Isovectors for the 
Hamilton-Jacobi-Bellman Equation, 
Formal Stochastic Differentials and First 
Integrals in Euclidean Quantum Mechanics

Paul Lescot and Jean-Claude Zambrini

Abstract. The study of Riemannian stochastic differential geometry is almost 
as old as the theory of stochastic differential equations themselves. But almost 
nothing is known about stochastic symplectic geometry. Using only the 
geometrical content of the heat equation (in a sense inspired by E. Cartan) and 
its relations with quantum dynamics, we present here the first steps toward 
such a stochastic version of symplectic geometry.

Mathematics Subject Classification (2000). 60H30, 70G65.

Keywords. Hamilton-Jacobi-Bellman equation, exterior differential systems, 
diffusion processes, martingales, Lie algebra of isovectors.

1. Introduction

The purpose of this paper is to provide proofs for all the statements made in our 
Comptes Rendus note [7].

The Schrödinger equation for a particle of mass $m$ in a potential $V$ in $\mathbb{R}^n$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V \psi,$$

can be written, for $n=1$, $V=0$ and $m=1$, as:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \frac{\partial^2 \psi}{\partial q^2} \equiv H_0 \psi.$$

In Euclidean Quantum Mechanics ([3, 10, 11, 12, 13]) this equation splits 
into:

$$\frac{\partial \psi}{\partial t} = -\frac{\hbar}{2} \frac{\partial^2 \psi}{\partial q^2} \quad (C_1)$$
and
\[ \frac{\partial \psi}{\partial t} = \frac{h}{2} \frac{\partial^2 \psi}{\partial q^2}, \] (C2)

the probability density being given, not as usual, by \( \psi \tilde{\psi} \), but by \( \eta \eta \), \( \eta \) et \( \eta \), denoting respectively an everywhere strictly positive solution of \((C_1)\) and an everywhere strictly positive solution of \((C_2)\). A Bernstein process \( z \) is then naturally associated with the situation; it satisfies the stochastic differential equation:
\[ dz(t) = \sqrt{\eta} d\eta(t) + \tilde{B}(t, z(t))dt \] (B1)

relatively to the canonical increasing filtration for the Brownian \( w \), and the stochastic differential equation
\[ d_{\eta} z(t) = \sqrt{\eta} dw_{\eta}(t) + \tilde{B}_{\eta}(t, z(t))dt \] (B2)

relatively to the canonical decreasing filtration for the Brownian \( w_{\eta} \), where
\[ \tilde{B} \equiv_{def} \frac{\frac{\partial}{\partial \eta}}{\eta}, \quad \text{and} \quad \tilde{B}_{\eta} \equiv_{def} -h \frac{\partial}{\partial \eta}. \]

Images of Bernstein processes under symmetries of the heat equation can be obtained by a purely algebraic concept of stochastic quadrature: if we consider an isovector (see \( \S 2 \)) \( N \), then \( U_N = e^{\eta N} \) maps \( \eta \) to a solution \( \eta_\alpha = U_N^\alpha(\eta) \) of \((C_1)\), and \( z \) to a process \( z_\alpha \), associated with \( \eta_\alpha \), whose law \( P_\alpha \) is absolutely continuous with respect to \( P \), with density \( \frac{dP_\alpha}{dP} = h_{\alpha} = \frac{\eta_\alpha}{\eta} \); more precisely:

**Theorem 1.1 (Theorem 4 of \([7]\)).** Let \( N = -V_N \) correspond to an isovector, so that \( U_N = e^{\eta N} \) preserve positivity, and \( \eta \) be the particular solution of \((C_1)\) defining the process \( z(t), t \in [0, T] \), with law \( P \). By definition of the symmetry group of \((C_1)\), \( \eta_\alpha(q, t) = (U_N \eta)(q, t) \) solves the same equation. Then \( h_{\alpha} = \frac{\eta_\alpha}{\eta} \) is a strictly positive martingale of \( z(t) \), and, if \( z^\alpha(t) \) denotes the process associated with \( \eta_\alpha \), its law \( P_\alpha \) is absolutely continuous with respect to \( P \) with \( \frac{dP_\alpha}{dP} = h_{\alpha} \). If
\[ E(\exp\left(\frac{1}{2} \int_0^T \frac{1}{2} d\tilde{B} \right)) < +\infty, \]
then \( z^\alpha(t) \) is an \( h \)-transform of \( z(t) \), with drift
\[ \tilde{B}^\alpha(q, t) = \tilde{B}(q, t) + h \frac{\partial}{\partial q} \ln h_{\alpha}(q, t). \]

**Proof.** The only property to check is that \( h_{\alpha} \) is a (strictly positive) martingale. Now, according to \((B_1)\) and the definition of the drift \( B \), the generator of the process can be written
\[ \tilde{B} = \frac{\partial}{\partial t} + \mathcal{L}, \quad \text{with} \quad \mathcal{L} = \tilde{B} \frac{\partial}{\partial q} + \frac{1}{2} \frac{\partial^2}{\partial q^2}. \]

Equivalently, for \( f \) of \( \mathcal{C}^\infty \)-class,
\[ \tilde{D} f(q, t) = \frac{1}{\eta} \frac{\partial}{\partial t} f(q) + \frac{1}{h} \frac{\partial^2}{\partial q^2} f(q). \]

In particular
\[ \tilde{D} h_{\alpha} = \frac{1}{\eta} \frac{\partial}{\partial t} h_{\alpha} - \frac{1}{h} H_\alpha \frac{\partial}{\partial \eta} h_{\alpha} = 0 \]
by construction.

