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THE MARTINGALE PROBLEM FOR PSEUDO-DIFFERENTIAL OPERATORS ON INFINITE-DIMENSIONAL SPACES

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Abstract. A martingale problem for pseudo-differential operators on infinite dimensional spaces is formulated and the existence of a solution is proved. Applications to infinite dimensional "stable-like" processes are presented.

§0. Introduction

The purpose of this paper is to formulate and solve a martingale problem for pseudo-differential operators on infinite dimensional state space. We thus provide a routine machinery to construct (non-trivial) infinite dimensional processes which are merely càdlàg and have not been obtained before by other means. We emphasize, however, that this work is only on existence of solutions to these martingale problems, not on uniqueness. The uniqueness, which is known to be already extremely difficult in infinite dimensions if we are merely dealing with "differential" (i.e., local) operators, will be studied in a forthcoming paper. The main motivation of the present work (see [FuR 97] and also [BRS 96]) is to make a contribution to the development of a theory of pseudo-differential operators in infinite dimensions, since this theory has proved to be so powerful in finite dimensions, e.g. in proving index theorems.

The state spaces E treated here are duals to countably nuclear spaces and the definition of a pseudo-differential operator $p(\cdot, D)$ on E is taken from [Hr82, Hr87]. There are several possibilities to introduce pseudo-differential operators in infinite dimensions. One of them is based on the

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idea to use Fourier transforms of measures as test functions. The approach to Feynman integrals developed in [AH-K76, AH-K77] was based on this idea. Earlier in [Fom68] infinite-dimensional differential operators were considered in the framework of the duality between spaces of functions and spaces of measures (replacing the finite-dimensional duality between the Schwartz test function space S and its dual S'). This latter approach was then further developed by several authors. The most convenient version for us is that due to A. Yu. Hrennikov (cf. [Hr82] and also [Hr87] and the references therein). In particular, the domain of $p(\,\cdot\,, D)$ is hence a space of Fourier transforms of measures. Thus one can avoid the use of a reference measure, as Lebesgue measure is in finite dimensions (which does not exist on our state space E if $\dim E = \infty$). We refer to Sections 1 and 2 for precise details, but emphasize here that it definitely is a natural and direct way to extend the definition of a pseudo-differential operator to infinite dimensions. A further justification for this from a probabilistic point of view is given by the results of this paper.

After discussing examples (cf. Examples 2.2 below) and giving the necessary probabilistic definitions at the end of Section 2, we then prove the existence of the solution to the martingale problem for $p(\,\cdot\,, D)$ on E in Section 3. We thus extend fundamental work of D.W. Stroock (cf. [St75]) and, in particular, the more recent work by W. Hoh (cf. [H92, H94], see also [H95a, H95b]). The problems in the proof arise in part from the non-metrizability of E and hence of D_E (i.e., the space of càdlàg paths from $\mathbb{R}_+$ to E equipped with the Skorohod topology). But here we benefit a lot from the tightness results in [M83] resp. [J86], i.e., the characterization of tightness of laws of càdlàg processes on E in terms of tightness of the laws of their "one-dimensional component processes." By these results the approximation method in [H92, H94] can be shown to extend to the infinite dimensional case. We particularly concentrate on the problems which arise from the measure theory on D_E which a-priori is much more difficult than that on $D_{\mathbb{R}^n}$ (but was well-analyzed in [J86]). Another substantial difficulty we had to face in our proof is the lack of "localizing" functions on E with *uniformly* bounded images under $p(\,\cdot\,, D)$. The existence of such functions is crucial in the proof of Hoh (cf. [H92, Lemma 2.10]) in the case $E = \mathbb{R}^n$. We overcome this difficulty by proving the existence of suitable localizing functions for "one-dimensional components of the processes" (cf. Claim 1 in Section 3) which turns out to be sufficient. Finally, it should be emphasized that the reduction to one-dimensional components in order to prove tightness according to [M83], [J86], unfortunately does not take us back to the finite (or even one-) dimensional situation which was solved in [H92, H94]. The reason is that these components do *not* solve a one-dimensional martingale problem, since the operator $p(\,\cdot\,, D)$ obviously introduces interactions between the different components and since the underlying filtration $(\mathcal{F}_t)_{t\geq 0}$ has to be the one generated by the full process and not the one generated by the respective component.

The results of this paper have been announced at conferences in Warwick and Bielefeld in Summer 1994, as well as in several invited talks e.g. at the University of California, San Diego, October 1994, and the Mittag-Leffler-Institute, Stockholm, March 1995.

§1. Definitions, notation and preliminary results

In this paper E will denote the topological dual F' of a real countably nuclear Fréchet space F endowed with the strong topology. We equip the topological dual E' of E with the topology of uniform convergence on bounded sets. Then $E' = F$ (as topological vector spaces; cf. e.g. [GV 64, p. 61] or [S71, Chap. III, §7 and Chap. IV, §5]). Let $\mathcal{B}(E)$, $\mathcal{B}(E')$ denote the Borel σ-algebras of E, E' respectively.

Remark 1.1. Since E' $(= F)$ is separable (see e.g. [GV 64, p. 73]), there exist $\xi_n \in E'$, $n \in \mathbb{N}$, separating the points of E. Since E is a Lusin space (i.e., the continuous one-to-one image of a Polish space, cf. [Sch73]), by [Sch73, Lemma 18, p. 108] we have that $\mathcal{B}(E) = \sigma(\xi_n | n \in \mathbb{N})$.

