times \Omega)}}{\prod_{j=0}^{p-1} (1 + j \rho)} t^{\rho p} \int_0^t u^{\rho p} \|du\)
\leq \frac{2^p \|\mathcal{L}^{n+1}g\|_{L^p(\mathcal{K} \times \Omega)}}{\prod_{j=0}^{p-1} (1 + j \rho)} t^{\rho p + 1} \int_0^t u^{\rho p} \|du\)
\leq \frac{2^p \|\mathcal{L}^{n+1}g\|_{L^p(\mathcal{K} \times \Omega)}}{\prod_{j=0}^{p-1} (1 + j \rho)} t^{\rho p + 1}.
\]
The result is now established at rank \(n + 1\). \(\blacksquare\)

4. MAJORATION OF THE OSCILLATION OF AN ABSTRACT FUNCTIONAL

In the above paragraph an oscillation theorem has been proved for the case of a \(W^{-r}\)-functional. We want to prove an oscillation result for a Gevrey functional having a critical set of large measure in the fiber. For
each $\alpha > 0$ and each Gaussian space $Y$ we define $\mathcal{G}_\alpha(Y)$ to be the space of $g: Y \to \mathbb{R}$ satisfying the hypothesis ($\mathcal{H}'$) of Theorem 1.1 for some $\gamma \geq 1$ and some $c$.

**Lemma 4.1.** Given a Gaussian probability space $Y$ and $g \in \mathcal{G}_\alpha(Y)$, then, for each integer $m \geq 1$, $u_m = \|\nabla g\|^m \in \mathcal{G}_\alpha(Y)$.

**Remark.** This would not be true for $u = \|\nabla g\|$, according to the fact that the square root is not a smooth functional.

**Proof.**

\[ u_1(y) = \|\nabla g(y)\|^2 \]

\[ = \int_0^1 \left( \frac{d}{dt} \langle \nabla g(y)(t) \rangle \right)^2 dt \]

\[ = \int_0^1 D_\gamma g(y)^2 dt. \]

whence

\[ D_\gamma u_1(y) = 2 \int_0^1 D_\gamma g(y) \cdot D_\gamma g(y) dt. \]

Thus $u_1 \in \mathcal{G}_\alpha(Y)$, whence the result because $u_m = u_1^m$. □

**Lemma 4.2.** For each $p > 1$ there is a constant $K_p$ such that, for each Hilbert space $G$ and each $f \in \mathcal{H}^{p,\alpha} (X, G)$, one has, $\forall n \in \mathbb{N}$,

\[ \| \mathcal{L}^p f \|_{L^n(X, G)} \leq K_p n! \sum_{j=0}^{2n} \| \nabla^j f \|_{L^n(X, H^{j\alpha} \otimes G)}. \]

**Proof.** By Meyer's inequalities [12] there are constants $c_p (p > 1)$ such that, $\forall f \in \mathcal{H}^{p,\alpha} (X, G)$,

\[ \| \mathcal{L} f \|_{L^n(X, G)} \leq c_p \| f \|_{L^n(X, G)} + \| \nabla f \|_{L^n(X, H^{\alpha} \otimes G)}. \]

We set $K_p = 2(1 + c_p)$ and proceed by induction on $n$. For $n = 0$ it is obvious. Let us assume the hypothesis true at rank $n-1$ ($n \geq 1$); then we have

\[ \| \mathcal{L}^n f \|_{L^n(X, G)} = \| \mathcal{L}^{n-1} (\mathcal{L} f) \|_{L^n(X, G)} \]

\[ \leq K_p^{n-1} (n-1)! \sum_{j=0}^{2n-2} \| \nabla^j \mathcal{L} f \|_{L^n(X, H^{j\alpha} \otimes G)}. \]
But it follows easily from Lemma 1.2 that $\nabla^j L = (L - j) \nabla^j$, whence
\[
\| L^m f \|_{L^p(X,G)} \leq K_p^{n-1}(n-1)! \sum_{j=0}^{2n-2} \left( \| L \nabla^j f \|_{L^p(X,H^+ \otimes \mathbb{R} \otimes G)} + j \| \nabla^j f \|_{L^p(X,H^+ \otimes \mathbb{R} \otimes G)} \right)
\leq K_p^{n-1}(n-1)! \sum_{j=0}^{2n-2} (c_p[n + \frac{1}{2} j] \| L \nabla^j f \|_{L^p(X,H^+ \otimes \mathbb{R} \otimes G)})
\leq K_p^{n-1}(n-1)! \left( c_p \sum_{j=2}^{2n} \| \nabla^j f \|_{L^p(X,H^+ \otimes \mathbb{R} \otimes G)} + \sum_{j=0}^{2n-2} (c_p + j) \| \nabla^j f \|_{L^p(X,H^+ \otimes \mathbb{R} \otimes G)} \right)
\leq K_p^{n-1}(n-1)! 2(c_p + n) \sum_{j=0}^{2n-2} \| \nabla^j f \|_{L^p(X,H^+ \otimes \mathbb{R} \otimes G)}
\leq K_p^n n! \sum_{j=0}^{2n} \| \nabla^j f \|_{L^p(X,H^+ \otimes \mathbb{R} \otimes G)}.
\]

The result then follows by induction. 

**Theorem 4.3.** Given $p < +\infty$ there exists $c_p > 0$ and $c_p^n > 0$ such that for all $m \in \mathbb{N}$,
\[
\int_Y \| (A^\otimes m)^{-1} \nabla^m g(y) \|_{L^p(H^+ \otimes G)} \, d\mu(y) \leq c_p^m e^{-c_p^{n+1}}
\]
whenever $\mu(\mathcal{E}_g) \geq \frac{1}{2}$ and $g \in \mathcal{G}_{m}(Y)$.

