Law of Large Numbers and Ergodic Theorem for convex weak star compact valued Gelfand-integrable mappings

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Abstract. We prove several results in the integration of convex weak star (resp. norm compact) valued random sets with application to weak star Kuratowski convergence in the law of large numbers for convex norm compact valued Gelfand-integrable mappings in the dual of a separable Banach space. We also establish several weak star Kuratowski convergence in the law of large numbers and ergodic theorem involving the subdifferential operators of Lipschitzean functions defined on a separable Banach space, and also provide an application to a closure type result arisen in evolution inclusions.
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1 Introduction

Several convergence problems in the dual of a separable Banach space have been treated with Fatou Lemma in Mathematical Economics [2, 9, 16], martingales [8] and ergodic theorem [11, 19, 48]. The aforementioned results lead naturally to the law of large numbers in the dual space. At this point, the law of large numbers for Pettis-integrable functions in locally convex spaces has been studied in [12], in particular, almost sure

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convergence for the law of large numbers in the weak star dual space for some classes of Gelfand-integrable mappings is also available. Some related results for the law of large numbers involving the subdifferential of Lipschitzean functions have been studied in [42, 43]. There are a plethore of results for the convergence in the law of large numbers for vector valued random variables and closed valued random sets in Banach spaces, see e.g. [14, 15, 31, 32, 33, 34, 35, 36, 37, 41] and the references therein. Here we provide new convergence (namely the weak star Kuratowski convergence) in the law of large numbers for convex weak star compact valued Gelfand-integrable mappings in a dual of a separable Banach space and, we also present some new versions of law of large numbers and ergodic theorem involving the subdifferential operator of a Lipschitzean function defined on a separable Banach space. The paper is organized as follows. In section 2 we give definitions and preliminaries on measurability properties for convex weak star compact valued mappings (alias multifunctions) in the dual of a separable Banach space. In section 3 we summarize the properties of conditional expectation for convex weak star compact valued Gelfand-integrable mappings, in particular we present a Jensen type inequality for convex weak star compact valued conditional expectation and a version of dominated Lebesgue convergence theorem for convex weak star compact valued Gelfand-integrable mappings. In section 4 we present several results on the integration of convex weak star (resp. convex norm compact) valued random sets with application to weak star Kuratowski convergence in the law large numbers for convex norm compact valued Gelfand-integrable mappings in the same vein as [14, 34, 35, 41] dealing with Wijsman and Mosco convergence in the law of large numbers for closed random integrable sets in separable Banach spaces. In section 5 we provide two weak star Kuratowski convergence results in the law of large numbers and ergodic theorem involving the subdifferential operators of Lipschitzean functions defined on a separable Banach space, and also an application to a closure type result arisen in evolution inclusions.

2 Notations and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let E be a separable Banach space, E^* the topological dual of E, $\overline{B}_E$ (resp. $\overline{B}_{E^*}$) the closed unit ball of E (resp. E^*), $D_1 = (e_k)_{k\in\mathbb{N}}$ a dense sequence in $\overline{B}_E$. We denote by E^*_s (resp. E^*_b) the vector space E^* endowed with the topology $\sigma(E^*, E)$ of pointwise convergence, alias w^* topology (resp. the topology s^* associated with the dual norm $||.||_{E^*_b}$), and by $E^*_{m^*}$ the vector space E^* endowed with the topology $m^* = \sigma(E^*, H)$, where H is the linear space of E generated by D_1, that is the Hausdorff locally convex topology defined by the sequence of semi-norms

$$P_n(x^*) = \max\{|\langle e_k, x^*\rangle| : k \leq n\}, \quad x^* \in E^*, n \in \mathbb{N}.$$

Recall that the topology m^* is metrizable, for instance, by the metric

$$d_{E^*_{m^*}}(x^*, y^*) := \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle e_k, x^*\rangle - \langle e_k, y^*\rangle|, \quad x^*, y^* \in E^*.$$