Let $M := \mathcal{M}_b^{\mathbb{C}}(E')$ be the space of complex-valued measures on $\mathcal{B}(E')$ with bounded total variation, and $\langle\ ,\ \rangle$ the dualization between E and E', i.e.,

$$\langle x, \xi\rangle = \xi(x), \qquad x \in E,\ \xi \in E'.$$

We recall that the Fourier transform of $\mu \in M$ is defined by

$$\mathcal{F}(\mu)(x) := \int_{E'} e^{i\langle x,\xi\rangle}\mu(d\xi), \qquad x \in E\ .$$

We set $W := \mathcal{F}(M)$. W will play the role of a "test-function" space (cf. [AH-K76, AH-K77] and also [Hr82, p. 779]). Since $\mathcal{F}$ is one-to-one $\mathcal{F}^{-1}\colon W \to M$ is well-defined. Let $C_b(E;\mathbb{C})$ denote the space of all bounded continuous complex valued functions on E. The following is standard for the type of state spaces considered here.

LEMMA 1.2.

$$W \subset C_b(E;\mathbb{C}).$$

Proof. see [Sch73, Theorem 3, p. 239 or Theorem 1, p. 193].

We define the set of cylinder functions $\mathcal{F}S(E;\mathbb{C})$ with base functions in $S(\mathbb{R}^m;\mathbb{C})$ (:= the space of complex-valued Schwartz test functions) for some $m \in \mathbb{N}$ by

$$\mathcal{F}S(E;\mathbb{C}) := \{f(\xi_1,\dots,\xi_m) \mid m \in \mathbb{N},\ f \in S(\mathbb{R}^m;\mathbb{C}), \xi_1,\dots,\xi_m \in E'\}.$$

LEMMA 1.3. *$\mathcal{F}S(E;\mathbb{C}) \subset W$.*

Proof. Let $\varphi \in \mathcal{F}S(E;\mathbb{C})$, $\varphi = f(\xi_1,\dots,\xi_m)$, $m \in \mathbb{N}$, $f \in S(\mathbb{R}^m;\mathbb{C})$ and $\xi_1,\dots,\xi_m \in E'$. Then there exists $g \in S(\mathbb{R}^m,\mathbb{C})$ such that

$$f(x) = \int_{\mathbb{R}^m} e^{i\langle x,y\rangle}\, g(y)\, dy$$

and $\nu_m := g\, dx$ is a finite measure on $\mathbb{R}^m$. Let ν be its image measure under the map $T_m\colon \mathbb{R}^m \to E'$ defined by

$$T_m(y_1,\dots,y_m) = \sum_{i=1}^m y_i\xi_i, \qquad (y_1,\dots,y_m) \in \mathbb{R}^m.$$

Then one has that $\nu \in M$ and that

$$\begin{aligned}\forall x \in E \quad \mathcal{F}(\nu)(x) &= \int_{E'} e^{i\langle x,\xi\rangle}\nu(d\xi)\\ &= \int_{\mathbb{R}^m} e^{i\langle x,T_m(y)\rangle}\nu_m(dy)\\ &= \int_{\mathbb{R}^m} e^{i\sum_{i=1}^m y_i\xi_i(x)}\nu_m(dy)\\ &= f(\xi_1(x),\dots,\xi_m(x)) = \varphi(x).\end{aligned}$$

Therefore, $\varphi = \mathcal{F}(\nu) \in \mathcal{F}(M) = W$.

The following result is more or less well-known. We include the proof for the reader's convenience.

LEMMA 1.4. *There exists a countable subset W_0 of $\mathcal{F}S(E;\mathbb{C})(\subset W)$ separating the points of E. In particular, $\sigma(W_0) = \mathcal{B}(E)$.*

Proof. By the Hahn-Banach theorem E' separates the points of E, hence so does $\mathcal{F}S(E;\mathbb{C})$. Since $E \times E$ is strongly Lindelöf (cf. [Sch73, Proposition 4, p. 105 and Proposition 3, p. 104.]), there exists a countable subset thereof with the same property. The last assertion follows by [Sch73, Lemma 18, p. 108].

§2. Pseudo-differential operators on E and corresponding martingale problems

We assume that we are given a $\mathcal{B}(E')$-measurable function $A\colon E' \to \mathbb{R}_+$ such that

(A1) $A(0) = 0$.

(A2) For each $\xi \in E'$, $t \mapsto A(t\xi)$ is continuous in $0 \in \mathbb{R}$.

(A3) For any $\xi_1, \dots, \xi_m \in E'$ there exist constants $c = c(\xi_1, \dots, \xi_m)$, $q = q(\xi_1, \dots, \xi_m) > 0$ such that

$$\forall t = (t_1, \dots, t_m) \in \mathbb{R}^m, \qquad A(t_1\xi_1 + \dots + t_m\xi_m) \le c(1 + |t|^q).$$

We then set

$$\mathcal{L}_A := \left\{ \varphi \in W \;\middle|\; \int_{E'} A(\xi)\, |\mathcal{F}^{-1}(\varphi)|\,(d\xi) < +\infty \right\}.$$

A map $p\colon E \times E' \to \mathbb{R}$ is called an *A-symbol* if the following conditions are satisfied:

(C1) For each $x \in E$, $\xi \mapsto p(x,\xi)$ is a continuous *negative definite function* on E' (cf. [BeF75], the local compactness assumption made there can be dropped), and, for each $\xi \in E'$, $x \mapsto p(x,\xi)$ is continuous.

(C2) $\forall x \in E$, $\forall \xi \in E'$ $|p(x,\xi)| \le A(\xi)$.