Applying this result first to \( N_4 \), and then to \(-\frac{1}{2} N_6 \) (with the notations of \( \S 2 \)), we recover, for the special case of the Brownian motion, i.e. the trivial solution \( \eta = 1 \) of \((C_1)\), firstly the scaling invariance of Brownian motion, and secondly the expression of the "Brownian bridge" in terms of Brownian motion; the details are given in \( \S 7 \).

Theorem 1.1 has a counterpart with respect to the decreasing filtration. This is the case of all the other results, formulated only for \((C_1)\), without lack of generality.

Both authors wish to thank the organizers of the Ascona Conference (May 2002) for the opportunity to lecture upon these matters. The first author heartily thanks the Grupo de Física-Matemática (Lisboa) for numerous invitations between 1999 and 2001, during which most of the work on this paper was done.

2. The complete Lie algebra of isovectors for the Hamilton-Jacobi-Bellman equation

We shall make explicit, in a slightly more general context, the result of calculations analogous (and in fact formally equivalent) to those contained in [6], pp. 657–659.

Let us consider the heat equation \((C_1)\) (for \( h \neq 0 \)):
\[ \frac{\partial \psi}{\partial t} = -h \frac{\partial^2 \psi}{\partial q^2}. \]

Setting \( S = -h \ln(\psi) \), one obtains the Hamilton-Jacobi-Bellman equation:
\[ \frac{\partial S}{\partial t} = -h \frac{\partial^2 S}{\partial q^2} + \frac{1}{2} \left( \frac{\partial S}{\partial q} \right)^2. \]

In order to deal with differential forms, we have to reduce that equation to a system of first order equations. For that purpose, let \( E = -\frac{\partial S}{\partial q} \) (formal energy) and \( B = -\frac{\partial S}{\partial q} \) (formal momentum; cf. the above definition of \( \tilde{B} \) ); then any solution of \((2)\) will annul the following differential forms on \( M = \mathbb{R}^5 \):
\[ \omega = dS + Edt + Bdq, \]
(by definition of \( B \) and \( E \));
\[ d\omega = dEdt + dBdq, \]
(by Schwarz' Theorem) and
\[ \beta = (E + \frac{1}{2} B^2)dq dt + \frac{h}{2} dBr \]
(because of \((2)\)).
3. The structure of the Lie algebra $\mathcal{G}$

With the notations of §2, an easy computation shows that:

$$[N_1, N_2] = 0, \quad [N_1, N_3] = 0, \quad [N_1, N_4] = N_1, \quad [N_1, N_5] = -2N_2,$$

$$[N_1, N_6] = 2N_4 - N_3, \quad [N_1, N_6'] = N_g, \quad (as \, g \, is \, a \, solution \, of \, (1), \, so \, is \, g_4),$$

$$[N_2, N_4] = 0, \quad [N_2, N_4] = N_2, \quad [N_2, N_5] = -\frac{2}{3}N_3,$$

$$[N_2, N_6] = -N_5, \quad [N_2, N_6'] = N_g, \quad (as \, g \, satisfies \, (1), \, so \, does \, g_4),$$

$$[N_3, N_4] = 0, \quad [N_3, N_4] = 0, \quad [N_3, N_6] = 0, \quad [N_3, N_6'] = -N_6 = N_g,$$

$$[N_4, N_5] = N_6, \quad [N_4, N_6] = 2N_6, \quad [N_4, N_6'] = N_g = N_g + 2qg + 2tg, \quad (2tg + qg_4 \, satisfies \, (1),$$

$$[N_5, N_6] = 0, \quad [N_5, N_6] = N_g, \quad [N_5, N_6'] = -2tg + 2qg, \quad [N_5, N_6'] = 0.$$  

Let $\mathcal{H}$ denote the subspace of $\mathcal{G}$ generated by $N_1, \ldots, N_6$, $\mathcal{J}$ the subspace of the $N_5$ ($g$ solution of (1)), $\mathcal{H}_1$ the subspace generated by $N_3$ and $\mathcal{H}_0$ the subspace generated by $N_2, N_3$ and $N_5$. Then it follows from the commutation relations that:

$\mathcal{H}$ is a subalgebra of $\mathcal{G}$, 

$\mathcal{J}$ is an abelian subalgebra of $\mathcal{G}$ and $[\mathcal{H}, \mathcal{J}] \subseteq \mathcal{J}$.

