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Perturbations of Generalized Mehler Semigroups and Applications to Stochastic Heat Equations with Levy Noise and Singular Drift

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Abstract. In this paper we solve the Kolmogorov equation and, as a consequence, the martingale problem corresponding to a stochastic differential equation of type

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + b(X_t)\,\mathrm{d}t + \mathrm{d}Y_t,$$

on a Hilbert space E, where $(Y_t)_{t\geqslant 0}$ is a Levy process on E, A generates a C_0-semigroup on E and $b : E \to E$. Our main point is to allow unbounded A and also singular (in particular, non-continuous) b. Our approach is based on perturbation theory of C_0-semigroups, which we apply to generalized Mehler semigroups considered on $L^2(\mu)$, where μ is their respective invariant measure. We apply our results, in particular, to stochastic heat equations with Levy noise and singular drift.

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1. Introduction

In this paper we study stochastic differential equations on a Hilbert space E of type

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + b(X_t)\,\mathrm{d}t + \mathrm{d}Y_t, \tag{1.1}$$

where $(Y_t)_{t\geqslant 0}$ is a Levy process on E, A generates a C_0-semigroup on E and $b : E \to E$. Our main point is to allow at the same time A to be unbounded and also singular (in particular, non-continuous) b. Therefore, we can only aim for weak solutions to (1.1). More precisely, we shall construct a strong (cadlag) Markov process solving (1.1) in the sense of a martingale problem (cf. [22] and Section 7 below).

There is quite a lot of literature on equation (1.1) in the finite-dimensional case (i.e. $\dim E < \infty$) with much more general types of processes in the noise part (cf., e.g., [1, 10, 11, 13, 21] and references therein). If $\dim E = \infty$ there are only a few papers on (1.1). We refer to [7] for the linear case $b \equiv 0$, [2] for the case with Lipschitz coefficients and Poisson noise (allowing also a non-constant diffusion coefficient) and [16, 17] for the case $b \equiv 0$ and a special non-constant coefficient in front of the noise. Already in the linear case (as also pointed out in [7, Section 8]) the infinite dimensional situation is drastically different from the finite-dimensional one. For example, as is well known, in particular in the case $(Y_t)_{t \geqslant 0}$ has a jump part, Girsanov's formula for solving (1.1) weakly does not work in general.

So, we will follow the classical approach of Kolmogorov and try to construct directly the transition semigroup of the solution to (1.1) by solving the (backward) Kolmogorov equations on $L^2(E; \mu)$ for some suitable probability measure μ on E. Since we have done a complete analysis of the case $b \equiv 0$ in [14], we will proceed by perturbation theory on $L^2(E; \mu)$, where μ is the invariant measure for the solution of the linear equation. We emphasize that this is not a straightforward modification of finite-dimensional or "Gaussian" techniques since only the Fourier transform of the reference measure μ is known and μ is "far away" from a Gaussian measure. In particular, it does not satisfy an integration by parts formula in general.

The two key ingredients to implement perturbation theory nevertheless are the following:

(1) the maximal dissipativity of the explicitely given infinitesimal generator L of the linear equation on a suitable space W of smooth cylinder functions on E. (This maximal dissipativity is one of the main results in [14] and will be summarized in Section 3 below.)

(2) an explicit formula for the square field operator given by L (cf. Sections 3 and 4, in particular Proposition 4.1 below) which immediately implies that for all $u \in W$

$$\int_E {}_{E'}\big\langle u'(x), R(u'(x))\big\rangle_E \mu(\mathrm{d}x) \leqslant -2(u, Lu)_{L^2(E;\mu)} \tag{1.2}$$

(where R is the symmetric operator associated with the quadratic form appearing in the Levy–Khinchin representation for the negative definite functional determining the Levy process $(Y_t)_{t \geqslant 0}$).

Both the above ingredients together by a standard result (e.g., from [19]) imply that the candiate for the inifinitesimal generator of (1.1) given by

$$L^b u := Lu + {}_{E'}\big\langle (u'(x), b(x)\big\rangle_E, \quad u \in W,$$

is dissipative on $L^2(E; \mu)$ and its closure $\overline{L^b}$ generates a C_0-semigroup $(P_t^b)_{t \geqslant 0}$ on $L^2(E; \mu)$ (cf. Section 5 below where also the assumptions on b are specified). In particular, $u(t, x) := P_t^b f(x)$ solves the Kolmogorov equation (or Cauchy problem)

$$\frac{\mathrm{d}}{\mathrm{d}t} u = \overline{L^b} u, \qquad u(0, \cdot) = f \in L^2(E; \mu).$$

We stress at this point that L is in general not sectorial on $L^2(E;\mu)$. So, perturbation theory in terms of sectorial forms is not applicable and we have to work with the operators directly.

We recall that if $b \equiv 0$, then $P_t^b = P_t^0 =: P_t,\ t \geqslant 0$, has an explicit form and is a so-called generalized Mehler semigroup. Generalized Mehler semigroups have been studied in detail in [5, 6, 9, 12, 14], and more recently in [8].

In order to show that $(P_t^b)_{t\geqslant 0}$ (as in the case $b \equiv 0$) really gives the transition probabilities of the solution to (1.1) one first has to prove that it is Markov, i.e. it is positivity preserving and satisfies $P_t^b 1 = 1$ for all $t > 0$. Because of our general drifts b, it seems impossible to check any version of a maximum principle for L^b (which would also imply the Markov property of $(P_t^b)_{t\geqslant 0}$), but instead we have to proceed via characterizations of the positivity preserving property of $(P_t^b)_{t\geqslant 0}$ (as $P_t^b 1 = 1$ for all $t > 0$ is obvious) through properties of L^b (e.g., proved in [15]). This turns out to be quite involved and requires knowledge about the invariant measure μ for the linear equation (cf. Section 6).

In Section 7 we construct the cadlag Markov process on E which solves (1.1) in the above sense. The price we have to pay for working in this quite general situation is that this process will possibly only live on an enlarged state space. The method of proof relies on results in [9], ensuring tightness of (r, p)-capacities related to the linear equation, i.e. (1.1) with $b \equiv 0$ (we need the case $r = 2,\ p = 2$). Furthermore, we use the existence results for processes associated with generalized Dirichlet forms from [20]. We also prove uniqueness in the sense of Markov selections under some constraint (see Section 7 for details).

Finally, in Section 8 we apply all this to a stochastic partial differential equation, namely to the stochastic heat equation on (an extension E of) $L^2(]0,1[)$

$$\mathrm{d}X_t = \mathrm{d}Y_t + [\Delta X_t + b(X_t)]\,\mathrm{d}t,$$

where Δ denotes the Laplacian on $]0,1[$ with Dirichlet boundary conditions and $(Y_t)_{t\geqslant 0}$ is a Levy process with determining negative definite functional

$$\lambda(\xi) := \|\xi\|^2_{L^2(]0,1[)} + \|\xi\|^{\alpha}_{L^2(]0,1[)}, \quad \xi \in L^2(]0,1[),$$

for some fixed $\alpha \in]0,2[$.

2. A General Lemma on Hilbert Spaces

Let E be a (real) separable Hilbert spaces. ${}_{E'}\langle\cdot,\cdot\rangle_E$ will denote the duality bracket between E and E':

$${}_{E'}\langle\xi, x\rangle_E = \xi(x), \quad \xi \in E',\ x \in E.$$

We shall also set $\langle\cdot,\cdot\rangle := {}_{E'}\langle\cdot,\cdot\rangle_E$ if there is no confusion possible. Let $(\cdot,\cdot)_E$ denote the Hilbertian inner product on E. Except for $E = \mathbb{R}^n$, we shall (as in [9, 14]) *not* identify E and E'. We need a technical, purely algebraic result concerning Hilbert spaces: let R_E denote the Riesz isomorphism $R_E : E \to E'$, defined by

$$_{E'}\langle R_E(v), w\rangle_E = R_E(v)(w) = (v, w)_E,$$

and let $J_{E'} : E' \to E$ denote its inverse isomorphism.* Let $R : E' \to E$ be a (continuous) linear operator such that

$$R \circ R_E : E \to E$$

is symmetric, positive and continuous.** We set $Q_0 := R \circ R_E$, and denote by $R_0 = Q_0^{1/2} : E \to E$ the positive, symmetric square root of Q_0. Since for all $v \in E$

$$\big(v, Q_0(v)\big)_E = \big(v, R_0^2(v)\big)_E = \big(v, R_0^*\big(R_0(v)\big)\big)_E = \big(R_0(v), R_0(v)\big)_E,$$

$Q_0(v) = 0$ implies $R_0(v) = 0$, i.e. $\ker Q_0 \subseteq \ker R_0$. Since, obviously, $\ker R_0 \subseteq \ker R_0^2 = \ker Q_0$, one has $\ker R_0 = \ker Q_0$. Let $G := (\ker R_0)^\perp = (\ker Q_0)^\perp$. Then

$$\begin{aligned} E &= \ker Q_0 \oplus^\perp G \\ &= \ker R_0 \oplus^\perp G, \end{aligned}$$

hence R_0 induces a continuous isomorphism

$$S_0 := R_0|_G : G \to R_0(G) = R_0(E).$$

Let $H_0 := R_0(E) = R_0(G) = S_0(G)$ be equipped with the scalar product defined by

$$(v, w)_{H_0} = \big(S_0^{-1}(v), S_0^{-1}(w)\big)_E.$$

It is clear that $(H_0, (\cdot,\cdot)_{H_0})$ is a Hilbert space and that the inclusion $i : H_0 \hookrightarrow E$ is continuous (because S_0 is). As R_0 is symmetric, $R_0(G) = \operatorname{Im} R_0 \subseteq (\ker R_0)^\perp = G$.

LEMMA 2.1. (i) *For all $l' \in E'$ one has*

$$S_0^{-1}\big(J_{H_0'}\big(i^*(l)\big)\big) = R_0\big(J_{E'}(l)\big).$$

(*It is easy to see that both sides of the above equality belong to G.*)

(ii) *For all $v, w \in E'$, one has*

$$\big(J_{H_0'}(i^*(v)), J_{H_0'}(i^*(w))\big)_{H_0} = {}_{E'}\langle v, Rw\rangle_E.$$

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* Modulo the canonical identification between E and E'' one has, of course, $J_{E'} = R_{E'}$.

** Whenever we shall use Lemma 2.1 in this paper, $R \circ R_E$ will actually be of trace class.

Proof. (i) Let w be an arbitrary element of G. Then

$$\begin{aligned}
&\big(R_0\big(J_{E'}(l)\big), w\big)_E \\
&\quad = \big(J_{E'}(l), R_0(w)\big)_E \quad \text{(by symmetry of } R_0) \\
&\quad = {}_{E'}\big\langle l, R_0(w)\big\rangle_E \quad \text{(by definition of } J_{E'}) \\
&\quad = {}_{E'}\big\langle l, i\big(R_0(w)\big)\big\rangle_E \\
&\quad = {}_{H_0'}\big\langle i^*(l), R_0(w)\big\rangle_{H_0} \quad \text{(since } R_0(w) \in R_0(E) = H_0) \\
&\quad = \big(J_{H_0'}\big(i^*(l)\big), R_0(w)\big)_{H_0} \quad \text{(by definition of } J_{H_0'}) \\
&\quad = \big(S_0^{-1}\big(J_{H_0'}\big(i^*(l)\big)\big), S_0^{-1}\big(R_0(w)\big)\big)_E \quad \text{(by definition of } (\cdot,\cdot)_{H_0}) \\
&\quad = \big(S_0^{-1}\big(J_{H_0'}\big(i^*(l)\big)\big), w\big)_E \quad \text{(since } w \in G \text{ and } S_0 = R_0|_G).
\end{aligned}$$

Therefore, $R_0(J_{E'}(l)) - S_0^{-1}(J_{H_0'}(i^*(l))) \in G$ is orthogonal to all elements of G, hence it has to be 0.