**Proof.** Let $x > p$ be an integer, and let $s = 2x$; let then $r$ be defined by $s = p(1 + 1/r)$. Clearly $r \in ]0, 1[$. Setting $\beta = r/(r + 1)$, $\rho = 1 + 1/r = 1/\beta$ and $\lambda = r + 1$, one has
\[
\frac{1}{\lambda} + \frac{1}{\rho} = 1,
\]
which will allow us to use Hölder’s inequality with the exponents $\lambda$ and $\rho$. Setting $q_{\varepsilon',\varepsilon} = P, 1_{\varepsilon'}$, we can write

$$
\delta(1_{\varepsilon'}(y_{\varepsilon'}(t))) \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g \right\|_{\mathcal{H}^\varepsilon} \left( y_{\varepsilon'}(0) \right)
= \int_y \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g \right\|_{\mathcal{H}^\varepsilon} \left( y \right) q_{\varepsilon',\varepsilon}(y) \, dq(y).
$$

Now the Schwarz inequality gives us

$$
\int_y \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g(y) \right\|_{\mathcal{H}^\varepsilon} \, dq(y)
= \int_y \left( \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g(y) \right\|_{\mathcal{H}^\varepsilon} q_{\varepsilon',\varepsilon}(y) q_{\varepsilon',\varepsilon}(y) \right) ^{1/2} \left( \frac{dq(y)}{\sqrt[1/2]{q_{\varepsilon',\varepsilon}(y)}} \right) ^{1/2}
\leq \left( \int_y \left( \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g(y) \right\|_{\mathcal{H}^\varepsilon} q_{\varepsilon',\varepsilon}(y) \right) ^{1/2} \right) ^{1/2} \left( \int_y \left( \frac{dq(y)}{\sqrt[1/2]{q_{\varepsilon',\varepsilon}(y)}} \right) ^{1/2} \right) ^{1/2}
= \left( \int_y \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g(y) \right\|_{\mathcal{H}^\varepsilon} q_{\varepsilon',\varepsilon}(y) \, dq(y) \right) ^{1/2} E(q_{\varepsilon',\varepsilon})^{1/2}.
$$

But one has, because of the reversibility of the Ornstein–Uhlenbeck process,

$$
\int_y \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g(y) \right\|_{\mathcal{H}^\varepsilon} q_{\varepsilon',\varepsilon}(y) \, dq(y)
= E\left( \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g \right\|_{\mathcal{H}^\varepsilon} \right)_{\mathcal{H}^\varepsilon} \left( P, 1_{\varepsilon'} \right)
= E\left( P, \left( (\mathcal{A}^{\otimes m})^{-1} \nabla^m g \right)_{\mathcal{H}^\varepsilon} \right) \left( 1_{\varepsilon'} \right)
\leq E\left( (P, u_m - u_m) \right) \left( 1_{\varepsilon'} \right),
$$

where $u_m = \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g \right\|_{\mathcal{H}^\varepsilon}$. Thus one has

$$
\int_y \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g \right\|_{\mathcal{H}^\varepsilon} dq(y) \leq E\left( u_m - P, u_m \right) \left( 1_{\varepsilon'} \right) ^{1/2} E(q_{\varepsilon',\varepsilon})^{1/2}
$$

We then apply Lemma 3.2 (with $p = 1$) to $u_m$. We have

$$
\int_y \left\| (\mathcal{A}^{\otimes m})^{-1} \nabla^m g(y) \right\|_{\mathcal{H}^\varepsilon} dq(y) \leq \left( \frac{2t^n}{n!} \left\| \mathcal{A}^{m} u_m \right\|_{L(\mathcal{H})} \right) ^{1/2} E(q_{\varepsilon',\varepsilon})^{1/2}.
$$
Taking \( t \geq r \) we get, by Lemma 4.2, that
\[
\int_y \| (\mathcal{A} )^{\mathbb{R}^n} \phi \|_{L^p(\mathbb{R}^n)}^r \, \text{d} \mu(y) \\
\leq \left[ 2K^r_n \varepsilon_a \left( \sum_{j=0}^{2m} \| \nabla_j u_m \|_{L^p(\mathbb{R}^n)} \right) \right]^{1/p} \mathcal{E}(\gamma, \varepsilon_a)^{1/2}
\]
and we get the result thanks to Lemmas 3.1 and 4.1.

**Corollary 4.4.** The oscillation of \( g \) on each fiber satisfies
\[
\forall p \geq 1, \quad \int_y \left| g(y) - \int_y g(\tilde{y}) \, d\mu(\tilde{y}) \right|^p \, d\mu(y) \leq \varepsilon^m \gamma e^{-c \gamma m^2}.
\]

**Proof.** We use Ocone's formula, which is possible because \( Y \) is isomorphic to the classical Wiener space,
\[
g(y) = \int_y g(\tilde{y}) \, d\mu(\tilde{y}) + \int_0^1 E \, \mathcal{L}^0(D_g) \, dy(t),
\]
where \( D_g(y) = \mathcal{L}^0(\nabla g(y))(t) \). The Burkholder-Gundy inequalities [2, Theorem 5.1] tell us that, for each \( p > 1 \), there is a constant \( a_p \) such that, for each Hilbert-space valued adapted martingale \( A(y, t) \), one has
\[
E \left( \int_0^t \| A(y, t) \|_p^p \, dt \right) \leq a_p \int_0^t E(\| A(y, t) \|_p) \, dt.
\]
Applying this result to \( A(y, t) = E \, \mathcal{L}^0(\nabla g(y)) \), one finds
\[
\int_y \left| g(y) - \int_y g(\tilde{y}) \, d\mu(\tilde{y}) \right|^p \, d\mu(y) = E \left( \int_0^1 \left| E \, \mathcal{L}^0(D_g(y)) \, dy(t) \right|^p \right)
\leq a_p \int_0^1 E \left( \left| E \, \mathcal{L}^0(D_g(y)) \right|^p \right) \, dt
\leq a_p \int_0^1 E \left( \left| D_g(y) \right|_p^p \right) \, dt
\leq a_p \int_0^1 E \left( \left| D_g(y) \right|_p \right) \, dt
\leq a_p E \left( \int_0^1 \left| D_g(y) \right|_p \, dt \right)
\]
and one can then apply Theorem 4.3.