As, obviously, $\mathcal{G} = \mathcal{H} \oplus \mathcal{J}$, we have: $[\mathcal{G}, \mathcal{J}] \subseteq \mathcal{J}$, 

$\mathcal{G}$ therefore appears as the semi-direct product of a 6-dimensional subalgebra $\mathcal{H}$ and an infinite-dimensional abelian ideal $\mathcal{J}$.

It is easy to analyze the structure of $\mathcal{H}$: the commutation relations yield that:

$\mathcal{H}_1 = \mathbb{R} N_3 \subset Z(\mathcal{H})$,

and that

$\mathcal{H}_0$ is an ideal of $\mathcal{H}$.

Let $\tilde{h}$ denote the class modulo $\mathcal{H}_1$ of $h \in \mathcal{H}$; then:

$$[\tilde{N}_1, \tilde{N}_2] = 0, \quad [\tilde{N}_1, \tilde{N}_3] = 2\tilde{N}_3, \quad [\tilde{N}_1, \tilde{N}_4] = -2\tilde{N}_2, \quad [\tilde{N}_1, \tilde{N}_6] = 2\tilde{N}_4,$$

$$[\tilde{N}_2, \tilde{N}_4] = \tilde{N}_2, \quad [\tilde{N}_2, \tilde{N}_6] = 0, \quad [\tilde{N}_3, \tilde{N}_6] = -\tilde{N}_3,$$

$$[\tilde{N}_4, \tilde{N}_6] = \tilde{N}_5, \quad [\tilde{N}_4, \tilde{N}_6'] = 2\tilde{N}_6, \quad [\tilde{N}_5, \tilde{N}_6] = 0.$$  

In the natural semi-direct product $sl_2(\mathbb{R}) \times \mathbb{R}^2$, let:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = (1, 0), \quad y = (0, 1).$$

Then, as is well known:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h, \quad [h, x] = h(x) = x,$$

$$[h, y] = h(y) = y, \quad [e, x] = e(x) = 0, \quad [e, y] = e(y) = x,$$

$$[f, x] = f(x) = y, \quad [f, y] = f(y) = 0, \quad [x, y] = 0.$$
Therefore the application:
\[ N_1 \mapsto -e, N_2 \mapsto x, N_3 \mapsto -h, N_4 \mapsto 2y, N_5 \mapsto 2f \]
induces an isomorphism of Lie algebras:
\[ \mathcal{H}/\mathcal{H}_0 \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathbb{R}^2. \]
In this isomorphism, \( \mathfrak{h} \) corresponds to \( <x, 2y> = \mathbb{R}^2 \), whence:
\[ \frac{\mathcal{H}}{\mathcal{H}_0} \cong \mathfrak{sl}_2(\mathbb{R}), \text{ i.e. } \frac{\mathcal{H}}{\mathcal{H}_0} \cong \mathfrak{sl}_2(\mathbb{R}). \]
As
\[ [N_2, N_3] = 0, [N_2, N_5] = -\frac{2}{\hbar} N_3, [N_3, N_5] = 0, \]
onc sees that \( \mathcal{H}_0 \) is a three-dimensional Heisenberg algebra having for its center the subspace \( H_1 = \mathbb{R} N_3 \) generated by \( N_3 \). But
\[ \frac{\mathcal{H}}{\mathcal{H}_0} \cong \mathfrak{sl}_2(\mathbb{R}) \]
has trivial center, hence \( Z(\mathcal{H}) \subset \mathcal{H}_0 \), whence \( Z(\mathcal{H}) \subset Z(\mathcal{H}_0) = H_1 \subset Z(\mathcal{H}) \), and:
\[ Z(\mathcal{H}) = Z(\mathcal{H}_0) = H_1. \]
We set:
\[ \mathcal{H}_2 = Vect(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4); \]
this is a subalgebra of \( \mathcal{H} \), hence of \( G \).

4. The bilinear form
For \( N \in \mathcal{H} \), we shall set:
\[ \Phi_N = -N^S = k_0(q^2 - \hbar t) - 2k_0q + k_3h. \]  (8)
This \( \Phi_N \) coincides with the phase defined in [10], pp. 317–318. It is visible that \( \Phi_N \) depends only upon \( q \) and \( t \). We also set:
\[ n_N = N^t E + N^y B - \Phi_N. \]  (9)

**Definition 4.1.** For any \((\delta, \delta') \in (T^1(M)^*)^2\) (the space of pairs of \( C^\infty \)-vector fields on \( M = R^7 \)), let:
\[ \Omega(\delta, \delta') = (\delta(B)\delta'(q) - \delta(q)\delta'(B)) + (\delta(E)\delta'(t) - \delta(t)\delta'(E)). \]

It is clear that \( \Omega \) is a bilinear and antisymmetric mapping from \((T^1(M)^*)^2\) to \( C^\infty(M) \).