(ii) One has

$$\begin{aligned}
&\big(J_{H_0'}\big(i^*(v)\big), J_{H_0'}\big(i^*(w)\big)\big)_{H_0} \\
&\quad = \big(S_0^{-1}\big(J_{H_0'}\big(i^*(v)\big)\big), S_0^{-1}\big(J_{H_0'}\big(i^*(w)\big)\big)\big)_E \quad \text{(by definition of } (\cdot,\cdot)_{H_0}) \\
&\quad = \big(R_0\big(J_{E'}(v)\big), R_0\big(J_{E'}(w)\big)\big)_E \quad \text{(by (i))} \\
&\quad = \big(J_{E'}(v), R_0^2\big(J_{E'}(w)\big)\big)_E \quad \text{(by symmetry of } R_0) \\
&\quad = \big(J_{E'}(v), Q_0\big(J_{E'}(w)\big)\big)_E \\
&\quad = {}_{E'}\big\langle v, Q_0\big(J_{E'}(w)\big)\big\rangle_E \quad \text{(by definition of } J_{E'}) \\
&\quad = {}_{E'}\big\langle v, R\big(R_E\big(J_{E'}(w)\big)\big)\big\rangle_E \quad \text{(by definition of } Q_0) \\
&\quad = {}_{E'}\big\langle v, R(w)\big\rangle_E \quad \text{(since } J_{E'} = R_E^{-1}).
\end{aligned}$$

□

3. Review of the Linear Case

The framework of "generalized Mehler semigroups" has been established in [6, 9, 14]; for the convenience of the reader we recall its main features here.

Let E be a (real) separable Hilbert space with Borel σ-algebra $\mathcal{B}(E)$. Let $(T_t)_{t\geqslant 0}$ denote a strongly continuous (i.e. C_0–) semigroup of bounded linear operators on E, with generator A, and let $\lambda : E' \to \mathbb{C}$ satisfy the following hypothesis:

(H1) $\lambda : E' \to \mathbb{C}$ is negative-definite and Sazonov-continuous (cf., e.g., [24]) with $\lambda(0) = 0$.

Then, as is well-known (cf., e.g., [18, Theorem VI.4.10]), λ posesses a unique Levy–Khinchin representation of the form

$$\lambda(\xi) = -i\langle \xi, \alpha\rangle + \frac{1}{2}\langle \xi, R\xi\rangle$$

$$- \int_E \left(\mathrm{e}^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1 + \|x\|_E^2} \right) M(\mathrm{d}x), \quad \xi \in E', \tag{3.1}$$

where $\alpha \in E$, $R : E' \to E$ is such that $R \circ R_E : E \to E$ is a symmetric trace class operator, and M is a Levy measure on $\mathcal{B}(E)$, i.e. $M(\{0\}) = 0$ and $\int_E (1 \wedge \|x\|_E^2) M(\mathrm{d}x) < \infty$. By [14, Lemma 3.2], there is a constant $D > 0$ such that for all $\xi \in E'$

$$\left|\lambda(\xi)\right| \leqslant D\left(1 + \|\xi\|_{E'}^2\right). \tag{3.2}$$

The generalized Mehler semigroup $(P_t)_{t\geqslant 0}$ associated with λ and $(T_t)_{t\geqslant 0}$ is then given by

$$P_t f(x) = \int_E f(T_t x + y) \mu_t(\mathrm{d}y), \quad x \in E, \tag{3.3}$$

where the measures μ_t, $t \geqslant 0$, have Fourier transforms given by

$$\hat{\mu}_t(\xi) := \exp\left\{ -\int_0^t \lambda(T_s^* \xi)\, \mathrm{d}s \right\}, \quad \xi \in E'. \tag{3.4}$$

We make additional assumptions:

(H2) There exists a probability measure μ on $\mathcal{B}(E)$ which is invariant under P_t, i.e. such that for all $t \geqslant 0$ and all bounded, $\mathcal{B}(E)$-measurable functions $f : E \to \mathbb{R}$ one has

$$\int_E P_t f(x)\, \mathrm{d}\mu(x) = \int_E f(x)\, \mathrm{d}\mu(x).$$

(H3) There exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ in E', consisting of eigenvectors of A^* (i.e. the dual of the generator A on E) and separating the points of E.

REMARK 3.1. By [9, Theorem 3.1] the following constitutes a sufficient condition for (H2) to hold (which is also necessary if $\lim_{t\to\infty} T_t x = 0$ for all $x \in E$):

(H2)′ (i) $\displaystyle \sup_{t>0} \mathrm{Tr}\left(\int_0^t T_s R T_s^*\, \mathrm{d}s \right) < \infty,$

(ii) $\displaystyle \int_0^\infty \int_E \left(1 \wedge \|T_s x\|_E^2\right) M(\mathrm{d}x) < \infty,$

(iii) $a_\infty := \lim_{t\to\infty}(a_t^{(1)} + a_t^{(2)})$ exists in E, where for $t \geqslant 0$

$$a_t^{(1)} := \int_0^t T_s \alpha\, \mathrm{d}s,$$

$$a_t^{(2)} := \int_0^t \int_E T_s x \left(\frac{1}{1 + \|T_s x\|_E^2} - \frac{1}{1 + \|x\|_E^2} \right) M(\mathrm{d}x)\, \mathrm{d}s.$$

In this case

$$\hat{\mu}(\xi) = \mathrm{e}^{-\lambda_\infty(\xi)},$$

where λ_∞ is given by (3.1) with α, R, M replaced by a_∞, $R_\infty := \int_0^\infty T_s R T_s^* \,\mathrm{d}s$, and $M_\infty := \int_0^\infty M \circ T_s^{-1} \,\mathrm{d}s$, respectively.

REMARK 3.2. Condition (H3) is satisfied whenever A is self-adjoint with compact resolvent. We therefore have numerous easy examples.

$(P_t)_{t\geqslant 0}$ extends naturally to $L^2(E;\mu)$, and the domain $D(L)$ of its generator L (denoted by $\mathcal{A}$ in [14]) contains a space W of test functions which we describe now:

Let W_0 be the space of functions u that have a representation of the form

$$u(x) = f\big(\langle \xi_1, x\rangle, \dots, \langle \xi_m, x\rangle\big),$$

for all $x \in E$ and for $m \geqslant 1$ an integer and $f \in \mathcal{S}(\mathbb{R}^m, \mathbb{C})$ (i.e. the Schwartz space of complex-valued functions, "rapidly decreasing" at infinity as well as their derivatives). With the notations above, let $g_0 : \mathbb{R}^m \to \mathbb{C}$ denote the inverse Fourier transform of f, i.e. the function g_0, such that for all $y \in \mathbb{R}^m$

$$f(y) = \int_{\mathbb{R}^m} \mathrm{e}^{i\langle y, v\rangle} g_0(v)\,\mathrm{d}v,$$

and let $\nu_0(\mathrm{d}v) := g_0(v)\,\mathrm{d}v$, where $\mathrm{d}v$ denotes the Lebesgue measure on $\mathbb{R}^m$. Let $\Pi_m : \mathbb{R}^m \to E'$ be defined by

$$\Pi_m(v_1, \dots, v_m) := v_1\xi_1 + \cdots + v_m\xi_m,$$

and let $\nu = (\Pi_m)_*\nu_0$. Then a very classical computation [3, Lemma 1.3, p. 103] yields that $u = \hat{\nu}$. It is clear that W_0 is a ($\mathbb{C}$-) vector subspace of $C_b(E, \mathbb{C})$. Let W be the ($\mathbb{R}$-) vector space of $\mathbb{R}$-valued elements of W_0. With the notations above, and $u \in W_0$, it will be that $u \in W$ as soon as for all $\beta \in \mathbb{R}$

$$g_0(-\beta) = \overline{g_0(\beta)}.$$

From this and the hypothesis made on A^*, it is easy to see that W separates the points of E and is dense in (real) $L^p(E;\mu)$.

Note that by definition of the Fréchet derivative u' of u it follows that $u'(x) \in D(A^*)$ for all $x \in E$. One has:

LEMMA 3.3 ([14, Theorem 1.2] with $p = 2$). *For all $f = \hat{\nu} \in W$ and any $x \in E$*

$$Lf(x) = \int_{E'} \big(i\langle A^*\xi, x\rangle - \lambda(\xi)\big)\mathrm{e}^{i\langle \xi, x\rangle}\nu(\mathrm{d}\xi), \tag{3.5}$$

and also

THEOREM 3.4 ([14, Theorem 1.3(ii)] with $p = 2$). *(L, W) is maximally dissipative on $L^2(E; \mu)$, i.e. for all $u \in W$ $(Lu, u) \leqslant 0$ and $(1 - L)(W)$ is dense in $L^2(E; \mu)$. In particular, the closure of (L, W) is $(L, D(L))$.*

We shall need another expression for L (cf., e.g., [21] for the case $E = \mathbb{R}^d$).

PROPOSITION 3.5. *For each $u \in W$, one has for all $x \in E$*

$$\begin{aligned} Lu(x) \;&=\; \big\langle A^*(u'(x)), x\big\rangle + \big\langle u'(x), \alpha\big\rangle \\ &\quad + \frac{1}{2}\Delta^{H_0}u(x) + \int_E \left(u(x+y) - u(x) - \frac{\langle u'(x), y\rangle}{1 + \|y\|^2}\right) M(\mathrm{d}y), \end{aligned} \tag{3.6}$$

where for $x \in E$

$$\begin{aligned} \Delta^{H_0}u(x) \;&:=\; \sum_{j,k=1}^{m} \partial_j\partial_k f\big({}_{E'}\langle\xi_1, x\rangle_E, \ldots, {}_{E'}\langle\xi_m, x\rangle_E\big)\big(J_{H_0'}\big(i^*(\xi_j)\big), J_{H_0'}\big(i^*(\xi_k)\big)\big)_{H_0} \\ &=\; \sum_{k=1}^{m} {}_{E'}\big\langle\big(\partial_k f\big(\langle\xi_1, \cdot\rangle, \ldots, \langle\xi_m, \cdot\rangle\big)\big)', R\xi_k\big\rangle_E \\ &=\; -\int_{E'} {}_{E'}\langle\xi, R\xi\rangle_E \mathrm{e}^{i\,{}_{E'}\langle\xi, x\rangle_E}\nu(\mathrm{d}\xi). \end{aligned} \tag{3.7}$$

Here as usual $\partial_i f$ denotes the partial derivative with respect to the i-th coordinate.