5. A Covering Lemma

**Lemma 5.1.** For \( a < \frac{1}{2} \), let \( \mathcal{A} \) denote a compact operator on \( H \) having \( \{ m^{-a} \}_{m=1}^{\infty} \) for eigenvalues. Denote by \( \mathcal{A} \) the restriction of \( \mathcal{A} \) to the space
$E_n$ generated by its first $n$ eigenfunctions. We take on $E_n$ the euclidean metric. Denote
\[ \mathcal{C}_n = \{ \xi \in E_n \mid \| \mathcal{A}_{a,n}^{-1}(\xi) \| \leq n^\delta \}. \]

Then for all $\varepsilon > 0$ we have
\[ \text{vol}(\mathcal{C}_n) \leq -\frac{2\pi^{n-2}}{n \Gamma(n/2)} c_\varepsilon^m (n^\delta)^m, \]

where $c_\varepsilon$ is independent of $n$ and $\varepsilon$.

**Proof.** We may assume that $\varepsilon < 1$; let us then set $\delta = 1 - \varepsilon$ and note that
\[ \forall \varepsilon \in ]0, \delta], \quad \frac{-\log(1 - \varepsilon)}{\varepsilon} \leq \frac{-\log(1 - \delta)}{\delta}. \]

Denote
\[ \mathcal{A}_{a,n}^{-1} = \frac{1}{n \delta} \mathcal{A}_{a,n}. \]

Then
\[ \{ \xi \mid \| \mathcal{A}_{a,n}^{-1}(\xi) \| \leq 1 \} = \mathcal{A}_{a,n}^{-1}(\{ \eta \mid \| \eta \| \leq 1 \}), \]

and
\[ \text{vol}(\mathcal{C}_n) = \text{vol}(\{ \xi \mid \| \mathcal{A}_{a,n}(\xi) \| \leq 1 \}) \]
\[ = \det(\mathcal{A}_{a,n}^{-1}) \text{vol}(B_{\mathbb{R}^n}(0, 1)). \]

But the volume of the unit ball in $\mathbb{R}^n$ is
\[ \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \]

$\mathcal{A}_{a,n}^{-1}$ has for its eigenvalues the $n^m/m!$, for $1 \leq m \leq n$; thus
\[ \log \det(\mathcal{A}_{a,n}^{-1}) = a \sum_{m=1}^{n} \log \frac{n}{m} \]
\[ = a \sum_{s=1}^{E(n^{\delta})} \left( -\log \left( 1 - \frac{s}{n} \right) \right) + a \sum_{s=1}^{n} \log \left( \frac{n}{n - s} \right) \]
\[ \leq -\alpha \delta \log(1 - \delta)n + an \log n, \]

whence the result with
\[ c_\varepsilon = e^{-\delta \log(1 - \delta)} = e^{-\delta} = e^{\varepsilon^{-1}}. \]
Theorem 5.2. For every $\varepsilon > 0$ there exists a constant $c_{\varepsilon,n}$ independent of $n$ such that it is possible to cover $C_n$ by $[n^{\text{dim}_n}c_{\varepsilon,n}r^{-n}]$ balls of radius $r$ ($r < 1$).

Proof. For each $\psi : \{1, \ldots, n\} \rightarrow \mathbb{Z}$ we consider the hypercube $C_\psi$ consisting of the points $(\xi_1, \ldots, \xi_n)$ of $\mathbb{R}^n$ such that

$$
\forall j \in \{1, \ldots, n\}, \xi_j \in \left[\frac{2r\psi(j)}{\sqrt{n}}, \frac{2r(\psi(j) + 1)}{\sqrt{n}} \right].
$$

For each $\psi$ such that $C_\psi \cap C_n \neq \emptyset$ we consider a ball of radius $r$ containing $C_\psi$; such a ball is contained in $(2 + 8r^2)^{1/2}C_n$, whence their number is at most

$$
N(r) = (2 + 8r^2)^{n/2}\text{vol}(C_n) (2r/\sqrt{n})^{-n}.
$$

As

$$
\text{vol}(C_n) \leq \frac{2\pi^{n/2}}{n^2 n!} c_{\varepsilon,n} n^{\text{dim}_n} r^{n/2}
$$

we get

$$
N(r) \leq (2 + 8r^2)^{n/2} \frac{2\pi^{n/2}}{n^2 n!} c_{\varepsilon,n} n^{\text{dim}_n} r^{n/2} \left(\frac{\sqrt{n}}{2r}\right)^n.
$$

By Stirling’s formula, one has

$$
\frac{n}{2} \Gamma\left(\frac{n}{2}\right) \approx \sqrt{n\pi} \left(\frac{n}{2}\right)^{n/2} e^{-n/2},
$$

whence we have a constant $c > 0$ such that

$$
\Gamma\left(\frac{n}{2}\right) \geq \frac{c}{\sqrt{n}} n^{n/2} (2e)^{-n/2},
$$

from which follows:

$$
N(r) \leq 10^{n/2} \frac{2\pi^{n/2}}{n} \sqrt{n} e^{-n/2} (2e)^{n/2} c_{\varepsilon,n} n^{\text{dim}_n} 2^{n/2} r^{-n}.
$$

$$
= (10\pi)^{n/2} \left(\frac{e}{2}\right)^{n/2} \frac{1}{c} \sqrt{n} c_{\varepsilon,n} n^{\text{dim}_n} 2^{n/2} r^{-n}
$$

$$
\leq c_{\varepsilon,n} n^{\text{dim}_n} r^{-n}. \blacksquare
$$
6. SARD Theorem

**Theorem 6.1.** Let \( a > 0 \) and \( x \in ]0, a/4[ \) be given, and let \( g \in \mathcal{G}_a(X) \); define \( p_x = g \ast (1 \ast \mu) \). Then \( p_x \) is carried by a set of zero Hausdorff dimension.