Note that by (C2) $p(x,0) = 0$, and that $p \ge 0$ since it is real-valued, which also implies that $p(x,-\xi) = p(x,\xi)$ (cf. [BeF75, Chap. II, §7]). We then define $p(\,\cdot\,,D)\colon \mathcal{L}_A \to C_b(E;\mathbb{C})$ by

$$p(x,D)\varphi(x) = \int_{E'} p(x,\xi) e^{i\langle x,\xi\rangle} \mathcal{F}^{-1}(\varphi)(d\xi).$$

Remark 2.1. (i) Let λ be a probability measure on $\mathcal{B}(E)$. Then

$$A(\xi) := \mathrm{Real}\left(1 - \int_E e^{i\langle x,\xi\rangle}\lambda(dx)\right), \qquad \xi \in E'$$

is a bounded negative definite function on E' (cf. [BeF75, Chap. II, §7]) such that $A(0) = 0$. In particular, $A(\xi) \geq 0$ for all $\xi \in E'$ and also (A2), (A3) are satisfied. Clearly, A itself is then an A-symbol (more precisely, $(x,\xi) \mapsto A(\xi)$ is one).
(ii) Obviously, if A satisfying (A1)–(A3) is bounded, then $\mathcal{L}_A = W$. Moreover, by (A3) it follows immediately that in any case $\mathcal{F}S(E;\mathbb{C}) \subset \mathcal{L}_A$.

Examples. (i) The easiest examples for p are the following. Let $n \in \mathbb{N}$ and

$$p(x,\xi) := \sum_{i=1}^{n} f_i(x)\, A_i(\xi),$$

where A_i is as A in Remark 2.1 (i) above and $f_i \in C_b(E;\mathbb{C})$. Also, obviously, appropriate limits of such p provide examples.
(ii) There is an explicit class of symbols, intensively studied in finite dimensions (cf. [B88], [JaL93], [KN], [N94], [Ts92]) and also occuring in infinite dimensions, which provides interesting non-trivial examples to which our main result (cf. Section 3 below) applies. As a consequence we obtain natural generalizations of the familiar infinite dimensional stable processes in [W84]. Let $\alpha\colon E \to (0,2]$ be a continuous function and let σ be a finite positive measure with compact support K in E. Suppose that $\alpha(x) \geq \alpha_0$ for all $x \in E$ and for some $\alpha_0 \in (0,\infty)$.
Define

$$p(x,\xi) := \int_K |\langle y,\xi\rangle|^{\alpha(x)}\sigma(dy), \qquad x \in E,\ \xi \in E'.$$

Let us verify that all our conditions are satisfied. Clearly, $x \mapsto p(x,\xi)$ is continuous for every fixed ξ. In addition, $\xi \mapsto p(x,\xi)$ is continuous for every fixed x, since σ has compact support and the topology of E' is the topology of uniform convergence on bounded sets in E. We claim that $\xi \mapsto p(x,\xi)$ is negative definite for every fixed x. Indeed, take a finite collection $\xi_1,\dots,\xi_n \in E'$. Passing to the image of σ under the mapping $y \mapsto (\xi_1(y),\dots,\xi_n(y))$, we easily see that the claim follows from the negative definiteness of $s \mapsto |s|^{\alpha(x)}$ on $\mathbb{R}^1$. Finally, it has to be shown that there exists an admissible function A such that $|p(x,\xi)| \leq A(\xi)$ for all x and all

ξ. Denote by K_1 and K_2 the sets of y's such that $|\langle y, \xi\rangle| \geq 1$ and $|\langle y, \xi\rangle| \leq 1$ respectively. We have:

$$\begin{aligned}\int_K |\langle y,\xi\rangle|^{\alpha(x)}\sigma(dy) &\leq \int_{K_1} |\langle y,\xi\rangle|^{\alpha(x)}\sigma(dy) + \int_{K_2} |\langle y,\xi\rangle|^{\alpha(x)}\sigma(dy)\\ &\leq \int_K |\langle y,\xi\rangle|^2\sigma(dy) + \int_K |\langle y,\xi\rangle|^{\alpha_0}\sigma(dy).\end{aligned}$$

Clearly, the expression on the right as a function of ξ satisfies conditions (A1)–A(3). By Theorem 3.1 below we hence have an associated process as the solution of the corresponding martingale problem for $(-p(\cdot, D), \mathcal{L}_A)$ and any initial measure μ. If α is constant this process is one of the well-known infinite dimensional stable processes in [W84].

Let D_E be the set of all E-valued càdlàg paths, i.e.,

$$D_E := \left\{\omega\colon \mathbb{R}_+ \to E \;\middle|\; \forall t \geq 0 \;\lim_{\substack{s\to t\\ s>t}} \omega(s) = \omega(t) \text{ and } \forall t > 0 \;\lim_{\substack{s\to t\\ s<t}} \omega(s) \text{ exists}\right\}$$

equipped with the usual (projective limit) Skorohod topology described in [J86, Sect. 4]. For $t \geq 0$, we define

$$\begin{aligned}&X_t : D_E \longrightarrow E\\ &X_t(\omega) := \omega(t), \qquad \omega \in D_E,\end{aligned}$$

and then the filtration

$$\begin{aligned}\mathcal{F}_t &:= \sigma(X_s \mid 0 \leq s \leq t), \quad t \geq 0\\ \mathcal{F}_\infty &:= \sigma(X_s \mid s \geq 0).\end{aligned}$$

We denote the set of all probability measures on $(D_E, \mathcal{F}_\infty)$ by $\mathcal{M}_1(D_E)$.