Proof. From the formula

$$Lu(x) = \int_{E'} \big(i\langle A^*\xi, x\rangle - \lambda(\xi)\big)\mathrm{e}^{i\langle\xi, x\rangle}\nu(\mathrm{d}\xi)$$

it follows that

$$\begin{aligned} Lu(x) \;&=\; \int_{E'} \Bigg(i\langle A^*\xi, x\rangle + i\langle\xi, \alpha\rangle - \frac{1}{2}\langle\xi, R\xi\rangle \\ &\quad + \int_E \left(\mathrm{e}^{i\langle\xi, y\rangle} - 1 - \frac{i\langle\xi, y\rangle}{1 + \|y\|^2}\right) M(\mathrm{d}y)\Bigg)\mathrm{e}^{i\langle\xi, x\rangle}\nu(\mathrm{d}\xi) \\ &=\; {}_{E'}\left\langle A^*\left(\int_{E'} \mathrm{e}^{i\langle\xi, x\rangle} i\xi\,\nu(\mathrm{d}\xi)\right), x\right\rangle_E + {}_{E'}\left\langle \int_{E'} \mathrm{e}^{i\langle\xi, x\rangle} i\xi\,\nu(\mathrm{d}\xi), \alpha\right\rangle_E \\ &\quad - \frac{1}{2}\int_{E'} \langle\xi, R\xi\rangle \mathrm{e}^{i\langle\xi, x\rangle}\nu(\mathrm{d}\xi) \\ &\quad + \int_E \left(\int_{E'} \left(\mathrm{e}^{i\langle\xi, x+y\rangle} - \mathrm{e}^{i\langle\xi, x\rangle} - \frac{\langle i\xi, y\rangle \mathrm{e}^{i\langle\xi, x\rangle}}{1 + \|y\|^2}\right)\nu(\mathrm{d}\xi)\right) M(\mathrm{d}y) \\ &=\; \big\langle A^*\big(u'(x)\big), x\big\rangle + \big\langle u'(x), \alpha\big\rangle - \frac{1}{2}\int_{E'} \langle\xi, R\xi\rangle \mathrm{e}^{i\langle\xi, x\rangle}\nu(\mathrm{d}\xi) \\ &\quad + \int_E \left(u(x+y) - u(x) - \frac{\langle u'(x), y\rangle}{1 + \|y\|^2}\right) M(\mathrm{d}y), \end{aligned}$$

where the interchange of integrals in the last term can be justified via Fubini's Theorem and majorations similar to those in [14]. Therefore, we only have to check that

$$\Delta^{H_0} u(x) = -\int_{E'} \langle \xi, R\xi \rangle e^{i\langle \xi, x\rangle} \nu(d\xi).$$

But, by definition of Δ^{H_0} (see (3.7)) and since

$$\partial_j \partial_k f(u_1, \ldots, u_m) = -\int_{\mathbb{R}^m} e^{i(u,v)_{\mathbb{R}^m}} v_j v_k g_0(v)\, dv$$

and

$$\big(J_{H_0}\big(i^*(\xi_j)\big), J_{H_0}\big(i^*(\xi_k)\big)\big)_{H_0} = {}_{E'}\langle \xi_j, R\xi_k\rangle_E$$

by Lemma 2.1(ii), we have

$$\begin{aligned}
\Delta^{H_0} u(x) &= -\sum_{j,k=1}^m \left(\int_{\mathbb{R}^m} e^{i\, {}_{E'}\langle v_1\xi_1 + \cdots + v_m\xi_m, x\rangle_E} v_j v_k g_0(v)\, dv \right) {}_{E'}\langle \xi_j, R\xi_k\rangle_E \\
&= -\int_{\mathbb{R}^m} e^{i\, {}_{E'}\langle v_1\xi_1 + \cdots + v_m\xi_m, x\rangle_E} \ {}_{E'}\!\left\langle \sum_{j=1}^m v_j\xi_j, R\left(\sum_{j=1}^m v_j\xi_j \right) \right\rangle_E g_0(v)\, dv \\
&= -\int_{\mathbb{R}^m} e^{i\, {}_{E'}\langle \Pi_m(v), x\rangle_E}\, {}_{E'}\langle \Pi_m(v), R(\Pi_m(v))\rangle_E g_0(v)\, dv \\
&= -\int_{E'} e^{i\, {}_{E'}\langle \xi, x\rangle_E}\, {}_{E'}\langle \xi, R\xi\rangle_E \nu(d\xi),
\end{aligned}$$

where all computations are finite-dimensional by definition of ν. □

REMARK 3.6. By Theorem 5.3 in [9] we know that there always exists a larger separable Hilbert space E_1, such that $E \hookrightarrow E_1$ is Hilbert–Schmidt and the following holds: T_t as well as μ_t have extensions to E_1 so that (3.3) extends to all $x \in E_1$, and there exists a Levy process $(Y_t)_{t\geqslant 0}$ on some probability space $(\Omega, \mathbb{P})$ with values in E_1 starting at 0 with determining negative definite function λ such that

$$X_t^x := T_t x + \overline{Y}_t + \int_0^t T_{t-s} A \overline{Y}_s\, ds, \quad t \geqslant 0,$$

is a well-defined process on E_1 with transition semigroup given by (the extension to E_1 of) $(P_t)_{t\geqslant 0}$ in (3.3). Furthermore, for all $\omega \in \Omega$ and all $x \in E_1$

$$X_t^x(\omega) = x + Y_t(\omega) + A\left(\int_0^t X_s^x(\omega)\, ds \right), \quad t \geqslant 0.$$

By standard arguments it follows that $\mathbb{P}_x := \mathbb{P} \circ (X_t^x)^{-1}$, $x \in E_1$, solves the martingale problem corresponding to (1.1) (cf. Section 7).

4. A Formula for the Square Field Operator

Let us consider the situation described in the previous section. In particular, (H1)–(H3) are still in force.

PROPOSITION 4.1. *For all $u \in W$ and all $x \in E$ we have the following expression for the square field operator:*

$$\begin{aligned}\Gamma(u,u)(x) &:= L(u^2)(x) - 2u(x)L(u)(x)\\ &= {}_{E'}\big\langle u'(x), R\big(u'(x)\big)\big\rangle_E + \int_E \big(u(x) - u(x+y)\big)^2 M(\mathrm{d}y).\end{aligned}$$

In particular, Γ *maps* $W \times W$ *into* $L^\infty(E;\mu)$.

REMARK. This is analogous to the well-known property that for all $f \in C^2(\mathbb{R}^n;\mathbb{R})$

$$\Delta(f^2) = 2f\Delta f + 2|\nabla f|^2_{\mathbb{R}^n}.$$

Proof. Let $u \in W$. Then (cf. Section 3)

$$\begin{aligned}&u = f \circ \Pi'_m = \hat{\nu},\\ &\Pi'_m(x) = \big(\langle \xi_1, x\rangle, \ldots, \langle \xi_m, x\rangle\big), \quad x \in E,\end{aligned}$$

where

$$f(\alpha) = \int_{\mathbb{R}^m} \mathrm{e}^{i(\alpha,\beta)_{\mathbb{R}^m}} g_0(\beta)\,\mathrm{d}\beta, \quad \alpha \in \mathbb{R}^m,$$

for some $g_0 \in \mathcal{S}(\mathbb{R}^m, \mathbb{C})$ and $\nu := (\Pi_m)_*\nu_0,\ \nu_0(\mathrm{d}\beta) := g_0(\beta)\,\mathrm{d}\beta$.

Then $u^2 = f^2 \circ \Pi'_m$, where f^2 is the Fourier transform of $h_0 = g_0 * g_0$. Therefore, $u^2 = \hat{\theta}$, where

$$\theta = (\Pi_m)_*(\nu_0 * \nu_0) = \nu * \nu.$$

By definition of L, one may write for all $x \in E$

$$\begin{aligned}L(u^2)(x) &= \int_{E'} \big(i\langle A^*\xi, x\rangle - \lambda(\xi)\big)\mathrm{e}^{i\langle \xi, x\rangle}\theta(\mathrm{d}\xi)\\ &= \iint_{E'\times E'} \big(i\langle A^*(\xi'+\xi''), x\rangle - \lambda(\xi'+\xi'')\big)\mathrm{e}^{i\langle \xi'+\xi'', x\rangle}\nu(\mathrm{d}\xi')\nu(\mathrm{d}\xi'')\end{aligned}$$

(by definition of the convolution product) and one arrives at

$$\begin{aligned}L(u^2)(x) &= -\iint_{E'\times E'} \mathrm{e}^{i\langle \xi'+\xi'', x\rangle}\lambda(\xi'+\xi'')\nu(\mathrm{d}\xi')\nu(\mathrm{d}\xi'')\\ &\quad + \iint_{E'\times E'} i\langle A^*(\xi'+\xi''), x\rangle \mathrm{e}^{i\langle \xi'+\xi'', x\rangle}\nu(\mathrm{d}\xi')\nu(\mathrm{d}\xi'')\end{aligned} \tag{4.1}$$

(as shown in [14], the splitting of the integral is permitted here). But, by Fubini's Theorem* and symmetry

$$\begin{aligned}&\iint_{E'\times E'} i\big\langle A^*(\xi'+\xi''), x\big\rangle e^{i\langle \xi'+\xi'',x\rangle}\nu(d\xi')\nu(d\xi'')\\ &\quad = 2\iint_{E'\times E'} i\langle A^*\xi', x\rangle e^{i\langle \xi'+\xi'',x\rangle}\nu(d\xi')\nu(d\xi''),\end{aligned}$$

which, again by Fubini's Theorem, equals to

$$\begin{aligned}&2\int_{E'} e^{i\langle \xi'',x\rangle}\bigg(\int_{E'} i\langle A^*\xi', x\rangle e^{i\langle \xi',x\rangle}\nu(d\xi')\bigg)\nu(d\xi'')\\ &\quad = 2\bigg(\int_{E'} e^{i\langle \xi'',x\rangle}\nu(d\xi'')\bigg)\bigg(\int_{E'} i\langle A^*\xi', x\rangle e^{i\langle \xi',x\rangle}\nu(d\xi')\bigg)\\ &\quad = 2u(x)\int_{E'} i\langle A^*\xi', x\rangle e^{i\langle \xi',x\rangle}\nu(d\xi')\\ &\quad = 2u(x)\bigg(Lu(x)+\int_{E'}\lambda(\xi')e^{i\langle \xi',x\rangle}\nu(d\xi')\bigg),\end{aligned}$$

where the last equality holds by the definition of L. Therefore, using Fubini's Theorem again (since λ has at most quadratic growth)

$$\begin{aligned}&L(u^2)(x) - 2u(x)L(u)(x)\\ &\quad = 2u(x)\int_{E'}\lambda(\xi')e^{i\langle \xi',x\rangle}\nu(d\xi')\\ &\qquad - \iint_{E'\times E'}\lambda(\xi'+\xi'')e^{i\langle \xi'+\xi'',x\rangle}\nu(d\xi')\nu(d\xi'')\\ &\quad = 2\int_{E'} e^{i\langle \xi'',x\rangle}\nu(d\xi'')\int_{E'}\lambda(\xi')e^{i\langle \xi',x\rangle}\nu(d\xi')\\ &\qquad - \iint_{E'\times E'}\lambda(\xi'+\xi'')e^{i\langle \xi'+\xi'',x\rangle}\nu(d\xi')\nu(d\xi'')\\ &\quad = 2\iint_{E'\times E'}\lambda(\xi')e^{i\langle \xi'+\xi'',x\rangle}\nu(d\xi')\nu(d\xi'')\\ &\qquad - \iint_{E'\times E'}\lambda(\xi'+\xi'')e^{i\langle \xi'+\xi'',x\rangle}\nu(d\xi')\nu(d\xi'')\end{aligned}$$