**Proof.** In case \( \mu(\mathcal{C}_x) = 0 \), we have \( p_x = 0 \) and the statement is obvious; we shall henceforth assume that \( \mu(\mathcal{C}_x) > 0 \). Let \( \mathcal{A} \) be the operator that appears in the definition of \( \text{"hypersmoothness"} \) and let \( e_i \) (\( i \in \mathbb{N}^* \)) be an orthonormal basis of \( H \) in which \( \mathcal{A} \) is diagonalised: \( \mathcal{A} e_i = \lambda_i e_i \), with the sequence \( \lambda_i \) (\( i \in \mathbb{N}^* \)) decreasing. We denote by \( V_n \) the subspace generated by the \( e_i \) (1 \( \leq i \leq n \)). We denote by \( \pi_n \) the canonical orthogonal projection onto \( V_n \):

\[
\pi_n(x) = \sum_{k=1}^n \langle e_k, x \rangle e_k.
\]

It is clear that \( \mu_{\pi_n} = (\pi_n) \ast \mu \) is the canonical Gaussian measure on \( V_n \). Let

\[
Y_n = \ker \pi_n = \text{Im}(I - \pi_n)
\]

and

\[
\mu_{V_n} = (I - \pi_n) \ast \mu.
\]

Then we have a pseudo-direct sum decomposition \( X = V_n \oplus Y_n \), \( \mu = \mu_{V_n} \oplus \mu_{Y_n} \) and \( (Y_n, V_n^\perp, \mu_{Y_n}) \) is an abstract Wiener space. For \( \xi \in V_n \) we define a measure on \( X \) by

\[
v^{\nu_n}(A) = \mu_{\pi_n}(A - \xi) \cap Y_n.
\]

It is clear that we have here a disintegration of \( \mu \) along \( (\pi_n) \ast \mu = \mu_{\pi_n} \), i.e.,

\[
\forall f \in L^1(X) \quad \int_X f(x) \mu(dx) = \int_{Y_n} \left( \int_X f(x) v^{\nu_n}(dx) \right) \mu_{\pi_n}(d\xi).
\]

Let \( \mathcal{C}_n = \pi_n(\mathcal{C}_x) \). Then

\[
\mu_{\pi_n}(\mathcal{C}_n) = \left( (\pi_n) \ast \mu \right)(\mathcal{C}_n) = \mu((\pi_n)^{-1}(\mathcal{C}_n, \mathcal{C}_x))
\]

decreases to \( \mu(\mathcal{C}_x) \) when \( n \to +\infty \). We can therefore find \( n_0 \) such that:

\[
\forall n \geq n_0, \quad \mu_{\pi_n}(\mathcal{C}_n) \leq \frac{3}{2} \mu(\mathcal{C}_x).
\]
For each \( n \geq n_0 \) there is \( \mathcal{C}_n \subset \mathcal{C}_n \) such that
\[
\mu_{\frac{1}{2}}(\mathcal{C}_n) > \frac{3}{4} \mu_{\frac{1}{2}}(\mathcal{C}_n)
\]
and such that
\[
E^{\mathcal{C}}(\|\mathcal{A}^{-1} \nabla g\|)^2(x) \leq c_1 e^{-c_2 n^\alpha} \tag{*}
\]
for each \( x \in \pi_n^{-1}(\mathcal{C}_n) \). Furthermore, we know, by Section 5, that \( \|\nabla \xi g\| \) along each fiber is small. Let us denote \( g_n = E^{\mathcal{C}}(g) \); then one can write
\[
\nabla^{m} g_n = E^{\mathcal{C}}(P_{\frac{1}{2}} \nabla^{m} (\nabla^{m} g)),
\]
whence
\[
E(\|\nabla^{m} g_n\|_{H^{m+n}}) \leq E(\|\nabla^{m} g\|_{H^{m+n}}).
\]

Therefore \( g_n \) has its \( m \)th derivative on \( \mathcal{C}_n \) bounded by \( c_1 e^{-c_2 n^\alpha} \); by the Sobolev embedding theorem on the finite-dimensional space \( V_n \), \( g_n \in \mathcal{C}^{\alpha}(V_n) \). Let us define
\[
\mathcal{G}_n = \{ \xi \in V_n \mid \left( \nabla \xi \right)^{\ast}\|_{n \nabla \xi} \text{ vanishes on a set of measure } > \frac{1}{2} \}.
\]

**Lemma 6.2.** \( \mathcal{G}_n \) is a closed set.


**Lemma 6.3.** For each \( n \geq n_0 \), \( \mu_{\frac{1}{2}}(\mathcal{G}_n) \geq \mu(\mathcal{G}_n) \).

**Proof.** We have
\[
\mu(\mathcal{G}_n) = \int_{\mathcal{G}_n} \mu_{\frac{1}{2}}((\mathcal{C}_n \cap \pi_{n}^{-1}(\zeta)) d\zeta)
\]
\[
\leq \frac{1}{2} \mu_{\frac{1}{2}}(\mathcal{C}_n - \mathcal{G}_n) + \mu_{\frac{1}{2}}(\mathcal{G}_n)
\]
\[
\leq \frac{1}{2} \mu(\mathcal{C}_n) + \frac{1}{2} \mu_{\frac{1}{2}}(\mathcal{G}_n). \tag{**}
\]