Remark 2.2. Note that by [J86, Proposition 5.3 (ii)] $\mathcal{F}_\infty$ coincides with the Borel σ-algebra on D_E. Furthermore, it immediately follows from [J86, Propositions 5.3 (i) and 1.6 (ii), (iii)] and the definition of the topology of D_E that D_E is a Lusin space. Hence for any countably generated sub-σ-algebra Σ of $\mathcal{F}_\infty$ by [StV79, Theorems 1.1.6, 1.1.8] a *regular conditional probability distribution* exists.

Let $A\colon E' \to \mathbb{R}_+$ satisfy (A1)–(A3), let p be an A-symbol and μ a probability measure on $\mathcal{B}(E)$. By a *solution to the martingale problem for* $(-p(\,\cdot\,,D),\mathcal{L}_A)$ *with initial measure* μ we shall mean a measure $P \in \mathcal{M}_1(D_E)$ such that

(i) $(X_0)_*P$ $(:= P \circ X_0^{-1}) = \mu$.

(ii) For each $\varphi \in \mathcal{L}_A$,

$$\varphi(X_t) + \int_0^t (p(\,\cdot\,,D)\varphi)(X_s)\,ds, \qquad (t \geq 0)$$

is a P-martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

§3. The existence theorem

THEOREM 3.1. *Let $A\colon E' \to \mathbb{R}_+$ satisfy (A1)–(A3) in Section 2, let p be an A-symbol, and let μ be a probability measure on $\mathcal{B}(E)$. Then the martingale problem for $(-p(\,\cdot\,,D),\mathcal{L}_A)$ with initial measure μ has a solution.*

The proof of Theorem 3.1 will be carried out through a series of lemmas and propositions. We start with the following easy relation

$$\text{(1)} \qquad \forall y \in \mathbb{R}_+ \quad \frac{ky}{y+k} = \int_0^{+\infty} (1-e^{-sy})\,k^2 e^{-ks}\,ds$$

which is actually a particular case of the representation theorem on Bernstein functions [BeF75, p. 64, Theorem 9.8]. For each $x \in E$ and each $s \geq 0$, by Schoenberg's Theorem $e^{-sp(x,\,\cdot\,)}$ is a positive definite function on E' (see [BeF75, Theorem 7.8]; as before the local compactness hypothesis made in [BeF75] on E is not used there). Therefore, (since $\xi \mapsto p(x,\xi)$ is continuous and $p(x,0) = 0$) by the Bochner-Minlos Theorem, there is a probability measure $\mu_{s,x}$ on E such that

$$\forall \xi \in E', \qquad \int_E e^{i\langle y,\xi\rangle}\mu_{s,x}(dy) = e^{-sp(x,\xi)}.$$

By a monotone class argument we conclude that $\mu_{s,x}(dy)$, $s \in \mathbb{R}_+$, $x \in E$ defines a kernel from $\mathbb{R}_+ \times E$ to E. Therefore, for each $k \in \mathbb{N}$ we can define a kernel $\mu_k(x,dy)$, $x \in E$, from E to E by

$$\mu_k(x,\,\cdot\,) = \int_0^{+\infty} \mu_{s,x}\,k^2 e^{-ks}\,ds.$$

LEMMA 3.2. *Let $k \in \mathbb{N}$, $x \in E$. Then $\mu_k(x, E) = k$.*

Proof. Since each $\mu_{s,x}$ is a probability measure we have that

$$\mu_k(x, E) = \int_0^{+\infty} \mu_{s,x}(E) k^2 e^{-ks}\, ds = k.$$

LEMMA 3.3. *For each $k \in \mathbb{N}$, each $x \in E$ and each $\xi \in E'$, one has*

$$\int_E (1 - \cos\langle y, \xi\rangle)\, \mu_k(x, dy) = \frac{kp(x,\xi)}{k + p(x,\xi)}.$$

Proof.

$$\begin{aligned}
&\int_E (1 - \cos\langle y, \xi\rangle)\, \mu_k(x, dy) \\
&\quad = \int_0^{+\infty} \left(\int_E (1 - \cos\langle y, \xi\rangle)\, \mu_{s,x}(dy) \right) k^2 e^{-ks}\, ds \\
&\quad = \int_0^{+\infty} \left(\int_E \Big(1 - \frac{1}{2} e^{i\langle y,\xi\rangle} - \frac{1}{2} e^{-i\langle y,\xi\rangle}\Big) \mu_{s,x}(dy) \right) k^2 e^{-ks}\, ds \\
&\quad = \int_0^{+\infty} \Big(1 - \frac{1}{2} e^{-sp(x,\xi)} - \frac{1}{2} e^{-sp(x,-\xi)}\Big) k^2 e^{-ks}\, ds \\
&\quad = \int_0^{+\infty} (1 - e^{-sp(x,\xi)})\, k^2 e^{-ks}\, ds \\
&\quad = \frac{kp(x,\xi)}{k + p(x,\xi)}
\end{aligned}$$

where we used (1) in the last step.