We assume from now on that $d_{E^*_{m^*}}$ is held fixed. Further, we have $m^* \subset w^* \subset s^*$. On the other hand, the restrictions of m^* and w^* to any bounded subset of E^* coincide and the Borel tribes $\mathcal{B}(E^*_s)$ and $\mathcal{B}(E^*_{m^*})$ associated with E^*_s and $E^*_{m^*}$ are equal, but the consideration of the Borel tribe $\mathcal{B}(E^*_b)$ associated with the topology of E^*_b is irrelevant here. Noting that E^* is the countable union of closed balls, we deduce that the space E^*_s is a Lusin space, as well as the metrizable topological space $E^*_{m^*}$. Let $\mathcal{K}^* = cwk(E^*_s)$ be the set of all nonempty convex weak star compact subsets in E^*. A $\mathcal{K}^*$-valued multifunction (alias mapping for short) $X : \Omega \rightrightarrows E^*_s$ is scalarly $\mathcal{F}$-measurable if, $\forall x \in E$, the support function $\delta^*(x, X(.))$ is $\mathcal{F}$-measurable, hence its graph belongs to $\mathcal{F} \otimes \mathcal{B}(E^*_s)$. Indeed, let $(f_k)_{k\in\mathbb{N}}$ be a sequence in E which separates the points of E^*, then we have $x^* \in X(\omega)$ iff $\langle f_k, x^*\rangle \leq \delta^*(f_k, X(\omega))$ for all $k \in \mathbb{N}$. Consequently, for any Borel set $G \in \mathcal{B}(E^*_s)$, the set

$$X^- G = \{\omega \in \Omega : X(\omega) \cap G \neq \emptyset\}$$

is $\mathcal{F}$-measurable, that is, $X^- G \in \mathcal{F}$, this is a consequence of the Projection Theorem (see e.g. [17, Theorem III.23] and of the equality

$$X^- G = \mathrm{proj}_\Omega \{Gr(X) \cap (\Omega \times G)\}.$$

In particular if $u : \Omega \to E^*_s$ is a scalarly $\mathcal{F}$-measurable mapping, that is, if for every $x \in E$, the scalar function $\omega \mapsto \langle x, u(\omega)\rangle$ is $\mathcal{F}$-measurable, then the function $f : (\omega, x^*) \mapsto ||x^* - u(\omega)||_{E^*_b}$ is $\mathcal{F} \otimes \mathcal{B}(E^*_s)$-measurable, and for every fixed $\omega \in \Omega$, $f(\omega, .)$ is lower semicontinuous on E^*_s, i.e. f is a normal integrand. Indeed, we have

$$||x^* - u(\omega)||_{E^*_b} = \sup_{k\in\mathbb{N}} |\langle e_k, x^* - u(\omega)\rangle|.$$

As each function $(\omega, x^*) \mapsto \langle e_k, x^* - u(\omega)\rangle$ is $\mathcal{F} \otimes \mathcal{B}(E^*_s)$-measurable and continuous on E^*_s for each $\omega \in \Omega$, it follows that f is a normal integrand.

Consequently, the graph of u belongs to $\mathcal{F}\otimes\mathcal{B}(E^*_s)$. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. It is easy and classical to see that a mapping $u : \Omega \to E^*_s$ is $(\mathcal{B}, \mathcal{B}(E^*_s))$ measurable iff it is scalarly $\mathcal{B}$-measurable. A mapping $u : \Omega \to E^*_s$ is said to be scalarly integrable (alias Gelfand integrable), if, for every $x \in E$, the scalar function $\omega \mapsto \langle x, u(\omega)\rangle$ is $\mathcal{F}$-measurable and integrable. We denote by $G^1_{E^*}[E](\mathcal{F})$ the space of all Gelfand integrable mappings and by $L^1_{E^*}[E](\mathcal{F})$ the subspace of all Gelfand integrable mappings u such that the function $|u| : \omega \mapsto ||u(\omega)||_{E^*_b}$ is integrable. The measurability of $|u|$ follows easily from the above considerations. More generally, by $\mathcal{G}^1_{cwk(E^*_s)}(\Omega, \mathcal{F}, P)$ (or $\mathcal{G}^1_{cwk(E^*_s)}(\mathcal{F})$ for short) we denote the space of all scalarly $\mathcal{F}$- measurable and integrable $cwk(E^*_s)$-valued mappings and by $\mathcal{L}^1_{cwk(E^*_s)}(\Omega, \mathcal{F}, P)$ (or $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ for short) we denote the subspace of all $cwk(E^*_s)$-valued scalarly integrable and *integrably bounded* mappings X, that is, such that the function $|X| : \omega \to |X(\omega)|$ is integrable, here $|X(\omega)| := \sup_{y^*\in X(\omega)} ||y^*||_{E^*_b}$, by the above consideration, it is easy to see that $|X|$ is $\mathcal{F}$-measurable.