Let \( \beta > 0 \) and \( \varepsilon > 0 \) be so chosen that \( \beta(\varepsilon + \beta) > 4\alpha/\alpha \); this is possible because \( \alpha < \alpha/4 \). Let \( r_n = n^{-\beta} \); by Theorem 5.2, it is possible to cover
\[
\mathcal{G}_{n,M} = \mathcal{G}_n \cap \mathcal{F}(B_{1/2}(0, M)),
\]
using
\[
N_n \leq n^{m \varepsilon r_{n,M}} \left( \frac{n^{c_r n}}{M} \right)^{-n} \tag{***}
\]
balls of radius \( r_n \). According to Corollary 4.4, for each \( \xi \in \mathcal{N}_{n,M} \), the oscillation of \( g \) on \( V_\xi \) is less than \( c_\rho^n \exp(-c_\rho^n n^{\alpha-4}) \). Let us apply the finite-dimensional Taylor’s formula to \( g_n \) on \( V_\xi \); in a neighborhood of \( \xi \in \mathcal{N}_n \); we get

\[
g_n(x) - g_n(\xi) = \sum_{j=0}^{n} \frac{1}{j!} \nabla^j g_n(x - \xi, ..., x - \xi) + R_{n,\xi}(x).
\]

But, by (*), the oscillation of \( g_n \) on each of these balls is at most

\[
c_\rho^n \exp(-c_\rho^n n^{\alpha-4}) r_n.
\]

Furthermore, the oscillation of \( g \) on each fiber introduces an increase of the length of those intervals which is \( \exp(-n^{\alpha-4}) \). Therefore,

\[
g(\tau_n; \mathcal{N}_{n,M}) \subset \bigcup_{j=1}^{N_n} I_j,
\]

where

\[
h(I_j) \leq \exp(-n^{\alpha-4}) + c_\rho^n \exp(-c_\rho^n n^{\alpha-4}) r_n.
\]

We want that

\[
\log N_n = o(n^{\alpha-4}).
\]

But this is the case because of (***) and

\[
\frac{\beta}{\varepsilon + \beta} > \frac{4\pi}{\alpha}.
\]

**APPENDIX.** **An Example of Hyper-Gevrey Functionals Associated to an Itô SDE**

A1. **A Classical Majoration for a Multiplicative Stochastic Integral**

For \( A \in \mathbf{M}_n(\mathbb{R}) \), we shall denote by \( \|A\|_2 \) its Hilbert–Schmidt norm,

\[
\|A\|_2 = (\text{tr}(A'A))^{1/2},
\]

and by \( \|A\|_\infty \) its uniform norm,

\[
\|A\|_\infty = \sup_{x \neq 0} \frac{|Ax|}{\|x\|}.
\]
It is well known that
\[
\forall (A, B) \in \mathbf{M}_d(\mathbf{R})^2, \quad \|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty.
\]
\[
\forall A \in \mathbf{M}_d(\mathbf{R}), \quad \|A\|_{op} = \|A\|_\infty.
\]
\[
\forall A \in \mathbf{M}_d(\mathbf{R}), \quad |\text{tr}(A)| \leq |A|_\infty.
\]

The following result is close to Théorème 3 from Ibero [6].

**Lemma A1.1.** Given a stochastic differential equation defined on \(\mathbf{GL}_m(\mathbf{R})\) \((m \geq 1)\),
\[
dM = M \left( \sum_{i=0}^d A_i \, dx^i \right) \quad (d \geq 1)
\]
\[
M(0) = I
\]

(we have set \(dx^0(\tau) = d\tau\), where the \(A_i\) are adapted matrix-valued functionals with \(\|A_i\|_{op} \leq c_0\), then there exists constants \(c_1 > 0\) and \(c_2 > 0\) depending only on \(d, m\) and \(c_0\) such that:
\[
\forall R \geq 1, \quad P(\sup_{\tau \in [0,1]} \|M(\tau)\|_{op} \geq R) \leq c_1 \exp(-c_2(\log R)^2).
\]

**Proof.** Let us denote by \(\lambda_1, \ldots, \lambda_m\) the eigenvalues of \(M'M\). And let
\[
\forall r \in \mathbb{N}, \quad Q_r(x, \tau) = (M(\tau)'M(\tau))^r.
\]

Then let us compute the drift term with the help of Itô calculus.
\[
dQ_r = \left( \sum_{i=1}^d \left( \sum_{j=1}^{r-1} M(A_j \cdots A_i)' M \cdots + \sum_{j=1}^{r-1} MA_j'M \cdots MA_i'M \cdots \\
+ \sum_{j=1}^{r-1} A_i'M \cdots A_j'M \cdots \right) + \sum_{j=1}^{r-1} MA_j' \cdots MA_i' \cdots \right) dt,
\]
where the first inner sum has \(r^2\) terms, each of the next two \(r(r-1)/2\) and each of the outer sums \(r\). Defining \(q_r = \text{tr}(Q_r)\) then one has
\[
dq_r \leq \|M\|_{op}^{2r} \left( dmc^2_0 r^2 + 2dmc^2_0 \frac{r(r-1)}{2} + 2rc_0m \right) dt.
\]

Therefore,
\[
dq_r \leq \|M\|_{op}^{2r} 2mc_0(1 + dc_0) r^2 \, dt.
\]
\[ \|M\|_{L^p}^p = \sup_i \lambda_i^q \leq q_r. \]

Therefore \( q_r e^{-\tau f} \) is a submartingale, where

\[ c_r = 2mc_0(1 + dc_0) r^2 \]