As in [H94] we consider $p_k(x, \xi) := \frac{kp(x,\xi)}{k+p(x,\xi)}$ and for $x \in E$ let $\tau_x(z) := z + x$, $z \in E$. Define a kernel of probability measures $\widetilde{\mu}_k(x, dy)$ (cf. Lemma 3.2) by

$$\widetilde{\mu}_k(x, dy) = \frac{1}{k} (\tau_x)_* \mu_k(x, dy).$$

For any $\varphi \in W$ and $\nu := \mathcal{F}^{-1}(\varphi)$ we then have for all $x \in E$ that

$$\begin{aligned}
&-p_k(x, D)\, \varphi(x) \\
&\quad = - \int_{E'} e^{i\langle x,\xi\rangle} p_k(x, \xi)\, \nu(d\xi)
\end{aligned}$$

$$= -\int_{E'} e^{i\langle x,\xi\rangle}\left(\int_E (1-\cos\langle y,\xi\rangle)\mu_k(x,dy)\right)\nu(d\xi)$$

(by Lemma 3.3)

$$= -\int_E \left(\int_{E'} e^{i\langle x,\xi\rangle}(1-\cos\langle y,\xi\rangle)\nu(d\xi)\right)\mu_k(x,dy)$$

$$= -\int_E \left(\int_{E'} \left(e^{i\langle x,\xi\rangle} - \frac{1}{2}e^{i\langle x+y,\xi\rangle} - \frac{1}{2}e^{i\langle x-y,\xi\rangle}\right)\nu(d\xi)\right)\mu_k(x,dy)$$

$$= -\int_E \Big(\varphi(x) - \frac{1}{2}\varphi(x+y) - \frac{1}{2}\varphi(x-y)\Big)\mu_k(x,dy)$$

because $\varphi = \mathcal{F}(\nu)$. But $\mathcal{F}\mu_{s,x}(-\xi) = e^{-sp(x,-\xi)} = e^{-sp(x,\xi)} = \mathcal{F}\mu_{s,x}(\xi)$, therefore the $\mu_{s,x}$ are symmetric, and so is $\mu_k(x,\,\cdot\,)$. Thus

$$\begin{aligned}
-p_k(x,D)\varphi(x) &= -\int_E \Big(\varphi(x) - \frac{1}{2}\varphi(x+y) - \frac{1}{2}\varphi(x-y)\Big)\mu_k(x,dy)\\
&= \int_E \varphi(x+y)\,\mu_k(x,dy) - \mu_k(x,E)\,\varphi(x)\\
&= \int_E \varphi(y)\,k\,\widetilde{\mu}_k(x,dy) - k\,\varphi(x)\\
&= k\int_E (\varphi(y)-\varphi(x))\,\widetilde{\mu}_k(x,dy).
\end{aligned}$$

Therefore, $-p_k(x,D)$ maps real functions into real functions and extends to a bounded operator on $\mathcal{B}_b(E)$. By the arguments of [EK86, Chap. 4, Proposition 1.7 and Subsection 2, pp. 162-164] (note that the assumption that E is metrizable is not used here), one has a solution $P_k \in \mathcal{M}_1(D_E)$ to the martingale problem for $(-p_k(\,\cdot\,,D),\mathcal{L}_A)$ and μ. (Here $\mathcal{L}_A$ can even be replaced by W since p_k is bounded.)

PROPOSITION 3.4. *$(P_k)_{k\in\mathbb{N}}$ is tight.*

Proof. By a result of Mitoma ([M83, p. 993, Theorem 4.1], cf. also [J86, Theorem 5.5]) it is sufficient to prove that, for any $\xi \in E'$, the family $((U_\xi)_* P_k)_{k\in\mathbb{N}}$ of measures on $D_\mathbb{R}$ is tight, where $U_\xi\colon D_E \to D_\mathbb{R}$ is defined by:

$$\forall \omega \in D_E, \qquad U_\xi(\omega) := \xi\circ\omega.$$

We shall closely follow Hoh's arguments in [H92, Subsection 2.2] (see also [H94]). Let $\chi \in C_0^\infty(\mathbb{R};\mathbb{R})$ be a function such that $0\le\chi\le 1$, $\chi = 1$ on

$[-\frac{1}{2}, \frac{1}{2}]$, and $\chi = 0$ on $\mathbb{R} \setminus]-1,1[$. There is a measure ν_0 on $\mathbb{R}$ such that

$$\chi(s) = \int_{\mathbb{R}} e^{i\lambda s} \nu_0(d\lambda), \qquad s \in \mathbb{R}.$$

(Actually, $\nu_0(d\lambda) = g(\lambda) d\lambda$ for some $g \in S(\mathbb{R};\mathbb{C})$). Let $T_1: \mathbb{R} \to E'$ be defined by $T_1(\lambda) := \lambda\xi$. For $R > 0$ let $j_R: \mathbb{R} \to \mathbb{R}$ be defined by $j_R(s) := s/R$, $s \in \mathbb{R}$ and let $\nu_R := (T_1 \circ j_R)_* \nu_0$. Clearly, $\nu_R \in M$, hence $\varphi_R := \mathcal{F}(\nu_R) \in W$. Let $x_0 \in E$ and define $\varphi_{R,x_0}: E \to \mathbb{R}$, by $\varphi_{R,x_0}(x) := \varphi_R(x - x_0)$. Then

$$\begin{aligned} &\varphi_{R,x_0} = \mathcal{F}(\nu_{R,x_0}) \in W, \\ \text{where} \qquad &\nu_{R,x_0}(d\alpha) = e^{-i\langle x_0, \alpha\rangle} \nu_R(d\alpha). \end{aligned}$$

In fact, $\forall x \in E$

$$\begin{aligned} \varphi_{R,x_0}(x) &= \int_{E'} e^{i\langle x,\alpha\rangle} e^{-i\langle x_0,\alpha\rangle} \nu_R(d\alpha) \\ &= \int_{\mathbb{R}} e^{i\langle x - x_0, T_1(j_R(\lambda))\rangle} \nu_0(d\lambda) \\ &= \int_{\mathbb{R}} e^{i\xi(x-x_0)\frac{\lambda}{R}} \nu_0(d\lambda) \\ &= \chi\left(\frac{\xi(x) - \xi(x_0)}{R}\right). \end{aligned} \tag{2}$$

CLAIM 1.

$$\sup_{k\in\mathbb{N}} \sup_{x,x_0 \in E} |p_k(x,D)\, \varphi_{R,x_0}(x)| \le c_1 \sup_{|t| \le \frac{1}{\sqrt{R}}} A(t\xi) + c(\xi) \left(\frac{c_2}{R} + \frac{c_{p(\xi)}}{R^{p(\xi)}}\right)$$

where $c_p := \int |\lambda|^p g(\lambda)\, d\lambda$ *for* $p > 0$ *and* $c(\xi)$, $p(\xi)$ *is as in* (A3).