---

* We need to check the convergence of $\iint_{E'\times E'} |\langle A^*\xi', x\rangle|\,|\nu|(d\xi')|\nu|(d\xi'')$, which is just

$$\begin{aligned}&\iint_{\mathbb{R}^{2m}}\left|\left\langle\sum_{i=1}^m \beta_i' A^*\xi_i, x\right\rangle\right| |g_0(\beta')|\,|g_0(\beta'')|\,d\beta'\,d\beta''\\ &\quad\leqslant C_{x,\xi}\iint_{\mathbb{R}^{2n}}\|\beta'\|_{\mathbb{R}^m}|g_0(\beta')|\,|g_0(\beta'')|\,d\beta'\,d\beta'',\end{aligned}$$

where $C_{x,\xi} := (\sum_{i=1}^m\langle A^*\xi_i, x\rangle^2)^{1/2}$. But the right-hand side is finite, since $g_0 \in \mathcal{S}$.

$$= \iint_{E'\times E'} \big(\lambda(\xi') + \lambda(\xi'')\big) e^{i\langle \xi'+\xi'', x\rangle} \nu(d\xi')\nu(d\xi'')$$
$$- \iint_{E'\times E'} \lambda(\xi' + \xi'') e^{i\langle \xi'+\xi'', x\rangle} \nu(d\xi')\nu(d\xi'')$$

(by the same symmetry argument as above). Hence

$$\begin{aligned} &L(u^2)(x) - 2u(x)L(u)(x) \\ &\quad = \iint_{E'\times E'} \big(\lambda(\xi') + \lambda(\xi'') - \lambda(\xi'+\xi'')\big) e^{i\langle \xi'+\xi'', x\rangle} \nu(d\xi')\nu(d\xi'') \end{aligned}$$

for all $x \in E$. But by the Levy–Khinchin representation (3.1) and the symmetry of $(\xi', \xi'') \mapsto {}_{E'}\langle \xi', R\xi''\rangle_E$ (cf. Lemma 2.1(ii)) it follows that

$$\begin{aligned} &\lambda(\xi') + \lambda(\xi'') - \lambda(\xi'+\xi'') \\ &\quad = -{}_{E'}\langle \xi', R\xi''\rangle_E - \int_E \big(e^{i\langle \xi', y\rangle} + e^{i\langle \xi'', y\rangle} - e^{i\langle \xi'+\xi'', y\rangle} - 1\big) M(dy) \\ &\quad = -{}_{E'}\langle \xi', R\xi''\rangle_E + \int_E \big(1 - e^{i\langle \xi', y\rangle}\big)\big(1 - e^{i\langle \xi'', y\rangle}\big) M(dy). \end{aligned}$$

Thus,

$$\begin{aligned} &L(u^2)(x) - 2u(x)L(u)(x) \\ &\quad = -\iint_{E'\times E'} e^{i\langle \xi'+\xi'', x\rangle} \langle \xi', R\xi''\rangle \nu(d\xi')\nu(d\xi'') \\ &\qquad + \iint_{E'\times E'} e^{i\langle \xi'+\xi'', x\rangle} \left(\int_E \big(1 - e^{i\langle \xi', y\rangle}\big)\big(1 - e^{i\langle \xi'', y\rangle}\big) M(dy) \right) \nu(d\xi')\nu(d\xi''). \end{aligned}$$

But, due to the obvious bounds

$$\left|\big(1 - e^{i\langle \xi', y\rangle}\big)\big(1 - e^{i\langle \xi'', y\rangle}\big)\right| \leqslant \begin{cases} 4, & \text{and} \\ |\langle \xi', y\rangle|\,|\langle \xi'', y\rangle| \leqslant \|\xi'\|\,\|\xi''\|\,\|y\|^2, \end{cases}$$

and the finiteness of ν, $\int \|\xi\|\nu(d\xi)$ and $\int (1 \wedge \|y\|^2) M(dy)$, Fubini's Theorem applies here. The second term of our sum therefore equals

$$\begin{aligned} &\int_E \left(\int_{E'} e^{i\langle \xi', x\rangle} \big(1 - e^{i\langle \xi', y\rangle}\big) \nu(d\xi') \right) \left(\int_{E'} e^{i\langle \xi'', x\rangle} \big(1 - e^{i\langle \xi'', y\rangle}\big) \nu(d\xi'') \right) M(dy) \\ &\quad = \int_E \left(\int_{E'} e^{i\langle \xi', x\rangle} \nu(d\xi') - \int_{E'} e^{i\langle \xi', x+y\rangle} \nu(d\xi') \right)^2 M(dy) \\ &\quad = \int_E \big(u(x) - u(x+y)\big)^2 M(dy). \end{aligned}$$

As $u \in \mathcal{F}C_b^\infty(E)$, u is both globally Lipschitzian and bounded, so that there is a $C_u \geqslant 0$, such that for all $x, y \in E$

$$\big|u(x) - u(x+y)\big| \leqslant C_u\big(\|y\|_E \wedge 1\big),$$

so the convergence of the last integral follows from the very definition of a Levy measure. Furthermore, it follows from

$$u(x) = \int_{E'} \mathrm{e}^{i\langle \xi, x\rangle} \nu(\mathrm{d}\xi), \quad x \in E,$$

that for all $x \in E$

$$u'(x) = \int_{E'} i\mathrm{e}^{i\langle \xi, x\rangle} \xi \nu(\mathrm{d}\xi),$$

hence

$$_{E'}\big\langle u'(x), R\big(u'(x)\big)\big\rangle_E = -\iint_{E'\times E'} \mathrm{e}^{i\langle \xi'+\xi'', x\rangle} {}_{E'}\langle \xi', R\xi''\rangle_E \nu(\mathrm{d}\xi')\nu(\mathrm{d}\xi'')$$

and the formula in the assertion follows.* □

COROLLARY 4.2. *For each $u \in W$,*

$$\int_E {}_{E'}\big\langle u'(x), R\big(u'(x)\big)\big\rangle_E \mu(\mathrm{d}x) \leqslant -2(u, Lu)_{L^2(E;\mu)}.$$

Proof. By Proposition 4.1, one has, for any given $x \in E$,

$$_{E'}\big\langle u'(x), R\big(u'(x)\big)\big\rangle_E \leqslant \Gamma(u,u)(x) = L(u^2)(x) - 2u(x)L(u)(x).$$

Therefore,

$$\begin{aligned}\int_E {}_{E'}\big\langle u'(x), R\big(u'(x)\big)\big\rangle_E \mu(\mathrm{d}x) &\leqslant \int_E \big(L(u^2)(x) - 2u(x)L(u)(x)\big)\mu(\mathrm{d}x) \\ &= -2(u, Lu)_{L^2(E;\mu)} + \int_E L(u^2)(x)\mu(\mathrm{d}x).\end{aligned}$$

But $\int_E L(u^2)\,\mathrm{d}\mu = 0$, since μ is $(P_t)_{t\geqslant 0}$-invariant (and therefore infinitesimally $(P_t)_{t\geqslant 0}$-invariant) and $u^2 \in W \subseteq D(L)$. Hence the result follows. □

5. Existence of the Semigroup

In this section we shall prove the existence of a C_0-semigroup P_t^B on $L^2(E;\mu)$ with generator $L+B$, where B denotes a "first-order" drift term. By the results of the previous section, in particular Corollary 4.2, we shall see that now standard perturbation theory for generators of C_0-semigroups applies.

* All the gradient computations are in fact *finite-dimensional* due to the definition of ν and, therefore, no problem arises from them.

(H4) Let $\hat{b} : E \times E \to \mathbb{R}$ be $\mu \otimes M$-measurable, such that

$$s := \left(\int_E \hat{b}(\cdot, y)^2 M(\mathrm{d}y) \right)^{1/2} \in L^\infty(E; \mu),$$

and $b : E \to H_0$, μ-measurable and bounded.

Define now $B : W_0 \to L^\infty(E; \mu)$ by

$$Bu := {}_{E'}\langle u', b \rangle_E + \int_E \hat{b}(\cdot, y)\big(u - u(\cdot + y)\big) M(\mathrm{d}y).$$

PROPOSITION 5.1. *Let $\alpha := \|s\|_{L^\infty(E;\mu)}$ and $\beta := \||b|_{H_0}\|_{L^\infty(E;\mu)}$. Then for all $u \in W$*

$$\|Bu\|_{L^2(E;\mu)} \leqslant K\sqrt{-(Lu, u)_{L^2(E;\mu)}} \tag{5.1}$$

with $K := 2\max(\alpha, \beta)$.

Proof. By assumption, $b = S_0 \circ c$, for a certain bounded $c : E \to G$. Then

$$\beta = \|b\|_{L^\infty(E;H_0)} = \|c\|_{L^\infty(E;G)} = \|c\|_{L^\infty(E;E)}.$$

For a given $x \in E$, one has

$$\begin{aligned}
\big(Bu(x)\big)^2 &= \left[{}_{E'}\big\langle u'(x), b(x) \big\rangle_E + \int_E \hat{b}(x, y)\big(u(x) - u(x + y)\big) M(\mathrm{d}y) \right]^2 \\
&\leqslant 2\bigg[{}_{E'}\big\langle u'(x), b(x) \big\rangle_E^2 \\
&\quad + \left(\int_E \hat{b}(x, y)^2 M(\mathrm{d}y) \right)\left(\int_E \big(u(x) - u(x + y)\big)^2 M(\mathrm{d}y) \right)\bigg]
\end{aligned}$$

by the Cauchy–Schwarz inequality. But with the results and notation of Section 1,

$$\begin{aligned}
{}_{E'}\big\langle u'(x), b(x) \big\rangle_E &= {}_{E'}\big\langle u'(x), S_0\big(c(x)\big) \big\rangle_E \\
&= {}_{E'}\big\langle u'(x), R_0\big(c(x)\big) \big\rangle_E \\
&= \big(J_{E'}\big(u'(x)\big), R_0\big(c(x)\big) \big)_E \\
&= \big((R_0 \circ J_{E'})\big(u'(x)\big), c(x) \big)_E,
\end{aligned}$$

where we used the symmetry of R_0. Hence, using $R_0^2 = Q_0 = R \circ R_E$ and $J_{E'} = R_E^{-1}$, we obtain

$$\begin{aligned}
{}_{E'}\big\langle u'(x), b(x) \big\rangle_E^2 &= \big((R_0 \circ J_{E'})\big(u'(x)\big), c(x) \big)_E^2 \\
&\leqslant \big\|c(x)\big\|_E^2 \big\|(R_0 \circ J_{E'})\big(u'(x)\big)\big\|_E^2 \\
&\leqslant \beta^2 \big(R_0\big(J_{E'}\big(u'(x)\big)\big), R_0\big(J_{E'}\big(u'(x)\big)\big) \big)_E \\
&= \beta^2 \big(J_{E'}\big(u'(x)\big), R_0^2\big(J_{E'}\big(u'(x)\big)\big) \big)_E \\
&= \beta^2 \big(J_{E'}\big(u'(x)\big), R\big(u'(x)\big) \big)_E \\
&= \beta^2 \; {}_{E'}\big\langle u'(x), R\big(u'(x)\big) \big\rangle_E.
\end{aligned}$$

Therefore, by Proposition 4.1

$$\begin{aligned}\big(Bu(x)\big)^2 &\leqslant \gamma\left[{}_{E'}\big\langle u'(x), R\big(u'(x)\big)\big\rangle_E + \int_E \big(u(x)-u(x+y)\big)^2 M(\mathrm{d}y)\right]\\ &= \gamma\big(L(u^2)(x) - 2u(x)L(u)(x)\big),\end{aligned}$$

where we have set $\gamma := 2\max(\alpha^2, \beta^2)$. From this it follows that

$$\begin{aligned}\|Bu\|^2_{L^2(E;\mu)} &\leqslant \gamma\left(\int_E L(u^2)\,\mathrm{d}\mu - 2\int_E uLu\,\mathrm{d}\mu\right)\\ &= -2\gamma(u, Lu)_{L^2(E;\mu)},\end{aligned}$$

by the same argument as in the proof of Corollary 4.2, which completes the proof. □

Let us remind the reader of the definition of the graph norm: For $u \in D(L)$

$$\|u\|_{\mathrm{gr}} := \|u\|_{L^2(E;\mu)} + \|Lu\|_{L^2(E;\mu)}.$$

For the next result, we use the fact that W is a core for $(L, D(L))$ in an essential way.