For any $X \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$, we denote by $\mathcal{S}^1_X(\mathcal{F})$ the set of all Gelfand-integrable selections of X. The Aumann-Gelfand integral of X over a set $A \in \mathcal{F}$ is defined by

$$E[1_A X] = \int_A X\, dP := \{\int_A f\, dP : f \in \mathcal{S}^1_X(\mathcal{F})\}.$$

We will consider on $\mathcal{K}^*$, the Hausdorff distance $d_{H^*_{m^*}}$ associated with the metric $d_{E^*_{m^*}}$ in the Lusin metrizable space $(E^*_{m^*}, d_{E^*_{m^*}})$ and also the Hausdorff distance $d_{H^*_b}$ associated with the norm dual $||.||_{E^*_b}$ on E^*_b, namely

$$d_{H^*_b}(A, B) = \sup_{x\in\overline{B}_E} |\delta^*(x, A) - \delta^*(x, B)| \quad \forall A, B \in \mathcal{K}^*.$$

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of w^*-closed convex sets, the sequential weak* upper limit w^*-$ls\, X_n$ of $(X_n)_{n\in\mathbb{N}}$ is defined by

$$w^*\text{-}ls\, X_n = \{x^* \in E^* : x^* = \sigma(E^*, E)\text{-}\lim_{j\to\infty} x^*_j;\ x^*_j \in X_{n_j}\}.$$

Similarly the sequential weak* lower limit w^*-$li\, X_n$ of $(X_n)_{n\in\mathbb{N}}$ is defined by

$$w^*\text{-}li\, X_n = \{x^* \in E^* : x^* = \sigma(E^*, E)\text{-}\lim_{n\to\infty} x^*_n;\ x^*_n \in X_n\}.$$

The sequence $(X_n)_{n\in\mathbb{N}}$ *weak star Kuratowski (w$^*K$ for short) converges* to a w^*-closed convex set X_∞ if the following holds

$$w^*\text{-}ls\, X_n \subset X_\infty \subset w^*\text{-}li\, X_n \quad \text{a.s.}$$

Briefly

$$w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty \quad \text{a.s.}$$

When dealing with w^*-closed valued mappings, it is convenient to adopt the following terminology. A closed valued mapping $\Gamma : \Omega \Rightarrow E^*_s$ is a $\mathcal{F}$-*measurable random set* , if its graph belongs to $\mathcal{F} \otimes \mathcal{B}(E^*_s)$. Such a mapping is *integrable* if the set S^1_Γ of $L^1_{E^*}[E](\mathcal{F})$ selections of Γ is nonempty.

In the remainder of the paper, the terminology *weak* or *weakly* is related to the weak topology of Banach space. We denote by $cwk(E^*_b)$ the collection of all nonempty convex weakly compact subsets in E^*_b, $ck(E^*_b)$ the collection of all nonempty convex norm compact subsets in E^*_b and by $\mathcal{L}^1_{cwk(E^*_b)}(\mathcal{F})$ (resp. $\mathcal{L}^1_{ck(E^*_b)}(\mathcal{F})$) we denote the collection of all $cwk(E^*_b)$-valued (resp. $ck(E^*_b)$-valued) scalarly integrable and *integrably bounded* mappings.

3 Measurability and Conditional expectation in the dual space

We summarize some needed results on measurability and conditional expectation for convex weak star compact valued Gelfand-integrable mappings in the dual space. A $\mathcal{K}^*$-valued mapping $X : \Omega \to E^*$ is a *$\mathcal{K}^*$-valued random set* if $X(\omega) \in \mathcal{K}^*$ for all $\omega \in \Omega$ and if X is scalarly $\mathcal{F}$-measurable. We will show that $\mathcal{K}^*$-valued random sets enjoy good measurability properties.