Proof of Claim 1.

$$\begin{aligned} |p_k(x,D)\, \varphi_{R,x_0}(x)| &= \left| \int_{E'} e^{i\langle x - x_0, \alpha\rangle} p_k(x,\alpha)\, \nu_R(d\alpha) \right| \\ &\le \int_{E'} p_k(x,\alpha)\, \nu_R(d\alpha) \\ &\le \int_{E'} A(\alpha)\, \nu_R(d\alpha) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} A(T_1(j_R(\lambda)))\,\nu_0(d\lambda) \\
&= \int_{\mathbb{R}} A\Big(\frac{\lambda\xi}{R}\Big)\nu_0(d\lambda) \\
&= \int_{\mathbb{R}} A\Big(\frac{\lambda\xi}{R}\Big) g(\lambda)\,d\lambda.
\end{aligned}
$$

We split the last integral in two parts, and get

$$
\begin{aligned}
&|p_k(x,D)\varphi_{R,x_0}(x)| \\
&\quad \le \int_{|\lambda|\le\sqrt{R}} A\Big(\frac{\lambda\xi}{R}\Big)|g(\lambda)|\,d\lambda + \int_{\lambda>\sqrt{R}} A\Big(\frac{\lambda\xi}{R}\Big)|g(\lambda)|\,d\lambda \\
&\quad \le \Big(\sup_{|t|\le\frac{1}{\sqrt{R}}} A(t\xi)\Big)\int_{\mathbb{R}}|g(\lambda)|\,d\lambda + c(\xi)\int_{|\lambda|>\sqrt{R}}\Big(1+\frac{|\lambda|^{p(\xi)}}{R^{p(\xi)}}\Big)|g(\lambda)|\,d\lambda \\
&\quad \le c_1 \sup_{|t|\le\frac{1}{\sqrt{R}}} A(t\xi) + c(\xi)\int_{|\lambda|>\sqrt{R}}\Big(\frac{\lambda^2}{R}+\frac{|\lambda|^{p(\xi)}}{R^{p(\xi)}}\Big)|g(\lambda)|\,d\lambda.
\end{aligned}
$$

□

Now we are prepared to prove the following two claims which by [EK86, p. 128] imply the assertion.

CLAIM 2. *Let $T \ge 0$. Then* $\lim_{R\to\infty}\sup_{k\in\mathbb{N}} P_k[\sup_{0\le t\le T}|\xi(X_t)| > R] = 0$.

CLAIM 3. *Let $T \ge 0$. Then for each $\epsilon > 0$*

$$\lim_{\delta\to 0}\sup_{k\in\mathbb{N}} P_k[\{w'(\,\cdot\,,\delta,T) > \epsilon\}] = 0$$

where for $\omega \in D_E$

$$w'(\omega,\delta,T) := \inf_{\mathcal{Z}(\delta,T)} \sup_{\substack{s,t\in[t_i,t_{i+1}[\\ i=0,\dots,m\\ \{t_0,\dots,t_{m+1}\}\in\mathcal{Z}(\delta,T)}} |\xi(\omega(t)) - \xi(\omega(s))|$$

and $\mathcal{Z}(\delta,T)$ denotes the set of all partitions $\{t_0,\dots,t_m\}$, $0 = t_0 < t_1 < \dots < t_m \le T < t_{m+1}$, $m \in \mathbb{N}$, such that $\inf_{i=0,\dots,m}(t_{i+1}-t_i) > \delta$.

Proof of Claim 2. Let $\epsilon > 0$. Since on E every bounded measure is Radon (see e.g. [Sch73, Theorems 9, 10 on p. 122]), μ is tight. Hence there is a compact set $K \subset E$ with $\mu(E \setminus K) \le \frac{\epsilon}{2}$. Let $R_1 \ge 0$ be such that $\xi(K) \subset [-R_1/2, R_1/2]$, and set for each $R \ge 0$

$$\tau_R := \inf\{t \ge 0 \mid |\xi(X_t)| > R\}.$$

Then τ_R is an $(\mathcal{F}_{t+})_{t\ge 0}$ stopping time, therefore so is $\tau_R \wedge T$ (where as usual $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$). By the right continuity of the sample paths we can use the optional sampling theorem w. r. t. $(\mathcal{F}_{t+})_{t\ge 0}$ to obtain for each $R \ge R_1$ and all $k \in \mathbb{N}$ that

$$\begin{aligned}(3) \qquad & E^{P_k}\left[1 - \varphi_R(X_{\tau_R\wedge T}) - \int_0^{\tau_R\wedge T} (p_k(x,D)\varphi_R)(X_u)\,du\right] \\ &\quad = E^{P_k}[1-\varphi_R(X_0)] = \int_E (1-\varphi_R)\,d\mu \le \mu(E\setminus K) \le \frac{\epsilon}{2}.\end{aligned}$$

By assumptions (A1), (A2) we have that

$$\lim_{R\to+\infty} \sup_{|t|\le \frac{1}{\sqrt{R}}} A(t\xi) = 0.$$

Therefore, by Claim 1 one can find R_2 such that

$$\forall R \ge R_2 \quad \sup_k \sup_{x\in E} |p_k(x,D)\varphi_R(x)| \le \frac{\epsilon}{2T}.$$