COROLLARY 5.2. *For all $\varepsilon > 0$ there exists $c_\varepsilon \in]0, \infty[$ such that*

$$\forall \mu \in W \quad \|Bu\|_{L^2(E;\mu)} \leqslant \varepsilon\|Lu\|_{L^2(E;\mu)} + c_\varepsilon\|u\|_{L^2(E;\mu)}. \tag{5.2}$$

In particular, B extends uniquely to a bounded operator (again denoted by B) from $(D(L), \|\cdot\|_{\mathrm{gr}})$ to $L^2(E;\mu)$, and (5.1) and (5.2) hold for all $u \in D(L)$.

Moreover, W is a core for $(L+B, D(L))$, i.e. W is dense in $D(L)$ with respect to the graph norm of $L+B$.

Proof. From Proposition 5.1 it follows that for all $u \in W$ and $\delta > 0$

$$\begin{aligned}\|Bu\|^2_{L^2(E;\mu)} \leqslant K^2\big|(u, Lu)_{L^2(E;\mu)}\big| &\leqslant K^2\|u\|_{L^2(E;\mu)}\|Lu\|_{L^2(E;\mu)}\\ &\leqslant \frac{K^2}{2}\big[\delta^2\|Lu\|^2_{L^2(E;\mu)} + \delta^{-2}\|u\|^2_{L^2(E;\mu)}\big].\end{aligned}$$

In particular, B is a bounded operator from $(W, \|\cdot\|_{\mathrm{gr}})$ to $L^2(E;\mu)$. But as W is dense in $D(L)$ for the graph norm [14, Theorem 1.3(ii)], the existence and uniqueness of a bounded extension of B to $D(L)$ follows and (5.1) holds for all $u \in D(L)$ by the usual density argument. □

PROPOSITION 5.3. *Let $c := K^2/4$. For all $t \in [0,1]$, $L+tB-c\mathrm{Id}$ is dissipative.*

Proof. We have to prove that for all $u \in D(L)$

$$(u, Lu + tBu - cu)_{L^2(E;\mu)} \leqslant 0.$$

Let $t \in]0, 1]$ be fixed and $\alpha := \sqrt{Kt/2}$. Then, using Corollary 5.2, one has, if $\alpha > 0$, that

$$\begin{aligned}(u, Bu)_{L^2(E;\mu)} &\leqslant \|u\|_{L^2(E;\mu)}\|Bu\|_{L^2(E;\mu)} \leqslant K\|u\|_{L^2(E;\mu)}\sqrt{-(u, Lu)_{L^2(E;\mu)}}\\ &= K\big(\alpha\|u\|_{L^2(E;\mu)}\big)\bigg(\frac{1}{\alpha}\sqrt{-(u, Lu)_{L^2(E;\mu)}}\bigg)\\ &\leqslant \frac{K}{2}\bigg(\alpha^2\|u\|^2_{L^2(E;\mu)} - \frac{1}{\alpha^2}(u, Lu)_{L^2(E;\mu)}\bigg)\\ &= \frac{K^2t}{4}\|u\|^2_{L^2(E;\mu)} - \frac{1}{t}(u, Lu)_{L^2(E;\mu)}.\end{aligned}$$

This result is obviously also true if $\alpha = 0$ (i.e. $K = 0$). Hence,

$$t(u, Bu)_{L^2(E;\mu)} \leqslant ct^2(u, u)_{L^2(E;\mu)} - (u, Lu)_{L^2(E;\mu)}.$$

That last inequality obviously remains true when $t = 0$. Thus, in all cases

$$(u, Lu + tBu - cu)_{L^2(E;\mu)} \leqslant c(t^2 - 1)(u, u)_{L^2(E;\mu)} \leqslant 0. \qquad \square$$

COROLLARY 5.4. $(L+B, D(L))$ *generates a* C_0*-semigroup* (*henceforth denoted by* P_t^B) *on* $L^2(E; \mu)$.

Proof. Let $c := K^2/4$. Then $L - c\mathrm{Id}$ generates a contraction C_0-semigroup $(\mathrm{e}^{-ct}P_t)_{t\geqslant 0}$ on $L^2(E; \mu)$. By Corollary 5.2, Proposition 5.3, and Theorem 3.4, we can apply [19, Theorem 3.2, p. 81] to $(L-c\mathrm{Id}, D(L))$ and $(B, D(L))$ and conclude that $L-c\mathrm{Id}+B$ generates a contraction C_0-semigroup Q_t^B. Hence $L+B$ generates the C_0-semigroup $P_t^B = \mathrm{e}^{ct}Q_t^B$. $\square$

6. Markov Property

Define

$$\begin{aligned}\lambda_1(\xi) &:= -i\langle\xi, \alpha\rangle + \frac{1}{2}\langle\xi, R\xi\rangle, \quad \xi \in E',\\ \lambda_2(\xi) &:= -\int_E\bigg(\mathrm{e}^{i\langle\xi,x\rangle} - 1 - \frac{i\langle\xi, x\rangle}{1+\|x\|^2}\bigg)M(\mathrm{d}x), \quad \xi \in E'.\end{aligned} \tag{6.1}$$

From now on, in addition to (H1), (H3) and (H4), we are going to assume the following:

(H5) (i) (H2) holds with both limits $a_\infty^{(1)} := \lim_{t\to\infty} a_t^{(1)}$ and $a_\infty^{(2)} := \lim_{t\to\infty} a_t^{(2)}$ existing in E.
(ii) $R\xi_k \in \mathrm{Im}\, R_\infty$ for all $k \in \mathbb{N}$.
(iii) $\hat{b}$ in (H4) is identically equal to zero.

Then the measures γ and σ on $\mathcal{B}(E)$ with Fourier transforms given by

$$\hat{\gamma}(\xi) = \exp\left\{ i\langle \xi, a_\infty^{(1)}\rangle - \frac{1}{2}\langle \xi, R_\infty \xi\rangle \right\},$$
$$\hat{\sigma}(\xi) = \exp\left\{ i\langle \xi, a_\infty^{(2)}\rangle + \int_E \left(\mathrm{e}^{i\langle \xi, y\rangle} - 1 - \frac{i\langle \xi, y\rangle}{1+\|y\|_E^2} \right) M_\infty(\mathrm{d}y) \right\},$$

$\xi \in E'$, are the invariant measures for the generalized Mehler semigroups $(P_t^{(1)})_{t\geqslant 0}$ resp. $(P_t^{(2)})_{t\geqslant 0}$, associated with λ_1 and $(T_t)_{t\geqslant 0}$ resp. λ_2 and $(T_t)_{t\geqslant 0}$.

Since $\lambda = \lambda_1 + \lambda_2$, it follows from Remark 3.1 that $\hat{\mu}(\xi) = \hat{\gamma}(\xi)\hat{\sigma}(\xi)$ for all $\xi \in E'$, i.e.

$$\mu = \gamma * \sigma. \tag{6.2}$$

Let L_i, $i = 1, 2$, be defined on W by (3.5) with λ_i replacing λ. Then applying Theorem 3.4 to L_i we obtain that the closures $(L_i, D(L_i))$ generate C_0-semigroups $(\mathrm{e}^{tL_i})_{t\geqslant 0}$ on $L^2(\gamma)$ resp. $L^2(\sigma)$ such that $P_t^{(i)} f$ is a version of $\mathrm{e}^{tL_i} f$, $t > 0$.

Let us define, as usual, for $u \in C^1(E; E)$:

$$\nabla^{H_0} u(x) := J_{H_0'}\big(i^*\big(u'(x)\big)\big), \tag{6.3}$$

where i, as above, denotes the inclusion $i : H_0 \hookrightarrow E$.

LEMMA 6.1. *For all $u \in W$ and $x \in E$:*

(i) $Bu(x) = (\nabla^{H_0} u(x), b(x))_{H_0}$.

(ii) *For $u = \hat{\nu} \in W$ let*

$$\begin{aligned} Su(x) &:= -\int_{E'} \lambda_2(\xi)\mathrm{e}^{i\langle \xi, x\rangle}\nu(\mathrm{d}\xi) \\ &= \int_E \left(u(x+y) - u(x) - \frac{\langle u'(x), y\rangle}{1+\|y\|_E^2} \right) M(\mathrm{d}y). \end{aligned}$$

Then for all $y \in E$

$$Su(x+y) = S\big(u(x+\cdot)\big)(y)$$

and

$$L_2 u(x) = \big\langle A^*\big(u'(x)\big), x\big\rangle + Su(x).$$

(iii) $Lu(x) = \frac{1}{2}\Delta^{H_0} u(x) + L_2 u(x) + {}_{E'}\langle u'(x), \alpha\rangle_E$.

Proof. (i) holds since for all $x \in E$, one has:

$$\begin{aligned} &\big(\nabla^{H_0} u(x), b(x)\big)_{H_0} \\ &\quad = \big(J_{H_0'}\big(i^*\big(u'(x)\big)\big), b(x)\big)_{H_0} \end{aligned}$$

$$
\begin{aligned}
&= \big(S_0^{-1}\big(J_{H_0'}\big(i^*\big(u'(x)\big)\big)\big), S_0^{-1}\big(b(x)\big)\big)_E \quad \text{(by definition of } (\cdot,\cdot)_{H_0})\\
&= \big(R_0\big(J_{E'}\big(u'(x)\big)\big), c(x)\big)_E \quad \text{(by Lemma 2.1(i))}\\
&= \big(J_{E'}\big(u'(x)\big), R_0\big(c(x)\big)\big)_E \quad \text{(by the symmetry of } R_0)\\
&= \big(J_{E'}\big(u'(x)\big), S_0\big(c(x)\big)\big)_E \quad \text{(as } c \text{ is } G\text{-valued)}\\
&= \big(J_{E'}\big(u'(x)\big), b(x)\big)_E\\
&= {}_{E'}\big\langle u'(x), b(x)\big\rangle_E \quad \text{(by definition of } J_{E'}).
\end{aligned}
$$

(ii) and (iii) follow immediately by definition and Proposition 3.5. □

THEOREM 6.2. $(P_t^B)_{t\geqslant 0}$ *is sub-Markovian.*

Proof. Let $L^B = L + B$. Since $L^B 1 = 0$, one has

$$P_t^B 1 = \mathrm{e}^{tL^B} 1 = 1.$$

Therefore, we just need to check that $(P_t^B)_{t\geqslant 0}$ is positivity preserving, for which it suffices to find a $c' > 0$ such that $(\mathrm{e}^{-c't} P_t^B)_{t\geqslant 0}$ is.