Proposition 3.1 *Let $X : \Omega \to cwk(E^*_s)$ be a convex weak star compact valued mapping. The following are equivalent*
(a) $X^-V \in \mathcal{F}$ for all m^*-open subset V of E^*.
(b) $\mathrm{Graph}(X) \in \mathcal{F} \otimes \mathcal{B}(E^*_s) = \mathcal{F} \otimes \mathcal{B}(E^*_{m^*})$.
(c) X admits a countable dense set of $(\mathcal{F}, \mathcal{B}(E^*_s))$-measurable selections.
(d) X is scalarly $\mathcal{F}$-measurable.

Proof. $(a) \Rightarrow (b)$. Recall that any $K \in \mathcal{K}^*$ is m^*-compact and $m^* \subset w^*$ and the Borel tribes $\mathcal{B}(E^*_s)$ and $\mathcal{B}(E^*_{m^*})$ are equal. Recall also that $E^*_{m^*}$ is a Lusin metrizable space. By (a) X is a m^*-compact valued measurable mapping from Ω into the Lusin metrizable space $E^*_{m^*}$. Hence $\mathrm{Graph}(X) \in \mathcal{F} \otimes \mathcal{B}(E^*_{m^*})$ because

$$\mathrm{Graph}(X) = \{(\omega, x^*) \in \Omega \times E^*_{m^*} : d_{E^*_{m^*}}(x^*, X(\omega)) = 0\}$$

and the mapping $(\omega, x^*) \mapsto d_{E^*_{m^*}}(x^*, X(\omega))$ is $\mathcal{F} \otimes \mathcal{B}(E^*_{m^*})$-measurable. $(b) \Rightarrow (a)$ by applying the measurable Projection Theorem (see e.g. [17, Theorem III.23]) and the equality

$$X^- V = \text{proj}_\Omega \{\text{Graph}(X) \cap (\Omega \times V)\}.$$

Hence (a) and (b) are equivalent.
$(b) \Rightarrow (c)$. Since E^*_s is a Lusin space, by [17, Theorem III-22], X admits a countable dense set of $(\mathcal{F}, \mathcal{B}(E^*_s))$-measurable selections (f_n), that is, $X(\omega) = w^*\text{cl}\{f_n(\omega)\}$ for all $\omega \in \Omega$.
$(c) \Rightarrow (d)$. Indeed one has $\delta^*(x, X(\omega)) = \sup_n \langle x, f_n(\omega)\rangle$ for all $x \in E$ and for all $\omega \in \Omega$, thus proving the required implication.
$(d) \Rightarrow (b)$. We have already seen in Section 2 that (c) implies that $\text{Graph}(X) \in \mathcal{F} \otimes \mathcal{B}(E^*_s)$. As $\mathcal{B}(E^*_s) = \mathcal{B}(E^*_{m^*})$, the proof is finished. ∎

Corollary 3.2 *Let $X : \Omega \to cwk(E^*_s)$ be a convex weak star compact valued mapping. The following are equivalent:*
(a) $X^- V \in \mathcal{F}$ for all w^*-open subset V of E^*.
(b) $\text{Graph}(X) \in \mathcal{F} \otimes \mathcal{B}(E^*_s)$.
(c) X admits a countable dense set of $(\mathcal{F}, \mathcal{B}(E^*_s))$-measurable selections.
(d) X is scalarly $\mathcal{F}$-measurable.

Proof. $(a) \Rightarrow (d)$ is easy. The implications $(d) \Rightarrow (b)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (d)$, $(b) \Rightarrow (a)$ are already known. For further details on these facts, consult Proposition 5.2 and Corollary 5.3 in [9]. ∎

Remarks 3.3 Proposition 3.1 shows that a $\mathcal{K}^*$-valued random set can be viewed as a $\mathcal{K}^*$-valued measurable mapping from Ω into the Lusin metrizable space $E^*_{m^*}$. We will see in the next section the usefulness of the space $E^*_{m^*}$ in the study of independence of $\mathcal{K}^*$-valued random sets.

Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and let X be a $\mathcal{K}^*$-valued integrably bounded random set, let us define

$$S^1_X(\mathcal{B}) := \{f \in L^1_{E^*}[E](\Omega, \mathcal{B}, P) : f(\omega) \in X(\omega) \text{ a.s.}\}$$

and the multivalued Aumann-Gelfand integral (shortly esperance) $E[X, \mathcal{B}]$ of X

$$E[X, \mathcal{B}] := \{\int f dP : f \in S^1_X(\mathcal{B})\}.$$

As $S^1_X(\mathcal{B})$ is $\sigma(L^1_{E^*}[E](\mathcal{B}), L^\infty_E(\mathcal{B}))$ compact [13, Corollary 6.5.10], the expectation $E[X, \mathcal{B}]$ is convex $\sigma(E^*, E)$ compact. Before going further we need to recall and summarize the existence and uniqueness of the conditional expectation in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. See [47, Theorem 3], [16, Theorem 7.3]. For more information on the conditional expectation of multifunctions, we refer to [1, 8, 36, 47]. In particular, existence results for conditional expectation in Gelfand and Pettis integration can be derived from the multivalued Dunford-Pettis representation theorem, see [8]. A fairly general version of conditional expectation for closed convex integrable random sets in the dual of a separable Fréchet space is obtained by Valadier [47, Theorem 3]. Here we need only a special version of this result in the dual space E^*_s.

Theorem 3.4 *Let E be a separable Banach space and let Γ be a closed convex valued integrable random set in E^*_s. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. Then there exist a closed convex $\mathcal{B}$-measurable mapping Σ in E^*_s such that:*
1) Σ is the smallest closed convex $\mathcal{B}$-measurable mapping Θ such that $\forall u \in S^1_\Gamma$, $E^{\mathcal{B}}u(\omega) \in \Theta(\omega)$ a.s.
2) Σ is the unique closed convex $\mathcal{B}$-measurable mapping Θ such that $\forall v \in L^\infty_{\mathbb{R}}(\mathcal{B})$,

$$\int_\Omega \delta^*(v, \Gamma)dP = \int_\Omega \delta^*(v, \Theta)dP.$$

3) Σ is the unique closed convex $\mathcal{B}$-measurable mapping such that $S^1_\Sigma = cl\, E^{\mathcal{B}}(S^1_\Gamma))$ where cl denotes the closure with respect to $\sigma(L^1_{E^*}(\mathcal{B}), L^\infty_E(\mathcal{B}))$.

Theorem 3.4 allows to treat the conditional expectation of convex weakly compact valued integrably bounded mappings in E. Indeed if $F := E^*_b$ is *separable* and if Γ is a convex weakly compact valued measurable mapping in E with $\Gamma(\omega) \subset \alpha(\omega)\overline{B}_E$ where $\alpha \in L^1_{\mathbb{R}}$, then applying Theorem 3.4 to F^* gives $\Sigma(\omega) = E^{\mathcal{B}}\Gamma(\omega) \subset E^{**}$ with $\Sigma(\omega) \subset E^{\mathcal{B}}\alpha(\omega)\overline{B}_{E^{**}}$ where $\overline{B}_{E^{**}}$ is the closed unit ball in E^{**}. As S^1_Γ is $\sigma(L^1_E, L^\infty_{E^*})$ compact, $S^1_\Sigma = E^{\mathcal{B}}(S^1_\Gamma) \subset L^1_E$. Whence $\Sigma(\omega) \subset E$ a.s. See [47, Remark 4, page 10] for details.

The following existence theorem of conditional expectation for convex weak star compact valued Gelfand-integrable mappings follows from a version of multivalued Dunford-Pettis theorem in the dual space [8, Theorem 7.3]. In particular, it provides the weak star compactness of conditional expectation for integrably bounded weak star compact valued scalarly measurable mappings with some specific properties.