Let

\[ T_{r, R} = \inf\{ \tau \geq 0 \mid q_r(\tau) \geq R^{2r} \} \]

then

\[ E(\exp(-c_r T_{r, R}) R^{2r}) \leq q_r(0) = m, \]

from which we get

\[ P(T_{r, R} \leq 1) \exp(-c_r) R^{2r} \leq m; \]

that is,

\[ P(T_{r, R} \leq 1) \leq m \exp(c_r - 2r \log R). \]

Let \( \tilde{c}_r = 2mc_0(1 + dc_0) \) and let \( r_0 = E(\log R) \); then \( c_{r_0} = \tilde{c}_r r_0 \leq r_0 \log R \), whence

\[ c_{r_0} - 2r_0 \log R \leq -r_0 \log R \]

\[ \leq -\frac{(\log R)^2}{\lambda} + \log R \]

\[ \leq -\frac{(\log R)^2}{\lambda} + \frac{1}{2\lambda} \left[ (\log R)^2 + \tilde{c}_r^2 \right] \]

\[ = -\frac{(\log R)^2}{2\lambda} + \frac{\tilde{c}_r}{2} \]

and

\[ P(T_{r_0, R} \leq 1) \leq m \exp(c_{r_0} - 2r_0 \log R) \]

\[ \leq m \exp \left( \frac{\tilde{c}_r}{2} \right) \exp \left( -\frac{(\log R)^2}{2\lambda} \right). \]

But from \( q_r \geq \|M\|_{L^p}^p \) it follows that

\[ \{ \sup_{r \in [0,1]} \|M(\tau)\|_{L^p} \geq R \} \subset \{ \sup_{r \in [0,1]} q_r(\tau) \geq R^{2r} \} \]

\[ \subset \{ T_{r_0, R} \leq 1 \}, \]
whence we get the result with
\[ c_1 = m \exp \left( \frac{s}{2} \right) = m \exp(mc_0(1 + dc_0)) \]
and
\[ c_2 = \frac{1}{2} \frac{1}{4mc_0(1 + dc_0)}. \]

**A2. Best Approximation of an Itô Functional by Functions of a Finite Number of Variables**

Given an increasing sequence \( V_n (n \in \mathbb{N}) \) of finite-dimensional subspaces of \( H \), then for every \( f \in H^2(X) \) the martingale \( f_n = E^T(f) \) converges towards \( f \) in \( H^2(X) \). The reduction of \( f \) consists in choosing a good sequence \( V_n (n \in \mathbb{N}) \) such that

1. \( f_n \) "approximates well" \( f \)
2. \( \dim(V_n) \) has a controlled growth.

For a general smooth functional \( f \), it is not possible to ensure a priori the control stated in (2). But in the case of Itô functionals this control is possible. We shall obtain it through the machinery introduced to prove the compactness of certain subsets of \( L^2(X) \) in \([4]\). We shall denote by \( e_s \) \((s \in \mathbb{N})\) the Haar basis of \( L^2([0, 1]) \) defined by

\[ \forall k \in \mathbb{N}, \forall j \in [0, 2^k - 1], \forall r \in [0, 1] \]
\[ e_{2^k + j}(r) = 2^{k/2}(1_{\{r \in [1/2^k, 1/2^{k-1})\}} - 1_{\{r \in [1/2^{k-1}, 1/2^{k-2})\}}) \]

and

\[ e_0 = 1. \]

Let
\[ A_\sigma : L^2([0, 1]) \rightarrow L^2([0, 1]) \]

be given by \( A_\sigma e_s = 2^se_s \) whenever \( s = 2^k + j \) with \( j \in [0, 2^k - 1] \) and \( A_\sigma e_0 = e_0 \). Let us define \( \bar{A}_\sigma : H \rightarrow H \) by

\[ \bar{A}_\sigma = j^{-1} \circ A_\sigma \circ j. \]

We shall work under the following hypothesis:
(\mathcal{H}) Let us consider on \( \mathbb{R}^n \) the following Stratonovich stochastic differential equation:

\[
dv_t(v) = \sum_{i=0}^d \sigma_i(v(t)) \, dv^i(t), \quad v(0) = 0
\]

(we have again set \( dx^i(t) = dt \)), and let \( g(x) = v_{\tau}(1) \). We assume that the \( \sigma_i^\tau \) \((1 \leq \tau \leq n)\) and all their partial derivatives of order at most three are bounded in absolute value by a constant \( M \) and we denote \( g(x) = v_{\tau}(1) \).

A2.1. Computation of the First Derivative

The first derivative is given (see [11]) by

\[
D_{\tau} g = \mathcal{J}_{1-\tau} = \mathcal{J}_{1-\tau}^{-1} (\mathcal{J}_{1-\tau}^{-1})^{-1}
\]

where \( \mathcal{J}_{1-\tau} \) is obtained by solving the linear stochastic differential equation,

\[
d\mathcal{J}_{1-\tau} = \left( \sum_{i=0}^d A_i(v(t)) \, dx^i(t) \right) \mathcal{J}_{1-\tau}, \quad \mathcal{J}_{1-0} = I.
\]

where \( A \) is defined by

\[
A_{\alpha_{\tau}}^\beta(\xi) = \frac{\partial \sigma_{\tau}}{\partial v^\beta(\xi)} \quad (\alpha, \beta \in \{1, \ldots, n\}).
\]

Then \( M(\tau) = (\mathcal{J}_{1-\tau})^{-1} \) is given by the following stochastic differential equation:

\[
dM(\tau) = -M(\tau) \left( \sum_{i=0}^d A_i(v(\tau)) \, dx^i(\tau) + A_0(\tau)) \, dt \right), \quad M(0) = I.
\]

In coordinates (ii) takes the form:

\[
d_t J_{1-\tau} = \sum_{i=0}^d \sum_{\alpha=1}^n A_i^\alpha(v(\tau)) \, J_{1-\tau}^{-1} \, dx^i(\tau), \quad J_{0-0} = \delta_{\tau,0}.
\]