According to [15, Theorem 1.7], this will be the case whenever

$$\int u^+\big(L^B u - c'u\big)\,\mathrm{d}\mu \leqslant 0$$

for all $u \in D(L^B)$. Here, $u^+ := \sup(u, 0)$ denotes the positive part of u.

By a density argument, it further suffices to check that for all $u \in W$

$$\int_E u^+ L^B u\,\mathrm{d}\mu \leqslant c' \int_E u^+ u\,\mathrm{d}\mu.$$

So, fix $u = \hat{\nu} = f(\langle \xi_1, \cdot\rangle, \dots, \langle \xi_m, \cdot\rangle) \in W$. Let $\varepsilon > 0$ and let $\varphi_\varepsilon : \mathbb{R} \to [-\varepsilon, \infty[$ such that $\varphi_\varepsilon \in C^\infty(\mathbb{R})$, $\varphi_\varepsilon(t) = t$ for all $t \in [0, \infty[$, $0 \leqslant \varphi_\varepsilon' \leqslant 1$, $\varphi_\varepsilon(t) = -\varepsilon$ for $t \in]-\infty, -2\varepsilon]$. Then

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon(t) = 1_{[0,\infty[} \cdot t, \qquad \lim_{\varepsilon\to 0} \varphi_\varepsilon'(t) = 1_{[0,\infty[}. \tag{6.4}$$

Hence, using Lebesgue's Dominated Convergence Theorem we get for $\varepsilon_n := 1/n$, $n \in \mathbb{N}$, by Lemma 6.1(ii) and (iii):

$$
\begin{aligned}
&(u^+, L^B u)_{L^2(E;\mu)}\\
&\quad = \lim_{n\to\infty} \iint_{E\times E} \varphi_{\varepsilon_n}(u)(x+y)\bigg(\frac{1}{2}\Delta^{H_0} u(x+y)\\
&\qquad + {}_{E'}\big\langle A^*\big(u'(x+y)\big), x\big\rangle_E\\
&\qquad + {}_{E'}\big\langle (u'(x+y), \alpha + b(x+y)\big\rangle_E\bigg)\gamma(\mathrm{d}x)\sigma(\mathrm{d}y)\\
&\qquad + \iint_{E\times E} u^+(x+y)\big(Su(x+y) + {}_{E'}\big\langle A^*\big(u'(x+y)\big), y\big\rangle_E\big)\gamma(\mathrm{d}x)\sigma(\mathrm{d}y).
\end{aligned}
$$

For $x \in E$, let $u^x : E \to E$ be defined by $u^x(y) := u(x+y),\ y \in E$. Then by Lemma 6.1(ii) and Fubini's Theorem the second integral can be written as

$$\int_E \int_E (u^x)^+(y) L_2 u^x(y) \sigma(\mathrm{d}y) \gamma(\mathrm{d}x),$$

which by [15, Theorem 1.7] is non-positive, since $P_t^{(2)}$ is positivity preserving. We therefore have

$$\begin{aligned}
&\big(u^+, L^B u\big)_{L^2(E;\mu)} \\
&\quad \leqslant \liminf_{n\to\infty} \iint_{E\times E} \varphi_{\varepsilon_n}(u)(x+y) \bigg(\frac{1}{2} \Delta^{H_0} u(x+y) \\
&\qquad + {}_{E'}\big\langle A^*\big(u'(x+y)\big), x\big\rangle_E + {}_{E'}\big\langle u'(x+y), \alpha + b(x+y)\big\rangle_E \bigg) \gamma(\mathrm{d}x)\sigma(\mathrm{d}y).
\end{aligned}$$

Defining for $k \in \mathbb{N}$

$$\beta_k^\gamma(x) := {}_{E'}\big\langle R_\infty^{-1}(R\xi_k), x - a_\infty^{(1)}\big\rangle_E$$

(which is well-defined by (H5)(ii)) we have the following integration by parts formula (cf., e.g., [4, Proposition 5.1.6]) for all $u \in W$

$$\int_E {}_{E'}\big\langle u'(x), R\xi_k\big\rangle_E \gamma(\mathrm{d}x) = -\int_E u(x)\beta_k^\gamma(x)\gamma(\mathrm{d}x).$$

We set

$$\beta_u^\gamma(x+y) := \sum_{k=1}^m \beta_k^\gamma(x) \partial_k f\big(\langle \xi_1, x+y\rangle, \dots, \langle \xi_m, x+y\rangle\big).$$

Then by (3.7) one has *for fixed* $y \in E$ and all $n \in \mathbb{N}$

$$\begin{aligned}
&\int_E \varphi_{\varepsilon_n}(u)(x+y) \frac{1}{2} \Delta^{H_0} u(x+y) \gamma(\mathrm{d}x) \\
&\quad = -\frac{1}{2} \int_E \big(\nabla^{H_0} \varphi_{\varepsilon_n}(u)(x+y), \nabla^{H_0} u(x+y)\big)_{H_0} \gamma(\mathrm{d}x) \\
&\qquad - \int_E \varphi_{\varepsilon_n}(u)(x+y) \beta_u^\gamma(x+y) \gamma(\mathrm{d}x).
\end{aligned}$$

Hence, using Lebesgue's Dominated Convergence Theorem again we obtain by (6.4), since $\nabla^{H_0}\varphi_{\varepsilon_n}(u)(x+y) = \varphi'_{\varepsilon_n}(u)(x+y)\nabla^{H_0}u(x+y)$, that

$$\begin{aligned}
&\big(u^+, L^B u\big)_{L^2(E;\mu)} \\
&\quad \leqslant \iint_{E\times E} 1_{[0,\infty[}\big(u(x+y)\big) \Big[-\frac{1}{2} \big(\nabla^{H_0} u(x+y), \nabla^{H_0} u(x+y)\big)_{H_0}
\end{aligned}$$

$$
\begin{aligned}
&- u(x+y)\bigg(\frac{1}{2}\beta_u^\gamma(x+y) - \big\langle A^*\big(u'(x+y)\big), x\big\rangle \\
&- \big\langle u'(x+y), \alpha + b(x+y)\big\rangle\bigg)\bigg]\gamma(\mathrm{d}x)\sigma(\mathrm{d}y) \\
= {}& \lim_{n\to\infty} \iint_{E\times E} \bigg[-\varphi'_{\varepsilon_n}\big(u(x+y)\big)^2 \frac{1}{2}\big(\nabla^{H_0}u(x+y), \nabla^{H_0}u(x+y)\big)_{H_0} \\
&- \varphi_{\varepsilon_n}\big(u(x+y)\big)\varphi'_{\varepsilon_n}\big(u(x+y)\big)\bigg(\frac{1}{2}\beta_u^\gamma(x+y) - \big\langle A^*\big(u'(x+y)\big), x\big\rangle \\
&- \big\langle u'(x+y), \alpha + b(x+y)\big\rangle\bigg)\bigg]\gamma(\mathrm{d}x)\sigma(\mathrm{d}y) \\
= {}& \lim_{n\to\infty} \bigg[-\iint_{E\times E} \frac{1}{2}\big[\big(\nabla^{H_0}\big(\varphi_{\varepsilon_n}(u)(x+y)\big), \nabla^{H_0}\big(\varphi_{\varepsilon_n}(u)(x+y)\big)\big)_{H_0} \\
&+ \varphi_{\varepsilon_n}(u)(x+y)\beta^\gamma_{\varphi_{\varepsilon_n}}(x+y)\big]\gamma(\mathrm{d}x)\sigma(\mathrm{d}y) \\
&+ \iint_{E\times E} \varphi_{\varepsilon_n}(u)(x+y)\big(\big\langle A^*\big(\varphi_{\varepsilon_n}(u)'(x+y)\big), x\big\rangle \\
&+ \big\langle\varphi_{\varepsilon_n}(u)'(x+y), \alpha + b(x+y)\big\rangle\big)\gamma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg].
\end{aligned}
$$

Integrating by parts again and due to the fact that

$$
\frac{1}{2}\varphi_{\varepsilon_n}(u)\Delta^{H_0}\varphi_{\varepsilon_n}(u) = \frac{1}{4}\Delta^{H_0}\big(\varphi^2_{\varepsilon_n}(u)\big) - \frac{1}{2}\big(\nabla^{H_0}\varphi_{\varepsilon_n}(u), \nabla^{H_0}\varphi_{\varepsilon_n}(u)\big)_{H_0}
$$

we get

$$
\begin{aligned}
&\big(u^+, L^B u\big)_{L^2(E;\mu)} \\
&\quad \leqslant \varlimsup_{n\to\infty} \bigg[\frac{1}{2}\iint_{E\times E}\bigg(\frac{1}{2}\Delta^{H_0}\big(\varphi^2_{\varepsilon_n}(u^y)\big)(x) + \big\langle A^*\big(\big(\varphi^2_{\varepsilon_n}(u^y)\big)'(x)\big), x\big\rangle \\
&\quad + \big\langle\big(\varphi^2_{\varepsilon_n}(u^y)\big)'(x), a\big\rangle\bigg)\gamma(\mathrm{d}x)\sigma(\mathrm{d}y) \\
&\quad + \iint_{E\times E}\bigg[-\frac{1}{2}\big(\nabla^{H_0}\varphi_{\varepsilon_n}(u)(x+y), \nabla^{H_0}\varphi_{\varepsilon_n}(u)(x+y)\big)_{H_0} \\
&\quad + \varphi_{\varepsilon_n}(u)(x+y)\big\langle\varphi_{\varepsilon_n}(u)'(x+y), b(x+y)\big\rangle\bigg]\gamma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg].
\end{aligned}
$$

The first integral above is equal to zero, since the integrand is just equal to $L_1(\varphi^2_{\varepsilon_n}(u))$ and γ is invariant under $P_t^{(1)}$ for all $t > 0$. To estimate the second, we note that by Lemma 6.1(i)

$$
\begin{aligned}
&\big|\varphi_{\varepsilon_n}(u)(x+y)\big\langle\varphi_{\varepsilon_n}(u)'(x+y), b(x+y)\big\rangle\big| \\
&\quad = \big|\varphi_{\varepsilon_n}(u)(x+y)\big\langle b(x+y), \nabla^{H_0}\varphi_{\varepsilon_n}(u)(x+y)\big\rangle_{H_0}\big|
\end{aligned}
$$

$$\leqslant \big|\varphi_{\varepsilon_n}(u)(x+y)\big| \, \big\| \, |b|_{H_0} \big\|_\infty \big|\nabla^{H_0}\varphi_{\varepsilon_n}(u)(x+y)\big|_{H_0}$$
$$\leqslant \frac{1}{2}\big\| \, |b|_{H_0} \big\|_\infty^2 \varphi_{\varepsilon_n}^2(u)(x+y) + \frac{1}{2}\big|\nabla^{H_0}\varphi_{\varepsilon_n}(u)(x+y)\big|_{H_0}^2.$$

Hence by dominated convergence it follows that

$$\begin{aligned}\big(u^+, L^B u\big)_{L^2(E;\mu)} &\leqslant \frac{1}{2}\big\| \, |b|_{H_0} \big\|_\infty^2 \varlimsup_{n\to\infty} \int_E \varphi_{\varepsilon_n}^2\big(u(x)\big)\mu(\mathrm{d}x) \\ &= \frac{1}{2}\big\| \, |b|_{H_0} \big\|_\infty^2 (u^+, u^+)_{L^2(E;\mu)} = c'(u^+, u)_{L^2(E;\mu)},\end{aligned}$$

where $c' = \frac{1}{2}\| \, |b|_{H_0} \|_\infty^2$. □

7. Construction of the Associated Process

We still consider the situation of the previous section, so assumptions (H1), (H3)–(H5) are still in force. We want to use a general result from [20] to construct a process whose transition probabilities are given by $(P_t^B)_{t\geqslant 0}$.

