FAMILIES OF SOLVABLE FROBENIUS SUBGROUPS IN FINITE GROUPS

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Abstract. We introduce the notion of abelian system on a finite group $G$, as a particular case of the recently defined notion of kernel system (see this Journal, September 2001). Using a famous result of Suzuki on $CN$-groups, we determine all finite groups with abelian systems. Except for some degenerate cases, they turn out to be special linear group of rank 2 over fields of characteristic 2 or Suzuki groups. Our ideas were heavily influenced by [1] and [8].

§0. Introduction

The purpose of this paper is to classify all abelian Frobenius systems on finite groups. By a Frobenius system on a finite group $G$, we mean Frobenius a kernel system in the sense of [4], i.e. a mapping $F$ from the set $\mathcal{MS}(G)$ of maximal solvable subgroups of $G$ to the power set $\mathcal{P}(G)$ of $G$, such that the following conditions are satisfied, for all $M \in \mathcal{MS}(G)$:

(FS1) $F(M)$ is a normal subgroup of $M$;
(FS2) $\forall a \in M \setminus F(M), \ C_{F(M)}(a) = 1$;
(FS3) $\forall g \in G \setminus M, \ F(M) \cap F(M)^g = \{1\}$.

The Frobenius system is said to be abelian if one has in addition:

(FS4) $\forall M \in \mathcal{MS}(G), \ M/F(M)$ is abelian.

As seen in [4, Lemma 1.2], if $G$ is a nonidentity finite CA-group, then $G$ possesses a canonical Frobenius system. The proof of the aforementioned lemma even yields that this Frobenius system is abelian. In particular $SL_2(K)$, for $K$ a finite field of characteristic 2, does possess a canonical abelian Frobenius system $F_K$.

Let $n \geq 1$ be an integer; then Theorem 9 (pp. 137–138) of [6] implies that the Suzuki group $Sz(2^{2n+1})$ (there denoted by $G(q)$, where $q = 2^{2n+1}$)
possesses a Frobenius subgroup $H$ of order $q^2(q-1)$, a dihedral subgroup $B_0$ of order $2(q-1)$, and cyclic subgroups $A_1, A_2$ of respective orders $q + r + 1$ and $q - r + 1$ ($r = \sqrt{2q} = 2^{r+1}$). By an easy application of the same Theorem, the elements of $\mathcal{MS}(G)$ are exactly the conjugates of $H, B_0, B_1$ and $B_2$, where $B_i = N_G(A_i)$ is a Frobenius group of order $4|A_i|$ ($i = 1, 2$). Let $A_0$ be the subgroup of $B_0$ of order $q-1$ and $N$ the Frobenius kernel of $H$; it is easily seen that by setting, for each $g \in S_z(2^{2n+1})$:

$$\mathcal{F}_n(H^g) = N^g,$$

$$\mathcal{F}_n(B_0^g) = A_0^g,$$

$$\mathcal{F}_n(B_i^g) = A_i^g (i = 1, 2),$$

one defines an abelian Frobenius system $\mathcal{F}_n$ on $S_z(2^{2n+1})$.

There is an obvious notion of isomorphism for groups with Frobenius systems: if $\mathcal{F}_1, \mathcal{F}_2$ are Frobenius systems respectively on $G_1, G_2$, an isomorphism between $(G_1, \mathcal{F}_1)$ and $(G_2, \mathcal{F}_2)$ is by definition an isomorphism $\alpha : G_1 \rightarrow G_2$ such that:

$$\forall M \in \mathcal{MS}(G_1), \mathcal{F}_2(\alpha(M)) = \alpha(\mathcal{F}_1(M)).$$

The purpose of this work is to prove the following:

**Theorem 0.1.** Let $\mathcal{F}$ be an abelian Frobenius system on the finite group $G$; then one of the following holds:

1. $G$ is abelian and $\mathcal{F}(G) = \{1\}$.
2. $G$ is a nonidentity solvable group and $\mathcal{F}(G) = G$.
3. $G$ is a solvable Frobenius group with cyclic complement, and $\mathcal{F}(G)$ is the Frobenius kernel of $G$.
4. $(G, \mathcal{F})$ is isomorphic to $(SL_2(\mathbb{F}_{2^n}), \mathcal{F}_{\mathbb{F}_{2^n}})$ for some $n \geq 2$.
5. $(G, \mathcal{F})$ is isomorphic to $(S_z(2^{2n+1}), \mathcal{F}_{(n)})$ for some $n \geq 1$.

Clearly these possibilities are mutually exclusive, and each of them yields an abelian Frobenius system.

The notations are mostly standard, and conform to those in [4].

§1. General preliminary lemmas

For the moment, let $G$ denote an arbitrary finite group.

**Lemma 1.1.** If some element $A$ of $\mathcal{MS}(G)$ is abelian, then $G = A$ is.
Proof. Let us proceed by induction on $|G|$ (the result being trivial for $G = \{1\}$). We may assume that $A \neq G$; then, for each subgroup $H$ with $A \subset H \subset G$, one has $A \in \mathcal{MS}(H)$, whence (by the induction hypothesis applied to $H$) $H = A$ – this means that $A$ is a maximal subgroup of $G$. By a Theorem of Herstein ([2]), $G$ is solvable; but then $\mathcal{MS}(G) = \{G\}$ and $A = G$, a contradiction.

From now on, let $\mathcal{F}$ denote an abelian Frobenius system on the finite group $G$.