The key to our subsequent computations is:

**Proposition 4.2.** For arbitrary \( N \in \mathcal{H} \) and \( \delta \in T^1(M)^* \), one has:
\[ \Omega(N, \delta) = n_N. \]

**Proof.** By Definition 4.1 and the definition (9) of \( n_N \), one has, with the notations of §3:
\[ \Omega(N, \delta) + \delta(n_N) = N(B)\delta(q) - N(q)\delta(B) + N(E)\delta(t) - N(t)\delta(E) \]
\[ + \delta(EN^t + B\dot{N} - \Phi_N) \]
\[ = N(B)\delta(q) - N(q)\delta(B) + N(E)\delta(t) - N(t)\delta(E) \]
\[ + \delta(E)N^t + B\delta(t) + B\delta(N^t) - \delta(\Phi_N) \]
\[ = N^B\delta(q) + N^B\delta(t) + E\delta(N^t) + B\delta(N^t) - \delta(\Phi_N) \]
\[ = -k_3B\delta(q) - 2k_3\delta(q) + 2k_3\delta(q) - 2k_3t\delta(q) - 2k_3qB\delta(t) \]
\[ -4k_3tE\delta(t) - 2k_3t\delta(t) + 2k_3B\delta(t) - 2k_3E\delta(t) + E\delta(N^t) \]
\[ + B\delta(N^t) - \delta(\Phi_N) \]
\[ = E\delta(k_3) + B\delta(k_3) + \delta(-k_3) \]
\[ = 0. \]
\[ \square \]

**Lemma 4.3.** For each \( N \in \mathcal{H} \), one has:
\[ (i) \quad N^S = -B \frac{dN^S}{dt} - E \frac{dN^t}{dt} - \frac{\hbar}{4} \frac{d^2N^t}{dt^2} \quad \text{and} \quad N^B = -B \frac{dN^t}{dt} + \frac{dN^S}{dt}. \]
\[ (ii) \quad \frac{d^2N^S}{dt^2} = 2 \frac{dN^S}{dt}. \]

**Proof.** This is apparent from formulas (3), (4), (5) and (6).
\[ \square \]

5. Sectioning

**Definition 5.1.** We shall call
\[ \omega_{PC} = Edt + Bdq = \omega - dS \]
the Poincaré-Cartan form, and
\[ L = \frac{B^2}{2} \]
the formal free Lagrangian.
Proposition 5.2. Let \( \eta \) be a given, everywhere > 0, solution of (1). Then there exists a unique morphism of graded differential algebras \( \theta_\eta : A \rightarrow B = \wedge T^*(\mathbb{R}^2) \) such that, denoting by \((t,q)\) the generic point of \( \mathbb{R}^2 \), one has:
\[
\theta_\eta(t) = t, \quad \theta_\eta(q) = q, \quad \theta_\eta(S) = -h \ln(\eta),
\]
\[
\theta_\eta(E) = \frac{h}{\eta} \frac{\partial \eta}{\partial t}, \quad \theta_\eta(B) = \frac{h}{\eta} \frac{\partial \eta}{\partial q}.
\]
In addition \( I \subseteq \ker(\theta_\eta) \), whence \( \theta_\eta \) induces a mapping \( \tilde{\theta}_\eta : \frac{1}{h} \rightarrow B \).

Proof. The existence and uniqueness of \( \theta_\eta \) follows from the usual universal properties of the exterior algebra. In order to show that \( \mathcal{I} \subseteq \ker(\theta_\eta) \), it is enough to see that \( \omega \) and \( \beta \) belong to \( \ker(\theta_\eta) \). For \( \omega \) it is obvious, and for \( \beta \) it follows from the equivalence of (1) and (2), and the definitions of \( \tilde{E} \) and \( \tilde{B} \). \( \Box \)

We shall call \( \theta_\eta \) the section map relative to \( \eta \).

From now on, we shall fix a solution \( \eta > 0 \) of (1), and denote \( \tilde{S} = \theta_\eta(S), \tilde{E} = \theta_\eta(E) \) and \( \tilde{B} = \theta_\eta(B) \).

Corollary 5.3.

\[
\theta_\eta(\omega_{PC}) = \tilde{B} dq + \tilde{E} dt.
\]

Proof. One has, by the definitions of \( \omega_{PC} \) and \( \theta_\eta \):
\[
\theta_\eta(\omega_{PC}) = \theta_\eta(\omega - dS) = \theta_\eta(\omega) - \theta_\eta(dS) = \theta_\eta(\omega) - d(\theta_\eta(S)).
\]
But \( \theta_\eta(\omega) = 0 \) by the proof of Proposition 5.2, whence:
\[
\theta_\eta(\omega_{PC}) = -d(\theta_\eta(S)) = -d(-h \ln(\eta)) = \frac{h}{\eta} \frac{\partial \eta}{\partial t} = \tilde{B} dq + \tilde{E} dt. \quad \Box
\]

Lemma 5.4.

(i) For each \( N \in \mathcal{H} \), \( \theta_\eta(n_N) = h \frac{\ln(N)}{\eta} \), where
\[
\mathcal{V}_N \equiv \frac{1}{h} \frac{N^q}{\partial t} + N^q \frac{\partial}{\partial q} - \frac{1}{h} \Phi_N \in T^*(\mathbb{R}^2)
\]
acts on functions of \((q,t)\).