According to [19, Corollary 10.6, p. 41], the adjoint semigroup $(\hat{P}_t^B)_{t\geqslant 0}$ of $(P_t^B)_{t\geqslant 0}$ is a C_0-semigroup on $L^2(E;\mu)$. Let $\hat{L}^B$ denote its generator. Define

$$\mathcal{E}(u,v) := \begin{cases} -(L^B u, v) & \text{for } u \in D(L^B),\ v \in L^2(E;\mu), \\ -(u, \hat{L}^B v) & \text{for } u \in L^2(E;\mu),\ v \in D(\hat{L}^B). \end{cases} \tag{7.1}$$

Then by [20, Example I.4.9(ii), p. 26] $\mathcal{E}$ is a generalized Dirichlet form. Since obviously W is an algebra and since W is a core for the generator of $(P_t^B)_{t\geqslant 0}$ (cf. Corollaries 5.2 and 5.4), we can apply Proposition IV.21 with $\mathcal{Y} := W$ and Theorem IV.2.2 from [20] to conclude that an associated cadlag strong Markov process exists, if we can prove that $\mathcal{E}$ is *quasi-regular* in the sense of generalized Dirichlet forms [20, Definition IV.1.7, p. 77].

It is easy to see that W contains a countable subset separating the points of E. Since, as already mentioned, $W \subseteq \mathcal{F}C_b^\infty(E)$ is *dense* in $D(L^B)$, it remains to show the existence of an $\mathcal{E}$*-nest* (again in the sense of [20, Definition III.2.3(i), p. 66]) of compact sets.

To prove this, we need an extra condition on the (2,2)-capacity of our initial generalized Mehler semigroup $(P_t)_{t\geqslant 0}$ (see (3.3)) which, however, is not so restrictive (cf. Remark 7.1 below). First, we recall the definition of the (r,p)-capacities determined by $(P_t)_{t\geqslant 0}$.

Let $r > 0$, $p \geqslant 1$. The gamma transform of $(P_t)_{t\geqslant 0}$ is defined by

$$V_r f := \frac{1}{\Gamma(\frac{r}{2})}\int_0^\infty t^{\frac{r}{2}-1}\mathrm{e}^{-t}P_t f\,\mathrm{d}t, \quad f \in L^p(E;\mu),\ r > 0,$$

where the $L^p(E;\mu)$-valued integral is taken in the sense of Bochner. Define the Banach space $(F_{r,p}, \|\cdot\|_{r,p})$ by

$$F_{r,p} := V_r\big(L^p(E;\mu)\big), \qquad \|V_r f\|_{r,p} := \|f\|_{L^p(E;\mu)}.$$

The (r,p)-capacity $C_{r,p}$ is defined for an open set $U \subseteq E$ by

$$C_{r,p}(U) := \inf\big\{\|u\|_{r,p}^p \mid u \in F_{r,p},\ u \geqslant 1 \text{ on } U\big\} \tag{7.2}$$

and for arbitrary $A \subseteq U$ by

$$C_{r,p}(A) := \inf\big\{C_{r,p}(U) \mid A \subseteq U,\ U \subseteq E,\ U \text{ open}\big\}. \tag{7.3}$$

$C_{r,p}$ is called *tight* if there exist compact $K_n \subseteq E,\ n \in \mathbb{N}$, such that $C_{r,p}(E\backslash K_n) \to 0$ as $n \to \infty$ (cf. [9] for classical references on these capacities). We now assume:

(H6) $C_{r,p}$ is tight.

REMARK 7.1. Let $\widetilde{E}$ be the Hilbert space defined in [9, Remark 6.5] which contains E as a dense subspace. Then Corollary 6.4 in [9] gives very general conditions on $(T_t)_{t\geqslant 0}$ and λ so that $C_{r,p}$ is tight on $\tilde{E}$ for all $r > 0,\ p \geqslant 1$. Note that if A is self-adjoint, then, as is easy to check, $\xi_k \in (\tilde{E})'$ $(\subseteq E')$ and (H2) will still be valid for $\tilde{E}$. So, under the assumptions of [9, Corollary 6.4] we can pass to $\tilde{E}$ and (H6) would be fulfilled. This works, in particular, in the case of the concrete example in the next section.

From now on we shall follow very closely the reasoning in [20, p. 67 ff.]. For $U \subseteq E, U$ open let

$$g_U := \inf\big\{u \in L^2(E;\mu) \mid u \geqslant 1 \text{ on } U,\ u \geqslant 0 \text{ and } \mathrm{e}^{-t} P_t^B u \leqslant u \text{ for all } t > 0\big\}.$$

Let $(G_\lambda^B)_{\lambda>0}$ denote the resolvent of $(P_t^B)_{t\geqslant 0}$, i.e. $G_\lambda^B = (\lambda\,\mathrm{Id} - L^B)^{-1}$. Consider the increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets in E from (H6). By [20, Proposition IV.2.10, p. 69], $(K_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest if

$$\mathrm{Cap}_1(K_n^c) \underset{n\to\infty}{\longrightarrow} 0,$$

where

$$\mathrm{Cap}_1(K_n^c) := (g_{K_n^c}, 1)_{L^2(E;\mu)}.$$

Therefore, it suffices to prove:

PROPOSITION 7.2. $g_{K_n^c} \underset{n\to\infty}{\longrightarrow} 0$ *in* $L^2(E;\mu)$.

Proof. By definition, there are $u_n \in F_{2,2},\ n \in \mathbb{N}$, with $u_n \geqslant 1_{K_n^c}$, such that

$$\|u_n\|_{2,2} \leqslant \frac{1}{2^n} + C_{2,2}(K_n^c).$$

But since $(\mathrm{Id} - L)^{-1} = V_2$, we have $\|u_n\|_{2,2} = \|(\mathrm{Id} - L)u_n\|_{L^2(E;\mu)}$, so that from Proposition 5.1 and Corollary 5.2 we obtain

$$\begin{aligned}\big\|(L - L^B)u_n\big\|_{L^2(E;\mu)} &= \|Bu_n\|_{L^2(E;\mu)}\\ &\leqslant K(-Lu_n, u_n)^{1/2}_{L^2(E;\mu)}\\ &\leqslant K\big((\mathrm{Id} - L)u_n, u_n\big)^{1/2}_{L^2(E;\mu)}\\ &\leqslant K\big\|(\mathrm{Id} - L)u_n\big\|^{1/2}_{L^2(E;\mu)}\|u_n\|^{1/2}_{L^2(E;\mu)}\\ &\leqslant K\big\|(\mathrm{Id} - L)u_n\big\|_{L^2(E;\mu)},\end{aligned}$$

since for all $v \in D(L)$, $\|v - Lv\|_{L^2(E;\mu)} \geqslant \|v\|_{L^2(E;\mu)}$ since $(v, Lv)_{L^2(E;\mu)} \leqslant 0$ as $(P_t)_{t\geqslant 0}$ are contractions on $L^2(E;\mu)$. Thus,

$$\begin{aligned}\big\|(\mathrm{Id} - L^B)u_n\big\|_{L^2(E;\mu)} &\leqslant (K+1)\big\|(\mathrm{Id} - L)u_n\big\|_{L^2(E;\mu)}\\ &\leqslant (K+1)\left(\frac{1}{2^n} + C_{2,2}(K_n^c)\right).\end{aligned}$$

Let $v_n := G_1^B(((\mathrm{Id} - L^B)u_n)^+)$. Then v_n is 1-excessive and

$$v_n \geqslant G_1^B\big((\mathrm{Id} - L^B)u_n\big) = u_n \geqslant 1_{K_n^c},$$

hence $v_n \geqslant g_{K_n^c}$ $(\geqslant 0)$, and for some constant $c > 0$

$$\begin{aligned}\mathrm{Cap}_1(K_n^c) &= (g_{K_n^c}, 1)_{L^2(E;\mu)}\\ &\leqslant (v_n, 1)_{L^2(E;\mu)}\\ &\leqslant \|v_n\|_{L^2(E;\mu)}\\ &= \big\|G_1^B\big(\big((\mathrm{Id} - L^B)u_n\big)^+\big)\big\|_{L^2(E;\mu)}\\ &\leqslant c\big\|\big((\mathrm{Id} - L^B)u_n\big)^+\big\|_{L^2(E;\mu)}\\ &\leqslant c\big\|(\mathrm{Id} - L^B)u_n\big\|_{L^2(E;\mu)}\\ &\leqslant c(K+1)\left(\frac{1}{2^n} + C_{2,2}(K_n^c)\right) \underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$ □

We now obtain:

THEOREM 7.3. (i) *There exists a conservative strong Markov process*

$$\mathbb{M} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (X_t)_{t\geqslant 0}, (\mathbb{P}_x)_{x\in E}\big)$$

with cadlag sample paths such that its transition probabilities, defined by[⋆]

$$p_t^B f(x) := \int_\Omega f(X_t)\,\mathrm{d}\mathbb{P}_x, \quad f \in \mathcal{B}_b(E),\ x \in E,$$

⋆ $\mathcal{B}_b(E)$ denotes the set of all bounded Borel measurable functions on E.

are given by $(P_t^B)_{t\geqslant 0}$*, i.e.* $p_t^B f$ *is a* μ*-version of* $P_t^B f$ *for all* $t > 0$, $f \in \mathcal{B}_b(E)$.

(ii) *For* $\mathcal{E}$*-q.e.* $x \in E$ (*i.e. every* x *outside a fixed* $\mathcal{E}$*-nest, cf. above*) *and all* $u \in W$

$$u(X_t) - u(X_0) - \int_0^t L^B u(X_s)\,\mathrm{d}s, \quad t \geqslant 0,$$

is an $(\mathcal{F}_t)$*-martingale under* $\mathbb{P}_x$ *with* $X_0 = x\mathbb{P}_x$*-a.s.*

Proof. Since we proved above that $\mathcal{E}$ as given by (7.1) is a generalized Dirichlet form, [20, Theorem IV.2.2] implies (i) apart from the conservativity, which in turn follows immediately, since $P_t^B 1 = 1$ for all $t \geqslant 0$.