A2.2. Computation of the Second Derivative

We shall use the mechanism of prolongation introduced by Malliavin [9, p. 228]. We have

\[
D_{\tau'} D_{\tau} g = \left[ (D_{\tau'} D_{1-\tau}) - D_{1-\tau}^{-1} D_{\tau'} (D_{1-\tau}^{-1}) \right] D_{1-\tau}^{-1}.
\]

We therefore have to compute, \( \tau' \) being fixed, \( D_{\tau'} D_{1-\tau} \). Then we have

\[
D_{\tau'} D_{1-\tau} = 0 \quad \text{if} \quad \tau' > \tau.
\]
Let us write $V^0 = \mathbb{R}^n$ and let us denote by $V^1$ the principal bundle $V^0 \times \text{GL}_n(\mathbb{R})$. Then let us define vector fields $\sigma_i^1$ on $V^1$ by

$$\sigma_i^1(v_i, \gamma) = (\sigma_i(v_i), A_i(v_i) \gamma).$$

This makes sense because $Z_i = A_i(v_i) \gamma$ defines a vector field on $\text{GL}_n(\mathbb{R})$ (the tangent space $\mathfrak{gl}_n(\mathbb{R})$ to $\text{GL}_n(\mathbb{R})$ being naturally identified to $\mathbb{M}_n(\mathbb{R})$). Then, setting $v_i^1(\tau) = (v_i(\tau), J_{\tau, -\theta})$, the conjunction of Eqs. (i) and (ii) can be written as

$$dv_i^1(\tau) = \sum_{r=0}^d \sigma_i^1(v_i^1(\tau)) \, dx^r(\tau), \quad v_i^1(0) = (0, I). \quad (iii)$$

We shall treat the computation of the second derivative by the procedure we have already used for (i); we have to differentiate along the vertical component $\text{GL}_n(\mathbb{R})$ of the fibre bundle. We choose a basis $(e_{s, \beta})_{1 \leq s \leq n, 1 \leq \beta \leq n}$ of the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ and we differentiate on the right:

$$\partial_s u(v, \gamma) = \frac{d}{d\epsilon} u(v, \gamma \exp(\epsilon e)).$$

Then the vector fields $(A_i(v) \gamma)_{1 \leq i \leq n}$ are invariant under this differentiation because it is performed on the right; therefore, defining $A^1 = \partial u/v_s$, we have

$$A_i^1 \in \text{End}(\mathbb{R}^n \times \mathfrak{gl}_n(\mathbb{R})), \quad A_i^1 = \begin{pmatrix} A_i^0 & Q_i \\ 0 & I_{\mathfrak{gl}_n(\mathbb{R})} \end{pmatrix}, \quad (iv)$$

where $Q_i \in \text{End}(\mathbb{R}^n \times \mathfrak{gl}_n(\mathbb{R}))$ is defined by

$$Q_i^s, \beta = \frac{\partial^2 \sigma_i^s}{\partial v^\beta \partial v^\gamma}.$$

Then we introduce the jacobian matrix

$$J_{\tau, -\theta}^1 \in \text{End}(\mathbb{R}^n \times \mathfrak{gl}_n(\mathbb{R}))$$

defined by

$$d_\tau J_{\tau, -\theta}^1 = \left( \sum_{r=0}^d A_i^1(v_i(\tau')) \, dx^r(\tau') \right) J_{\tau, -\theta}^1, \quad J_{0, -\theta}^1 = I. \quad (v)$$
Taking (iv) into account we have

\[
J_{\tau}^{(1)} a = \begin{pmatrix} J_{\tau}^{(1)} & 0 \\ 0 & I_{g_{\alpha}(\tau)} \end{pmatrix}.
\]

Then

\[
J_{\tau}^{(1)} a J_{\tau}^{(1)} a^{-1} = \begin{pmatrix} * & D_{\tau} D_{\tau} g \\ * & * \end{pmatrix}.
\]

Denoting \( M_{\tau} = (J_{\tau}^{(1)} a)^{-1} \) we have again that \( M_{\tau} \) is given by

\[
d_{\tau} M_{\tau} = -M_{\tau} \left( \sum_{i=0}^{d} A_{\tau}^*(\tau') \right) d\tau' (\tau'), \quad M_{0} = I. \tag{vi}
\]

A2.3. A Tensor Product Norm

Let \( G \) be a Hilbert space; for each \( a \in [0, \frac{1}{2}] \) we shall define

\[
\mathcal{H}_{a}(G) = \{ p : [0, 1] \rightarrow G \mid p(0) = 0 \text{ and } \| p \|_{\mathcal{H}_{a}(G)} < +\infty \}
\]

with

\[
\| p \|_{\mathcal{H}_{a}(G)}^2 = \int_{[0, 1]^2} \frac{\| p(t) - p(t') \|_{G}^2}{|t - t'|^{1 + 2a}} dt dt'.
\]

with the natural scalar product

\[
(p, q)_{\mathcal{H}_{a}(G)} = \int_{[0, 1]^2} \frac{(p(t) - p(t')) \cdot (q(t) - q(t'))_G}{|t - t'|^{1 + 2a}} dt dt'.
\]

We shall abbreviate \( \mathcal{H}_{a}(R) \) in \( \mathcal{H}_{a} \). For \( h : [0, 1]^2 \rightarrow R \), we shall denote, for each \( \chi \in [0, 1] \), by \( h_{\chi} \) the partial function,

\[
h_{\chi} : [0, 1] \rightarrow R
\]

\[y \mapsto h(\chi, y).\]