(ii) For each \( N \in \mathcal{H}_2 \), \( \theta_\eta(N(B)) = \delta_N(\tilde{B}) \) and \( \theta_\eta(N(E)) = \delta_N(\tilde{E}) \), where \( \delta_N \) denotes the variation associated with \( N \) in the sense of [13], §6, formula (60).

Proof. (i) One has:
\[
\theta_\eta(n_N) = \theta_\eta(N^qE + N^qB - \Phi_N) = \theta_\eta(N^q)\theta_\eta(E) + \theta_\eta(N^q)\theta_\eta(B) - \theta_\eta(\Phi_N)
\]
\[
= N^q\tilde{E} + N^q\tilde{B} - \Phi_N.
\]

Lemma 5.5. For \( N \in \mathcal{H}_2 \) and \( N' \in \mathcal{H}_2 \), \( \Omega_\eta(N,N') \) coincides with the bilinear form \( \Omega_\eta(\delta_N, \delta_{N'}) \) as defined in [13], Definition 3, p. 402.

Proof. One has:
\[
\Omega_\eta(N,N') = \theta_\eta(\Omega(N,N'))
\]
\[
= \theta_\eta(\Omega(N'(q) - N(q)N'(B)) + (N'(E)N'(t) - N(t)N'(E))
\]
(by Definition 4.1)
\[
= (\theta_\eta(N')\theta_\eta(N^q) - \theta_\eta(N^q)\theta_\eta(N') + (\theta_\eta(N'^q)\theta_\eta(N') - \theta_\eta(N'^q))\theta_\eta(N'E).
\]

But \( N' \) depends only upon \( q \) and \( t \), therefore \( \theta_\eta(N') = N^t = \delta_N(t) \), and similarly \( \theta_\eta(N^q) = N^q = \delta_N(q) \). By Lemma 5.4(ii) we have \( \theta_\eta(N'(B)) = \delta_N(\tilde{B}) \) and \( \theta_\eta(N'(E)) = \delta_N(\tilde{E}) \), and similarly for \( N' \), whence:
\[
\Omega_\eta(N,N') = (\delta_N(\tilde{B}) \delta_{N'}(q) - \delta_N(q) \delta_{N'}(\tilde{B})) + (\delta_N(\tilde{E}) \delta_{N'}(t) - \delta_N(t) \delta_{N'}(\tilde{E})) = 
\]
\[
\Omega_\eta(\delta_N, \delta_{N'}). \quad \Box
\]

Our main result connects the canonical stochastic differential 2-form and the Lie algebra structure of \( \mathcal{G} \):

Theorem 5.6. For \( (N', N') \in \mathcal{H}_2 \), one has:
\[
\omega_\eta(N,N') = h \frac{\mathcal{V}_N N'}{\eta}.
\]
Proof. One has:

\[ [N, N']^t = [N, N'](t) = (NN' - N'N)(t) \]
\[ = N(N'(t)) - N'(N(t)) = N(N'^t) - N'(N^t) \]

and similarly

\[ [N, N']^q = N(N'^q) - N'(N^q) \quad \text{and} \quad [N, N']^S = N(N'^S) - N'(N^S), \]

that is:

\[ \Phi_{[N, N']} = N(\Phi_{N'}) - N'(\Phi_N). \]

But, by Proposition 4.2, one may write:

\[ \Omega(N, N') = -hN'(n_N), \quad \text{where} \quad n_N = N^1E + N^qB - \Phi_N. \]

Therefore:

\[ \Omega(N, N') = N'(\Phi_N - N^1E - N^qB) \]
\[ = N'(\Phi_N) - N'(N^1E) - N'^t(N^tE) - N'^q(N^qE) - N'^qN'(B) \]
\[ = N'(\Phi_N) - N'(N^1E) - N'^t(-E\frac{dN'^t}{dt} - \frac{h}{4}\frac{d^2N'^t}{dt^2}) - N'^qB - N'^q(-B\frac{dN'^q}{dt} + \frac{\partial N'^q}{\partial t}), \]

according to Lemma 4.3(i) applied to \( N' \).

Whence:

\[ 2\Omega(N, N') = \Omega(N, N') - \Omega(N', N) \]
\[ = N'(\Phi_N) - N(N') - N'(N^1E + EN'(N^1) + EN'(N^1)) \]
\[ + E(N'\frac{dN'^t}{dt} - \frac{h}{4}\frac{d^2N'^t}{dt^2}) + B(N\frac{\partial N'^q}{\partial t} - N'^t\frac{\partial N'^q}{\partial t}) + \frac{h}{4}(N'^q\frac{d^2N'^q}{dt^2} - N'^t\frac{d^2N'^q}{dt^2}) \]
\[ - B(N'^t\frac{\partial N'^q}{\partial t} + N'^q\frac{\partial N'^q}{\partial t}) + B(N'\frac{\partial N'^q}{\partial t} + N'^q\frac{\partial N'^q}{\partial t}) \]
\[ + \frac{B}{2}(N'^q\frac{dN'^q}{dt} - N'^q\frac{dN'^q}{dt} - N'^q\frac{\partial N'^q}{\partial t} + N'^q\frac{\partial N'^q}{\partial t}) \]
\[ = N'(\Phi_N) - N(N') + 2E[N, N']^t + \frac{h}{4}\frac{d[L, N']^t}{dt} \]
\[ + 2BN'^q - 2BN'(N^q) + (N'^q\frac{\partial N'^q}{\partial t} - N'^q\frac{\partial N'^q}{\partial t}) \]
\[ + \frac{B}{2}(N'^q\frac{dN'^q}{dt} - N'^q\frac{dN'^q}{dt} + B(N'\frac{\partial N'^q}{\partial t} - N'^q\frac{\partial N'^q}{\partial t}), \]