Theorem 3.5 *Given $\Gamma \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ and a sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$, there exists a unique (for equality a.s.) mapping $\Sigma := E^{\mathcal{B}}\Gamma \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{B})$, that is the conditional expectation of Γ with respect to $\mathcal{B}$, which enjoys the following properties:*

a) $\int_\Omega \delta^*(v,\Sigma)dP = \int_\Omega \delta^*(v,\Gamma)dP$ *for all* $v \in L^\infty_E(\mathcal{B})$.
b) $\Sigma \subset E^{\mathcal{B}}|\Gamma|\overline{B}_{E^*}$ *a.s.*
c) $\mathcal{S}^1_\Sigma(\mathcal{B})$ *is* $\sigma(L^1_{E^*}[E](\mathcal{B}), L^\infty_E(\mathcal{B}))$ *compact (here* $\mathcal{S}^1_\Sigma(\mathcal{B})$ *denotes the set of all* $L^1_{E^*}[E](\mathcal{B})$ *selections of* Σ*) and satisfies*

$$\delta^*(v, E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F})) = \delta^*(v, \mathcal{S}^1_\Sigma(\mathcal{B}))$$

for all $v \in L^\infty_E(\mathcal{B})$.
d) $E^{\mathcal{B}}$ *is increasing:* $\Gamma_1 \subset \Gamma_2$ *a.s. implies* $E^{\mathcal{B}}\Gamma_1 \subset E^{\mathcal{B}}\Gamma_2$ *a.s.*
e) For any $B \in \mathcal{B}$*, and for any* $X, Y \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ *we have*

$$\int_B d_{H^*_{m^*}}(E^{\mathcal{B}}X, E^{\mathcal{B}}Y)dP \leq \int_B d_{H^*_{m^*}}(X,Y)dP$$

Proof. Properties $a) - b)$ are classical, see e.g. [8, 17, 16, 47]. c) follows from a weak compactnes result [13, Corollary 6.5.10]. $e)$ can be proved as in [36], nevertheless this needs a bit more details. For technical consideration, we may assume $\mathcal{B}$ is complete. Since $E^{\mathcal{B}}X$ and $E^{\mathcal{B}}Y$ are scalarly $\mathcal{B}$-measurable, by Proposition 3.1 they are viewed as compact valued measurable mapping in the Lusin metric space $(E^*_{m^*}, d_{E^*_{m^*}})$. Consequently the function $d_{E^*_{m^*}}(x^*, E^{\mathcal{B}}X)$ and $d_{E^*_{m^*}}(x^*, E^{\mathcal{B}}Y)$ are separately $\mathcal{B}$-measurable on Ω and separately continuous on $(E^*_{m^*}, d_{E^*_{m^*}})$. Whence the function

$$\sup_{x^* \in E^{\mathcal{B}}X} d_{E^*_{m^*}}(x^*, E^{\mathcal{B}}Y)$$

is $\mathcal{B}$-measurable and so is the function

$$\sup_{y^* \in E^{\mathcal{B}}Y} d_{E^*_{m^*}}(y^*, E^{\mathcal{B}}X)\}.$$

It follows that $d_{H^*_{m^*}}(E^{\mathcal{B}}X, E^{\mathcal{B}}Y)$ is $\mathcal{B}$-measurable because

$$d_{H^*_{m^*}}(E^{\mathcal{B}}X, E^{\mathcal{B}}Y) = \max\{\sup_{x^* \in E^{\mathcal{B}}X} d_{E^*_{m^*}}(x^*, E^{\mathcal{B}}Y), \sup_{y^* \in E^{\mathcal{B}}Y} d_{E^*_{m^*}}(y^*, E^{\mathcal{B}}X)\}.$$

Let us set

$$A = \{\omega \in \Omega : \sup_{x^* \in E^{\mathcal{B}}X(\omega)} d_{E^*_{m^*}}(x^*, E^{\mathcal{B}}Y(\omega)) \geq \sup_{y^* \in E^{\mathcal{B}}Y(\omega)} d_{E^*_{m^*}}(y^*, E^{\mathcal{B}}X(\omega))\}.$$

Then by the above consideration A is $\mathcal{B}$-measurable. By integration on B we have

$$\int_B d_{H^*_{m^*}}(E^{\mathcal{B}}X, E^{\mathcal{B}}Y)dP = \int_{B\cap A} \sup_{x^*\in E^{\mathcal{B}}X(\omega)} d_{E^*_{m^*}}(x^*, E^{\mathcal{B}}Y(\omega))dP(\omega)$$
$$+ \int_{B\setminus A} \sup_{y^*\in E^{\mathcal{B}}Y(\omega)} d_{E^*_{m^*}}(y^*, E^{\mathcal{B}}X(\omega))dP(\omega).$$