We consider the space \( \mathcal{H}_{a}[0, 1] : \mathcal{H}_{a}([0, 1]) \). Its norm is defined by

\[
\| h_{\chi} - h_{\chi'} \|_{\mathcal{H}_{a}}^2 = \int_{[0, 1]^2} \frac{dx dx'}{|x - x'|^{1 + 2a}} = \| h \|_{\mathcal{H}_{a} \otimes \mathcal{H}_{a}}^2.
\]

Thus it is easily seen that

\[
\| h \|_{\mathcal{H}_{a}[0, 1] : \mathcal{H}_{a}([0, 1])}^2 = \int_{[0, 1]} A_{h}(s, s', t, t') \frac{ds ds' dt dt'}{|s - s'|^{1 + 2a} |t - t'|^{1 + 2a}}.
\]
where
\[ A_{\beta}(s, s', t, t') = (h(s, t) - h(s, t') - h(s', t) + h(s', t'))^2. \]

**Theorem A2.3.1.** For any \( a < \frac{1}{2} \) there exists a constant \( c_a \) depending only on \( n, d \) and the uniform norm of the first three derivatives of the \( \sigma \), such that:
\[ E(\|D_{t_{r_{i-1}}t}g\|_{\mathcal{L}(\mathbb{R}^n; \mathbb{R}^d\otimes \mathbb{R}^n)}^2) \leq c_a. \] (vii)

**Remark.** \( D_{t_{r_{i-1}}t}g \) is \( \mathbb{R}^n \)-valued, which explains the presence of \( \mathbb{R}^n \) in the tensor product.

**Proof.** By the above, we have
\[ D_t g = J_{r_{i-1}}(J_{r_{i-1}})^{-1}, \]
whence
\[ D_{t_{r_{i-1}}}D_t g = (D_t J_{r_{i-1}}) J_{r_{i-1}}^{-1} - J_{r_{i-1}} J_{r_{i-1}}^{-1} (D_t J_{r_{i-1}}) J_{r_{i-1}}^{-1}. \]
The finiteness of the norm in (vii) results by a direct computation from 2.1, the Burkholder–Gundy inequalities [2, Theorem 5.1] and Lemma A.1.1.

**A2.4. The Reduction Theorem**

We denote by \( V_n \) the subspace of \( L^2([0, 1]) \) generated by the \( e_k \) for \( 0 \leq k \leq n \). The Hilbert space splitting,
\[ H = V_n \oplus V_n^\perp, \]
duces the following decomposition of \( X \):
\[ X = \text{Seg}(V_n) \otimes \text{Seg}(V_n^\perp). \]
Let \( u \) be a \( G \)-valued function on \( X \), \( G \) being an Hilbert space, and let \( x \) correspond to \((x, y)\) as above; then we shall set
\[ u(x, y) = u_x(y). \]

**Lemma A2.4.1.** Let \( G \) be an Hilbert space, let \( u \) be a \( G \)-valued function on \( X \); then one has
\[ \int_{V_n} \|\nabla_x u_x\|_{L^2(\mathfrak{g}; \mathfrak{h} \oplus G)}^2 \, d\mu_x(v) = \int_X \left( \sum_{k=n+1}^{\infty} \|D_{a_k} u(x)\|_{\mathfrak{g}}^2 \right)^{p/2} \, d\mu(x). \]

**Proof.** That is a result of Cruzeiro [3, pp. 210–211].
Corollary A2.4.2. Let \( u = \nabla g \); then
\[
E(\|u_n\|^2_{H_n \otimes H}) \leq c_n n^{-2n},
\]
where \( c_n \) is the constant depending only on \( a \) and \( M \) appearing in the hypothesis (\( \mathcal{H} \)).

Proof. From Lemma A2.4.1 we get
\[
E\left( \sum_{k=n}^{\infty} \|D_k g\|^2 \right) \leq n^{-2n} \|D_{n+1} g\|^2_{H_n \otimes H}. \tag{\textit{\textbullet}}
\]

Lemma A2.4.3. Let \( Y \) be a Gaussian space, and let \( \phi \in \mathcal{W}_{-1}^2 Y \); then we have
\[
\|\phi - E(\phi)\|_{L^2(Y)} \leq \|\nabla \phi\|_{L^2(Y, H)}.
\]

Proof. Let \( C_n \) denote the \( n \)th Wiener chaos on \( Y \) (i.e., the closed subspace of \( L^2(Y) \) spanned by the Hermite polynomials of degree \( n \) in the elements of an orthonormal basis of \( H \) contained in \( Y \)). It is well known that \( L^2(Y) \) is the orthogonal direct sum of \( C_n \); let \( \phi = \sum_{n=0}^{\infty} \phi_n \) correspond to this decomposition. We can write
\[
\nabla \psi = \sum_{n=0}^{\infty} \nabla \phi_n
\]
with \( \nabla \phi_n \in C_{n-1} \otimes H \). By well-known facts concerning the Hermite polynomials, we have
\[
\forall n \in \mathbb{N}, \quad \|\nabla \phi_n\|^2_{L^2(Y, H)} = n \left\| \phi_n \right\|^2_{L^2(Y)}
\]
which implies the theorem. \( \textit{\textbullet} \)

Remark. The same inequality holds in each \( L^p \) \((p > 1)\) up to a multiplicative constant that depends on \( p \) by using the Clark–Ocone representation formula (see the proof of Corollary 4.4 above).

Theorem A2.4.4. Let \( g \) be as in Theorem 1.1, and let \( u = \nabla g \); then
\[
E(\|E^I g - g\|^2_{H}) \leq c_n n^{-2n}.
\]

Proof. Applying Corollary A2.4.2 and Lemma A2.4.3, we get
\[
E(\|E^I g - g\|^2_H) \leq E(\|u\|^2_{H_n \otimes H})
\]
\[
\leq c_n n^{-2n}. \tag{\textit{\textbullet}}
\]
REFERENCES