\[ = N'(\Phi_N) - N(N') + 2E[N, N']^t + 2B[N, N']^q + \frac{h}{4}\frac{d[L, N']^t}{dt} \]
\[ + (N'^q\frac{\partial N'^q}{\partial t} - N'^q\frac{\partial N'^q}{\partial t}), \]

where we have used, once more, that \( N' \) and \( N'^t \) depend only upon \( t \), and \( N'^q \) and \( N'^q \) only upon \( q \) and \( t \), and also Lemma 4.3(ii), applied to \( N \) and \( N' \). But:

\[ \frac{\partial N'^q}{\partial t} = \frac{\partial \Phi_N}{\partial q}, \quad \text{and} \quad \frac{d^2N'^t}{dt^2} = \frac{4}{h}\frac{\partial \Phi_N}{\partial t} \]

(this follows from (3), (4) and (8), and also from [10], Lemma 3.5, p. 318), whence:

\[ \frac{d[L, N']^t}{dt} = N'^t\frac{\partial N'^t}{\partial t} - N'^q\frac{\partial N'^q}{\partial t} \]
\[ = N'^t\left(\frac{4}{h}\frac{\partial \Phi_N}{\partial t}\right) - N'^q\left(-\frac{4}{h}\frac{\partial \Phi_N}{\partial t}\right) \]

and

\[ \frac{\partial N^q}{\partial t} = \frac{\partial N^q}{\partial q} - \frac{\partial \Phi_{N'}}{\partial q}, \]

from which follows:

\[ \frac{h}{4}\frac{d[L, N']^t}{dt} + (N'^q\frac{\partial N'^q}{\partial t} - N'^q\frac{\partial N'^q}{\partial t}) \]
\[ = -N'^q\frac{\partial \Phi_N}{\partial t} + N'^t\frac{\partial \Phi_N}{\partial t} + N'^q\frac{\partial \Phi_{N'}}{\partial q} - N'^q\frac{\partial \Phi_{N'}}{\partial q} \]
\[ = N'(\Phi_N) - N(\Phi_{N'}), \]

as \( \Phi_N \) and \( \Phi_{N'} \) depend only upon \( q \) and \( t \).

Therefore:

\[ 2\Omega(N, N') = 2(N'(\Phi_N) - N(\Phi_{N'})) + 2E[N, N']^t + 2B[N, N']^q \]
\[ = -2\Phi_{[N, N']} + 2E[N, N']^t + 2B[N, N']^q, \]

that is:

\[ \Omega(N, N') = [N, N']^tE + [N, N']^qB - \Phi_{[N, N']} \]

and:

\[ \Omega_{\eta}(N, N') = \theta_{\eta}(\Omega(N, N')) \]
\[ = \theta_{\eta}([N, N']^tE + [N, N']^qB - \Phi_{[N, N']}) \]
\[ = [N, N']^t\frac{\partial \Phi_N}{\partial t} + [N, N']^qB - \Phi_{[N, N']} \]
\[ = \frac{h}{\eta}\left(([N, N']^t\frac{\partial \Phi_N}{\partial t} + [N, N']^q\frac{\partial \Phi_{N'}}{\partial q} - \Phi_{[N, N']})\right) \]
\[ = \frac{\eta}{h}[N, N']^t\eta. \]

\[ \square \]
Corollary 5.7. For \((N, N') \in \mathcal{H}_2 \times \mathcal{H}_2\), one has:
\[
\Omega_{\eta}(\delta N, \delta N') = \frac{h}{\eta} [N, N']\eta.
\]

Proof. By the proof of Theorem 5.6, one has:
\[
\Phi_{[N,N']} = N(\Phi_{N'}) - N'(\Phi_N) = 0,
\]
as \Phi_N and \Phi_{N'} are constant (this can also be deduced from the commutation relations in §3, which imply that \([N, N'] \in Vect(N_1, N_2, N_4)\)). Therefore:
\[
\begin{align*}
\nabla_{[N,N']}\eta &= [N, N'] \frac{\partial}{\partial t} + [N, N'] \frac{\partial}{\partial q} - \Phi_{[N,N']} \\
&= [N, N'] \frac{\partial}{\partial t} + [N, N'] \frac{\partial}{\partial q} \\
&= [N, N']\eta,
\end{align*}
\]
as \eta depends only upon \(q\) and \(t\). The result now follows from Theorem 5.6. \(\square\)

Remark 5.8. The above Corollary is essentially equivalent to the formula used in the proof of Theorem 1, p. 402 in [13].