**Lemma 1.2.** Let us suppose that

$$\forall M \in \mathcal{MS}(G), \quad \mathcal{F}(M) \neq M.$$  

Then every nonabelian Sylow subgroup of $G$ is a TI-set.

Proof. Let us assume that the Sylow $q$-subgroup $Q$ of $G$ is not abelian. There is an $M \in \mathcal{MS}(G)$ with

$$Q \subseteq M;$$

as

$$Q/Q \cap \mathcal{F}(M) \simeq Q\mathcal{F}(M)/\mathcal{F}(M) \subseteq M/\mathcal{F}(M),$$

$Q/Q \cap \mathcal{F}(M)$ is abelian according to (FS4). Therefore $Q \cap \mathcal{F}(M) \neq \{1\}$, so $q$ divides $|\mathcal{F}(M)|$. As $\mathcal{F}(M)$ is a Hall subgroup of $M$ (see [4, Corollary 1.4]), it follows that $q$ does not divide the order of $M/\mathcal{F}(M)$. But $Q\mathcal{F}(M)/\mathcal{F}(M)$ is a $q$-subgroup of $M/\mathcal{F}(M)$, hence $Q\mathcal{F}(M)/\mathcal{F}(M) = \{1\}$ and:

$$Q \subseteq \mathcal{F}(M).$$

But $\mathcal{F}(M)$ is nilpotent ([4, Proposition 1.5]), therefore $Q = O_q(\mathcal{F}(M)) \triangleleft M$ and $M \subseteq N_G(Q)$. If $Q \cap Q^x \neq \{1\}$, then $\mathcal{F}(M) \cap \mathcal{F}(M)^x \neq \{1\}$ (because $Q \subseteq \mathcal{F}(M)$), hence $x \in M$ (FS3), whence $x \in N_G(Q)$ and $Q = Q^x$: $Q$ is a TI-set.

§2. The proof of Theorem 0.1

Let $\mathcal{F}$ be an abelian Frobenius system on the finite group $G$. If $\mathcal{F}(M) = \{1\}$ for some $M \in \mathcal{MS}(G)$, then $M \simeq M/\mathcal{F}(M)$ is abelian (by (FS4)). But Lemma 1.1 now yields that $G = M$, and we are in case $(1)$. If $\mathcal{F}(M) = M \neq \{1\}$ for some $M \in \mathcal{MS}(G)$, then either $G = M$ (hence we are in
case (2)), or (according to (FS3)) \(M\) is a Frobenius complement in \(G\). By Frobenius’ Theorem, \(G\) possesses a Frobenius kernel \(N\), and, by [5, 12.6.13, p. 354], \(N\) is nilpotent. Therefore \(N\) and

\[
G/N = MN/N \simeq M/M \cap N \simeq M
\]

are solvable, hence so is \(G\); but then \(\mathcal{M}(G) = \{G\}\) and \(M = G\), a contradiction. Therefore we may assume that:

\[
\forall M \in \mathcal{M}(G), \{1\} \neq F(M) \neq M.
\]

It now follows from (FS1) and (FS2) that \(F(M)\) is a Frobenius kernel in \(M\) for all \(M \in \mathcal{M}(G)\).

**Lemma 2.1.** \(G\) is a CN-group.

**Proof.** Let \(x \in G^2\), and let \(S \in \mathcal{M}(C_G(x))\); there is \(M \in \mathcal{M}(G)\) with \(S \subseteq M\). If \(S\) is abelian, then, by Lemma 1.1, \(C_G(x) = S\) is abelian, hence nilpotent. Else one has \(S \cap F(M) \neq \{1\}\) (because of (FS4)); let \(u \in (S \cap F(M))^2\). One has \(u \in S \subseteq C_G(x)\), whence \(x \in C_G(u)\); but \(C_G(u) \subseteq F(M)\) by Lemma 1.3 of [4], whence \(x \in F(M)\). A second application of the same Lemma now yields \(C_G(x) \subseteq F(M)\); but \(F(M)\) is nilpotent according to Proposition 1.5 of [4], hence so is \(C_G(x)\).

If \(G\) is solvable, then \(\mathcal{M}(G) = \{G\}\) and \(F(G)\) is a Frobenius kernel in \(G\), with abelian complement by (FS4); it is well-known that such a complement is necessarily cyclic (this follows from [5, 12.6.15, p. 356]), and we are in case (3). Otherwise, \(G\) is a nonsolvable CN-group; by the main results of [6] and [7], \(G\) is therefore isomorphic to \(SL_2(F_{2^n})\) for some \(n \geq 2\), \(Sz(2^{2n+1})\) for some \(n \geq 1\), or \(M_9\) (a nonsimple, nonsolvable group of order 1440). But the Sylow 2-subgroups of \(M_9\) are not abelian, otherwise so would be those of the alternating group \(A_6\) which is a section of \(M_9\) – a contradiction, as these last are dihedral of order 8; but they are not TI-sets either ([6]), whence, by Lemma 1.2, \(G\) is not isomorphic to \(M_9\). Therefore we may assume that \(G = SL_2(F_{2^n})\) \((n \geq 2)\) or \(G = Sz(2^{2n+1})\) \((n \geq 1)\). But we have seen that, for \(M \in \mathcal{M}(G)\), \(F(M)\) was a Frobenius kernel for \(M\), and a finite group does possess at most one Frobenius kernel ([5, (12.6.12), p. 354]), therefore \(F\) is uniquely determined, and so has to be \(F_{F_{2^n}}, \xi\) (resp. \(F_{(n)}, \xi\)).
REFERENCES


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