Let $R_3 := \max(R_1, R_2)$. Then it follows from (3.3) that for all $R > R_3$

$$\sup_k E^{P_k}[1 - \varphi_R(X_{\tau_R\wedge T})] \le \epsilon,$$

i.e., by (2) with $x_0 = 0$

$$\sup_k E^{P_k}\left[1 - \chi\Big(\frac{\xi(X_{\tau_R\wedge T})}{R}\Big)\right] \le \epsilon$$

whence

$$\sup_k P_k\Big[\sup_{0\le t\le T} |\xi(X_t)| > R\Big] \le \epsilon$$

and Claim 2 is proved. $\square$

Proof of Claim 3. Let $\epsilon > 0$. By Claim 1 there exists a constant K $(= K(\epsilon)) > 0$ such that

$$\sup_{k\in\mathbb{N}} \sup_{x,x_0\in E} |p_k(x, D)\varphi_{\frac{\epsilon}{4},x_0}(x)| \le K. \tag{4}$$

Now (as in [H92]) we define a sequence $(\tau_m)_{m\in\mathbb{N}\cup\{0\}}$ of $(\mathcal{F}_t)$-stopping times (depending on ϵ) as follows. Set $\tau_0 := 0$ and for $m \in \mathbb{N}\cup\{0\}$

$$\tau_{m+1} := \inf\Big\{t \ge \tau_m \;\Big|\; |\xi(X_t) - \xi(X_{\tau_m})| \ge \frac{\epsilon}{2} \quad \text{or} \\ \Big(t > \tau_m \text{ and } |\xi(X_{t-}) - \xi(X_{\tau_m})| \ge \frac{\epsilon}{2}\Big)\Big\}.$$

(For a proof that each τ_m is indeed an $(\mathcal{F}_t)$-stopping time see e.g. [H92, p. 38]). By [StV79, Lemma 1.3.3] (whose proof carries over directly to the case where E is replacing $\mathbb{R}^d$)

$$\mathcal{F}_{\tau_m} = \sigma(X_{t\wedge\tau_m} \mid t \ge 0), \qquad m \in \mathbb{N}.$$

The right continuity of $(X_t)_{t\ge0}$ and Remark 1.1 hence imply that each $\mathcal{F}_{\tau_m}$ is countably generated. Hence by Remark 2.2 regular conditional probabilities $P^k_{\cdot,m}$ of $E^{P_k}[\,\cdot \mid \mathcal{F}_{\tau_m}]$, $m \in \mathbb{N}\cup\{0\}$, exist. If we can show that for all $k \in \mathbb{N}$ and P_k-a.e. $\omega \in D_E$

$$P^k_{\omega,m}[\{\tau_{m+1} - \tau_m \le \delta,\ \tau_m < \infty\}] \le K\delta \tag{5}$$

for all $\delta > 0$ and all $m \in \mathbb{N}\cup\{0\}$, then the assertion of Claim 3 follows in exactly the same way as in [H92, Lemma 2.15, Satz 2.16] (see also [StV79, Lemma 1.4.5, Theorem 1.4.6] for the case of continuous sample paths and also [St75, Appendix]). Fix $k \in \mathbb{N}$ and let $\varphi \in \mathcal{L}_A$. Then by [StV79, Theorem 1.2.10]

$$\varphi(X_t) + \int_0^t (p_k(\,\cdot\,, D)\,\varphi)(X_s)\,ds, \qquad t \ge 0 \tag{6}$$

for P^k-a.e. $\omega \in D_E$ is an $(\mathcal{F}_t)$-martingale after $\tau_m(\omega)$ under the measure $P^k_{\omega,m}$ (with the zero set depending on φ). But since M is separable w.r.t. the weak topology (cf. [Sch73, Theorem 7, p. 385]) and since $p_k \le k$, we can choose a P^k-zero set $N_k \in \mathcal{F}_\infty$ such that for all $\omega \in D_E \setminus N_k$ (6) is an $(\mathcal{F}_t)$-martingale after $\tau_m(\omega)$ under $P^k_{\omega,m}$ for all $\varphi \in \mathcal{L}_A$, in particular, for

$$\Phi := \varphi_{\frac{\epsilon}{4},X_{\tau_m}(\omega)}.$$

Let $m \in \mathbb{N} \cup \{0\}$ and fix $\omega \in D_E \setminus N_k$ such that $\tau_m(\omega) < \infty$. Define

$$\sigma_m := \inf\Big\{t \geq \tau_m \;\Big|\; |\xi(X_t) - \xi(X_{\tau_m})| > \frac{\epsilon}{4}\Big\}.$$

Then each σ_m is an $(\mathcal{F}_{t+})$-stopping time, hence for each fixed $\omega \in D_E \setminus N_k$ by right-continuity and optional sampling

$$\begin{aligned}(7)\qquad & E^{P^k_{\omega,m}}\Big[\Phi(X_{(\sigma_m \wedge (\tau_m(\omega)+\delta)) \vee \tau_m(\omega)}) \\ &\qquad + \int_0^{(\sigma_m \wedge (\tau_m(\omega)+\delta)) \vee \tau_m(\omega)} (p_k(\,\cdot\,, D)\Phi)(X_s)\,ds\Big] \\ &= E^{P^k_{\omega,m}}\Big[\Phi(X_{\tau_m(\omega)}) + \int_0^{\tau_m(\omega)} (p_k(\,\cdot\,, D)\Phi)(X_s)\,ds\Big] \\ &= 1 - E^{P^k_{\omega,m}}\Big[\int_0^{\tau_m(\omega)} (p_k(\,\cdot\,, D)\Phi)(X_s)\,ds\Big]\end{aligned}$$