Theorem 5.9. The differential 2-form \(\Omega_{\eta}\) is closed, i.e. \(d\Omega_{\eta} = 0\).

Proof. As is well known ([4]), the 3-form \(d\Omega_{\eta}\) is defined by the following formula, for all \((N, N', N'') \in \mathcal{H}^3\):
\[
d\Omega_{\eta}(N, N', N'') = N(\Omega_{\eta}(N', N'')) - N'(\Omega_{\eta}(N, N'')) + N''(\Omega_{\eta}(N, N')).
\]

According to Theorem 5.6, the last three terms add up to:
\[
\frac{h}{\eta} [\nabla_{[N,N'],[N',N'']}\eta + \nabla_{[N',N''],[N,N']}\eta - \nabla_{[N',N'],[N,N'']}\eta]
\]
\[
= \frac{h}{\eta} \nabla_{[N,N',N''] + [N',N'',N] + [N'',N,N']\eta}
\]
\[
= \frac{h}{\eta} \nabla_{\eta} \eta \text{ (by the Jacobi identity)}
\]
\[
= 0.
\]

Therefore, by the antisymmetry of \(\Omega_{\eta}\), one finds:
\[
d\Omega_{\eta}(N, N', N'') = N(\Omega_{\eta}(N', N'')) - N'(\Omega_{\eta}(N, N'')) + N''(\Omega_{\eta}(N, N')).
\]
But this is easily computed to be zero. \(\square\)

Corollaries 5.3 and 5.7 and Theorem 5.9 together yield Theorem 1 of [7], modulo the correction of a misprint there: in the first line of “Théorème 1”, one has to read “\(\theta_{\eta}(\omega_{PC})\)” instead of “\(\theta_{\tilde{\eta}}(\omega)\)”.

6. An algebraic version of the Itô differential

The following two results provide, in a purely algebraic way, stochastic analogue of some classical formulas of Analytical Mechanics. Let us denote by \(D\) the formal total derivative along the Bernstein process associated with \(\eta\) (cf. the introduction and [10, 11]):
\[
D = \frac{\partial}{\partial t} + B \frac{\partial}{\partial q} + \frac{h}{2} \frac{\partial^2}{\partial q^2}.
\]

Theorem 6.1. For each \(N \in \mathcal{H}\), \(\mathcal{L}_N(\omega_{PC}) = d\Phi_N\).

Proof. By the well-known formal properties of the Lie derivative (see for instance [6], p. 654), one has:
\[
\mathcal{L}_N(\omega_{PC}) = \mathcal{L}_N (E dt + B dq)
\]
\[
= \mathcal{L}_N (N dt) + \mathcal{L}_N (B dq) + B \mathcal{L}_N (dq)
\]
\[
= N dt + E dt \mathcal{L}_N (t) + N dq + B (dt \mathcal{L}_N (q))
\]
\[
= N dt + E dt N' + N dq + B dN.
\]

But, using formulas (3), (4), (5) and (6), this is easily seen to equal:
\[
-h dt - 2k_0 dq + 2k_0 dq = d(k_0 q^2 - 2k_0 q - k_0 t) dt = \Phi_N.
\]

We have established yet another coincidence between our results and constructions using Itô’s formula in [10].

Theorem 6.2. For each \(N \in \mathcal{H}\),
\[
\mathcal{L}_N (L) + L \frac{dN}{dt} = D\Phi_N.
\]

Proof. One has:
\[
\mathcal{L}_N (L) = N(L) = N \left( \frac{1}{2} B^2 \right) = BN(B) = BN = B(-k_4 B - 2k_0 + 2k_0 q - 2k_0 t B)
\]
\[
= -k_4 B^2 - 2k_0 B + 2k_0 q B - 2k_0 B - 2k_0 t B^2 = -B^2 \left( 2k_4 + 4k_0 t \right) + 2k_0 s B - 2k_0,
\]
\[
= -B^2 \frac{d}{dt} (2k_0 t + k_1) - k_0 h + B (2k_0 q - 2k_0) + \frac{h}{2} (2k_0)
\]
\[
= -L \frac{dN}{dt} + \frac{d\Phi_N}{dt} + B \frac{\partial}{\partial q} \frac{\partial}{\partial q} + \frac{h}{2} \frac{\partial^2}{\partial q^2} = -L \frac{dN}{dt} + D\Phi_N.
\]

By Theorems 6.1 and 6.2 are justified all the assertions made in [7], Theorems 2 and 3, modulo a misprint in “Théorème 3”, instead of “\(B^2\)”, read “\(B^2\)”.