(ii) follows from [23, Section 2.2]. □

REMARK 7.4. In fact, the process $\mathbb{M}$ in Theorem 7.3 is even more regular, namely it is "special standard" and its resolvent maps functions from $\mathcal{B}_b(E)$ to $\mathcal{E}$-quasi-continuous functions (cf. [20, Theorem IV.2.2]).

We also have the following uniqueness result:

THEOREM 7.5. *Suppose that* $\widetilde{\mathbb{M}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geqslant 0}, (\widetilde{X}_t)_{t\geqslant 0}, (\widetilde{\mathbb{P}}_x)_{x\in E})$ *is another Markov process satisfying* 7.3(ii)*, but even only under* $\widetilde{\mathbb{P}}_\mu := \int \widetilde{\mathbb{P}}_x \mu(\mathrm{d}x)$ (*hence* $\widetilde{\mathbb{P}}_\mu \circ \widetilde{X}_0^{-1} = \mu$)*, such that its transition probabilities* $(\tilde{p}_t^B)_{t\geqslant 0}$*, considered as linear operators on* $L^2(E;\mu)$ (*with domain* $\mathcal{B}_b(E)$)*, are continuous.*

Then for μ*-a.e.* $x \in E$, $\widetilde{\mathbb{P}}_x$ *has the same finite-dimensional distributions as* $\mathbb{P}_x$ (*from Theorem* 7.3).

Proof. From the assumption that $\widetilde{\mathbb{P}}_\mu$ solves the martingale problem one concludes by a standard argument that the $L^2(E;\mu)$-generator of $(\tilde{p}_t^B)_{t\geqslant 0}$ must coincide with L^B on W. Since W is a core for $(L^B, D(L^B))$, it must coincide with L^B on all of $D(L^B)$. Hence, $(\tilde{p}_t^B)_{t\geqslant 0}$ and $(p_t^B)_{t\geqslant 0}$ (from Theorem 7.3) coincide μ-a.e. □

REMARK 7.6. Theorem 7.5 can be summarized as follows: We have μ-a.e. uniqueness of Markovian selections that solve our martingale problems under the constraint that their transition semigroups are continuous on $L^2(E;\mu)$.

REMARK 7.7. The link of the *martingale problem* with the *stochastic differential equation*

$$\mathrm{d}X_t = \mathrm{d}Y_t + \big(AX_t + b(X_t)\big)\,\mathrm{d}t$$

is worked out, in the finite-dimensional case, in the classical paper [13]. On an infinite-dimensional space, the reasoning is more elaborate, as will be explained in a subsequent paper.

8. A Stochastic Heat Equation

In this section we want to apply our results to a stochastic heat equation, i.e. to

$$\mathrm{d}X_t = \mathrm{d}Y_t + [\Delta X_t + b(X_t)]\,\mathrm{d}t, \tag{8.1}$$

where Δ denotes the Laplacian on $]0,1[$ with Dirichlet boundary conditions, and $(Y_t)_{t\geqslant 0}$ is a Levy process on (an extension E of) $L^2(]0,1[)$ with characteristic function

$$\lambda(\xi) := \|\xi\|^2_{L^2([0,1[)} + \|\xi\|^\alpha_{L^2([0,1[)}, \tag{8.2}$$

for some fixed $\alpha \in]0,2[$. In other words, we are studying a *stochastic heat equation with Levy noise,* the noise being composed of a (standard) white noise and an α-stable noise.

By Bochner's Theorem for all $t \in]0,\infty]$ there exists a cylindrical measure μ_t on (the Borel σ-algebra of) $L^2(]0,1[)$ with Fourier transform given by

$$\hat{\mu}_t(\xi) = \exp\left\{-\int_0^t \left(\|\mathrm{e}^{s\Delta}\xi\|^2_{L^2(]0,1[)} + \|\mathrm{e}^{s\Delta}\xi\|^\alpha_{L^2(]0,1[)}\right)\mathrm{d}s\right\} \tag{8.3}$$

for all $\xi \in L^2(]0,1[)$ ($\simeq L^2(]0,1[)'$). Moreover, one has:

PROPOSITION 8.1. *For every $t \in]0,\infty]$, μ_t extends to a Borel probability measure on $L^2(]0,1[)$.*

Proof. Let $t \in]0,\infty]$ be fixed. It suffices to show that $\xi \mapsto \hat{\mu}_t(\xi)$ is Sazonov continuous on $L^2(]0,1[)$. The eigenvalues of Δ on $L^2(]0,1[)$ are $\lambda_j = -j^2\pi^2$, $j \geqslant 1$, each with multiplicity one, thus the eigenvalues of $\mathrm{e}^{t\Delta}$ are $\mathrm{e}^{-tj^2\pi^2}$, $j \geqslant 1$. It will appear that $\xi \mapsto N(\xi) := -\ln(\hat{\mu}_t(\xi))$ is Sazonov continuous.

Let $(e_j)_{j\geqslant 1}$ be an othonormal basis of $L^2(]0,1[)$, consisting of eigenvectors of Δ (i.e. $e_j(x) = \sqrt{2}\sin(j\pi x)$). Then one has

$$\begin{aligned} N(\xi) &= \int_0^t \left[\sum_{j=1}^\infty \mathrm{e}^{-2sj^2\pi^2}\langle\xi, e_j\rangle^2 + \left(\sum_{j=1}^\infty \mathrm{e}^{-2sj^2\pi^2}\langle\xi, e_j\rangle^2\right)^{\alpha/2}\right]\mathrm{d}s \\ &=: N_1(\xi) + N_2(\xi). \end{aligned}$$

Obviously,

$$\begin{aligned} N_1(\xi) &= \sum_{j=1}^\infty \left(\int_0^t \mathrm{e}^{-2sj^2\pi^2}\,\mathrm{d}s\right)\langle\xi, e_j\rangle^2 \\ &= \sum_{j=1}^\infty \frac{1}{2j^2\pi^2}\left(1 - \mathrm{e}^{-2tj^2\pi^2}\right)\langle\xi, e_j\rangle^2 = \|U\xi\|^2, \end{aligned}$$

where $U : L^2(]0,1[) \to L^2(]0,1[)$ is defined by

$$U(e_j) := \frac{(1 - \mathrm{e}^{-2tj^2\pi^2})^{1/2}}{j\pi\sqrt{2}}e_j, \quad j \geqslant 1.$$

As U is Hilbert–Schmidt, N_1 is Sazonov continuous by definition of the Sazonov topology.

Therefore, we only have to show the same for N_2. But

$$\begin{aligned} N_2(\xi) &= \int_0^t \left(\sum_{j=1}^{\infty} \mathrm{e}^{-2sj^2\pi^2} \langle \xi, e_j \rangle^2 \right)^{\alpha/2} \mathrm{d}s \\ &\leqslant \frac{1}{\alpha\pi^2} \int_0^{\infty} \left(\sum_{j=1}^{\infty} \mathrm{e}^{-2s(j^2-1)\pi^2} \langle \xi, e_j \rangle^2 \right)^{\alpha/2} \alpha\pi^2 \mathrm{e}^{-\alpha\pi^2 s}\, \mathrm{d}s. \end{aligned}$$

Applying Jensen's Inequality to the probability measure $\rho(\mathrm{d}t) := \alpha\pi^2\mathrm{e}^{-\alpha\pi^2 t}\,\mathrm{d}t$ on $[0, +\infty[$ and the exponent $p = 2/\alpha > 1$, we therefore get

$$\begin{aligned} \left(\alpha\pi^2 N_2(\xi)\right)^{2/\alpha} &\leqslant \left[\int_0^{\infty} \left(\sum_{j=1}^{\infty} \mathrm{e}^{-2s(j^2-1)\pi^2} \langle \xi, e_j \rangle^2 \right)^{\alpha/2} \rho(\mathrm{d}s) \right]^{2/\alpha} \\ &\leqslant \int_0^{\infty} \left(\left(\sum_{j=1}^{\infty} \mathrm{e}^{-2s(j^2-1)\pi^2} \langle \xi, e_j \rangle^2 \right)^{\alpha/2} \right)^{2/\alpha} \rho(\mathrm{d}s) \\ &\leqslant \int_0^{\infty} \left(\sum_{j=1}^{\infty} \mathrm{e}^{-2s(j^2-1)\pi^2} \langle \xi, e_j \rangle^2 \right) \rho(\mathrm{d}s) \\ &= \sum_{j=1}^{\infty} \langle \xi, e_j \rangle^2 \int_0^{\infty} \mathrm{e}^{-2s(j^2-1)\pi^2} \alpha\pi^2 \mathrm{e}^{-\alpha\pi^2 s}\, \mathrm{d}s \\ &= \sum_{j=1}^{\infty} \alpha\pi^2 \langle \xi, e_j \rangle^2 \frac{1}{\pi^2(2j^2+\alpha-2)} = \alpha \| V\xi \|^2, \end{aligned}$$

where $V : L^2(]0, 1[) \to L^2(]0, 1[)$ is defined by

$$V(e_j) := \frac{1}{\sqrt{2j^2+\alpha-2}} e_j, \quad j \geqslant 1.$$

As V is Hilbert–Schmidt, N_2 is a norm on $L^2(]0, 1[)$, and

$$0 \leqslant N_2(\xi) \leqslant \frac{\alpha^{\alpha/2-1}}{\pi^2} \| V\xi \|^{\alpha},$$

it follows that also N_2 is Sazonov continuous. □

We now set

$$\mu := \mu_{\infty}. \tag{8.4}$$

By [9, Proposition 2.3] there exists a separable Hilbert space E such that $L^2(]0,1[) \hookrightarrow E$ is Hilbert–Schmidt and the C_0 semigroup on $L^2(]0,1[)$ given by

$$T_t := \mathrm{e}^{t\Delta}, \quad t \geqslant 0,$$

extends to a C_0-semigroup on E which we again denote by $(T_t)_{t\geqslant 0}$. For any such Hilbert space E by [9, Lemma 2.7] the restriction of λ to E' ($\subseteq H'$) is Sazonov continuous. So, we are in the situation described in Section 3. In particular, (H1) holds and we can consider the corresponding generalized Mehler semigroup $(P_t)_{t\geqslant 0}$. Considering μ as a measure on E via the natural embedding $H \hookrightarrow E$, it follows by [9, Lemma 6.2] that μ is invariant for $(P_t)_{t\geqslant 0}$, so (H2) holds.

Taking instead of E the enlargement $\widetilde{E}$ described in Remark 7.1 we have that (H6) holds. It is easy to check that (H3) and (H5)(ii) also hold. Note that for our special λ in (8.2) $a_t^{(1)}$ in (H2)$'$ is identically equal to zero for all $t \geqslant 0$. This easily follows from the fact that $\hat{\mu}_t$ is real-valued and the uniqueness of the Levy–Khinchin representation. The same holds for $a_t^{(2)}$ in (H2)$'$.

Hence, also (H5) is fulfilled if we take $\hat{b} \equiv 0$ and b satisfying (H4). So, all results in this paper apply to this case. In particular, we have:

COROLLARY 8.2. *The stochastic partial differential equation* (8.1) *has a solution in the sense of Theorem* 7.3.

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