where in the last step we used that

$$(8)\qquad \tau_m = \tau_m(\omega), \quad X_{\tau_m} = X_{\tau_m}(\omega), \qquad P^k_{\omega,m}\text{-a.s.}$$

by the regularity of $P^k_{\cdot,m}$. Let $\delta > 0$. Since $\tau_m \leq \sigma_m \leq \tau_{m+1}$ and $\{\sigma_m < \infty\} \subset \{|\xi(X_{\sigma_m}) - \xi(X_{\tau_m})| \geq \frac{\epsilon}{4}\}$, we have that

$$\{\tau_{m+1} - \tau_m \leq \delta,\ \tau_m < \infty\} \subset \{|\xi(X_{(\sigma_m \wedge (\tau_m+\delta)) \vee \tau_m}) - \xi(X_{\tau_m})| \geq \frac{\epsilon}{4},\ \tau_m < \infty\}.$$

Hence (7), (8) imply that

$$\begin{aligned}& P^k_{\omega,m}[\{\tau_{m+1} - \tau_m \leq \delta,\ \tau_m < \infty\}] \\ &\quad \leq P^k_{\omega,m}\Big[\Big\{|\xi(X_{(\sigma_m \wedge (\tau_m(\omega)+\delta)) \vee \tau_m(\omega)}) - \xi(X_{\tau_m(\omega)})| \geq \frac{\epsilon}{4}\Big\}\Big] \\ &\quad \leq E^{P^k_{\omega,m}}\Big[1 - \Phi(X_{(\sigma_m \wedge (\tau_m(\omega)+\delta)) \vee \tau_m(\omega)})\Big] \\ &\quad \leq E^{P^k_{\omega,m}}\Big[\int_{\tau_m(\omega)}^{(\sigma_m \wedge (\tau_m(\omega)+\delta)) \vee \tau_m(\omega)} (p_k(\,\cdot\,, D)\Phi)(X_s)\,ds\Big] \\ &\quad \leq \delta K\end{aligned}$$

where the last step follows by (4). Thus (5) has been shown if $\omega \in D_E \setminus N_k$ with $\tau_m(\omega) < \infty$. But if $\omega \in D_E$ such that $\tau_m(\omega) = \infty$, then by (8) inequality (5) is trivial, hence Claim 3 is proved, and the proof of Proposition 3.4 is complete. ∎

Now we can complete the proof of Theorem 3.1. By Proposition 3.4 and Prohorov's theorem (cf. [Sch73, p. 379] for the version needed here) it follows that $\{P^k \mid k \in \mathbb{N}\}$ is relatively compact w.r.t. the weak topology. Since $\mathcal{M}_1(D_E)$ is Lusin (cf. [Sch73, Theorem 7, p. 385]) and since, therefore, by [Sch73, Corollary 2, p. 106] $\{P^k \mid k \in \mathbb{N}\}$ is metrizable, there exists a subsequence of $(P^k)_{k\in\mathbb{N}}$, again denoted by $(P^k)_{k\in\mathbb{N}}$ for simplicity, such that it converges weakly to some probability measure P on $\mathcal{F}_\infty$ (cf. [J86, Theorem 4.6(ii)]). As in [J86] for $f \in C(E;\mathbb{R})$ we define the map

$$\widetilde{f}\colon D_E \to D_{\mathbb{R}}$$

by

$$\widetilde{f}(\omega)(t) := f(\omega(t)).$$

By the remarkable result [J86, Theorem 1.7] the Skorohod topology on D_E is generated by $\{\widetilde{f} \mid f \in C(E;\mathbb{R})\}$. By [EK86, Chap. III, Lemma 7.7] for any probability measure Q on $\mathcal{B}(D_{\mathbb{R}})$ there exist a countable set $T_Q \subset \mathbb{R}_+$ such that $\omega \mapsto \omega(t)$ is continuous for Q-a.e. $\omega \in D_{\mathbb{R}}$ for all $t \in \mathbb{R}_+ \setminus T_Q$. Hence if $f \in C(E;\mathbb{R})$ by [J86, Theorem 1.7] $\omega \mapsto f(\omega(t))$ is continuous for P-a.e. $\omega \in D_E$ for all $t \in \mathbb{R}_+ \setminus T_{\widetilde{f}_*(P)}$. In particular, this holds for φ, $p(\,\cdot\,, D)\varphi$ replacing f and any $\varphi \in \mathcal{L}_A$. By a version of the portemanteau theorem on not-necessarily metric spaces (cf. [VTC87, Theorem 3.5(d), p. 42], which applies here since D_E is Lusin (cf. Remark 2.2) hence every probability measure is Radon, cf. [Sch73, Theorem 9, p. 122]) we also know that

$$\int f\,dP_k \longrightarrow \int f\,dP, \qquad \text{as } k \to \infty$$

for every bounded $\mathcal{F}_\infty$-measurable $f\colon D_E \to \mathbb{R}$ which is P-a.e. continuous. Now a straightforward modification of the argument in [EK86, Chap. III, Lemma 5.1] (see also [H92, Satz 2.17]) implies that P is a solution for the martingale problem for $(-p(\,\cdot\,, D), \mathcal{L}_A)$ with initial measure μ and Theorem 3.1 is proved. $\square$

Remark 3.5. The reader should note that under some reasonable assumptions the main result of this paper has a natural extension to the case where E is replaced by an abelian topological group.

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