The probabilistic interpretation of all our results is founded on the observation that
\[
\mathcal{S}(q,t) = E_{\eta,t}(\int_t^T \omega_{PC})
\]
where \( E_{q,t} \) denotes the conditional expectation given that \( z(t) = q \), and \( z \) is the Bernstein process of the Introduction. Using Definition 5.1 we have, more explicitly, under integrability condition

\[
\tilde{S}(q,t) = E_{q,t}(\int_t^T L(Dz(\tau))d\tau) = E_{q,t}(\int_t^T \frac{1}{2} \tilde{B}^2(z(\tau),\tau)d\tau).
\]

Of course, \( \omega_{PC} \) when evaluated along the paths of \( z(\tau) \), has to be interpreted in the Stratonovich sense. Theorem 6.2, for instance, is a stochastic condition of invariance of the Lagrangian \( L \) under the symmetry group built from the one of \((\mathcal{C}_t)\). It was originally found [10, 11] at the expense of long sessions of Itô calculus, both in the flat and in the Riemannian case [12].

In conclusion, let us observe that Euclidean Quantum Mechanics is in no way restricted to diffusion processes [14]. So the geometrical methods introduced here should hold more generally.

7. Two interesting particular cases

7.1. Case 1: Scaled Brownian motion

Let us consider the generator \( N_q \) (cf. (7)), let \( \alpha \in \mathbb{R} \), let \( U_\alpha = e^{\alpha N_q} \) and let \( \eta \) be a solution of (1). Then, as

\[
\tilde{N}_t = -\mathcal{V}_N = -2t \frac{\partial}{\partial t} - q \frac{\partial}{\partial q},
\]

it is readily seen that \( U_\alpha^{N_q} = e^{\alpha \tilde{N}_t} \) maps \( \eta(q,t) \) to \( \eta(e^{-\alpha}q, e^{-2\alpha}t) \).

Moreover, \( e^{\alpha N_q} \) maps \((T,Q)\) to \((e^{2\alpha T}, e^{\alpha Q})\), whence

\[
z^\alpha(t) = e^\alpha z(e^{-2\alpha}t).
\]

In the case \( \eta = 1 \), we find \( \eta_t = 1 \). Applying (B1) to \( \eta \) and \( w \), we find that

\[
z(t) = \sqrt{\alpha}w(t);
\]

setting now \( e^{-2\alpha} = \epsilon \), it appears that, whenever \( w \) is a Brownian motion,

\[
w'(t) \equiv_{df} e^{-\frac{1}{2}}w(\epsilon t)
\]

is also one, for any \( \epsilon > 0 \). Thus have we (re)established the scaling invariance of the Brownian motion, as claimed in [7], p. 266, and in §2 above.

As a matter of fact, \( N_q \) provides the complete collection of Bernstein processes satisfying such a property. Of course for \( \eta \neq 1 \), we obtain a transformation of the starting diffusion and not an invariance.

7.2. Case 2: Brownian bridge

Let us here set \( \eta = 1 \) and

\[
N = -\frac{1}{2}N_q
\]

whence

\[
\tilde{N} = -\frac{1}{2}N_q = \frac{1}{2}Y_{N_q} = t^2 \frac{\partial}{\partial t} - q^2 \frac{\partial}{\partial q} - \frac{1}{2h}(q^2 - ht).
\]

Let \( \alpha \in \mathbb{R} \); then \( U_\alpha^{N_q} \) maps \( \eta(q,t) \) to

\[
\frac{1}{\sqrt{1 - \alpha t}} e^{-\frac{\alpha q^2}{2h(1 - \alpha t)}} = (1 - \alpha t)z(1 - \alpha t).
\]

Also, \( e^{\alpha N_q} \) maps \((T,Q)\) to \((\frac{T}{1 + \alpha T}, \frac{Q}{1 + \alpha T})\) whence, in particular:

\[
z^\alpha(t) = (1 - \alpha t)z(1 - \alpha t).
\]

In the case \( \alpha = 1 \), one has:

\[
h_\alpha = \frac{\eta_\alpha}{\eta} = \eta_\alpha = U_\alpha^{N_q} (1) = \frac{1}{\sqrt{1 - \alpha t}} e^{-\frac{\alpha q^2}{2h(1 - \alpha t)}},
\]

and the drift associated with \( z^\alpha \) is:

\[
\tilde{B}^\alpha = h \frac{\partial}{\partial q} (\ln \eta_\alpha) = h \frac{\partial}{\partial q} (-\frac{1}{2} \ln(1 - \alpha t) - \frac{\alpha q^2}{2h(1 - \alpha t)}) = -\frac{\alpha q}{1 - \alpha t},
\]

whence \( z^\alpha \) satisfies the stochastic differential equation:

\[
\frac{dz^\alpha(t)}{dt} = \sqrt{\alpha}dw(t) - \frac{2\alpha^2(t)}{1 - \alpha t} dt,
\]

and (for \( \alpha > 0 \)), this defines the usual Brownian bridge, i.e. the standard Brownian motion \( w(t) \) conditioned by \( w(\alpha^{-1}) = 0 \), often denoted by \( w^{0,\alpha^{-1}}_0(t) \).

References

Stochastic Methods in Financial Models