By a standard application of a measurable selection theorem (see e.g. [17]), we have

$$\int_{B\cap A} \sup_{x^*\in E^{\mathcal{B}}X(\omega)} d_{E^*_{m^*}}(x^*, E^{\mathcal{B}}Y(\omega))dP(\omega) = \sup_{g\in S^1_{E^{\mathcal{B}}X}} \int_{B\cap A} d_{E^*_{m^*}}(g, E^{\mathcal{B}}Y)dP$$

and similarly

$$\int_{B\setminus A} \sup_{y^*\in E^{\mathcal{B}}Y(\omega)} d_{E^*_{m^*}}(y^*, E^{\mathcal{B}}X(\omega))dP(\omega) = \sup_{h\in S^1_{E^{\mathcal{B}}Y}} \int_{B\setminus A} d_{E^*_{m^*}}(h, E^{\mathcal{B}}X)dP$$

and also

$$\sup_{g\in S^1_{E^{\mathcal{B}}X}} \int_{B\cap A} d_{E^*_{m^*}}(g, E^{\mathcal{B}}Y)dP = \sup_{g\in S^1_{E^{\mathcal{B}}X}} \inf_{h\in S^1_{E^{\mathcal{B}}Y}} \int_{B\cap A} d_{E^*_{m^*}}(g, h)dP$$
$$= \sup_{u\in S^1_X} \inf_{v\in S^1_Y} \int_{B\cap A} d_{E^*_{m^*}}(E^{\mathcal{B}}u, E^{\mathcal{B}}v)dP$$

and

$$\sup_{h\in S^1_{E^{\mathcal{B}}Y}} \int_{B\setminus A} d_{E^*_{m^*}}(h, E^{\mathcal{B}}X)dP = \sup_{h\in S^1_{E^{\mathcal{B}}Y}} \inf_{g\in S^1_{E^{\mathcal{B}}X}} \int_{B\setminus A} d_{E^*_{m^*}}(h, g)dP$$
$$= \sup_{v\in S^1_Y} \inf_{u\in S^1_X} \int_{B\setminus A} d_{E^*_m}(E^{\mathcal{B}}u, E^{\mathcal{B}}v)dP.$$

Taking into account the definition of $d_{E^*_{m^*}}$ and a classical property of real valued conditional expectation we have

$$\int_{B\cap A} d_{E^*_m}(E^{\mathcal{B}}u, E^{\mathcal{B}}v)dP \leq \int_{B\cap A} d_{E^*_{m^*}}(u, v)dP$$

and

$$\int_{B\setminus A} d_{E^*_m}(E^{\mathcal{B}}u, E^{\mathcal{B}}v)dP \leq \int_{B\setminus A} d_{E^*_{m^*}}(u, v)dP.$$

Whence we deduce that

$$\sup_{u\in S^1_X}\inf_{v\in S^1_Y}\int_{B\cap A} d_{E^*_{m^*}}(E^{\mathcal{B}}u, E^{\mathcal{B}}v)dP \le \sup_{u\in S^1_X}\inf_{v\in S^1_Y}\int_{B\cap A} d_{E^*_{m^*}}(u,v)dP$$

and

$$\sup_{v\in S^1_Y}\inf_{u\in S^1_X}\int_{B\setminus A} d_{E^*_{m^*}}(E^{\mathcal{B}}u, E^{\mathcal{B}}v)dP \le \sup_{v\in S^1_Y}\inf_{u\in S^1_X}\int_{B\setminus A} d_{E^*_{m^*}}(u,v)dP.$$

But we have

$$\begin{aligned}\sup_{u\in S^1_X}\inf_{v\in S^1_Y}\int_{B\cap A} d_{E^*_{m^*}}(u,v)dP &= \int_{B\cap A}\sup_{x^*\in X(\omega)} d_{E^*_{m^*}}(x^*, Y(\omega))dP(\omega)\\ &\le \int_{B\cap A} d_{H^*_{m^*}}(X,Y)dP\end{aligned}$$

and

$$\begin{aligned}\sup_{v\in S^1_Y}\inf_{u\in S^1_X}\int_{B\setminus A} d_{E^*_{m^*}}(u,v)dP &= \int_{B\setminus A}\sup_{y^*\in Y(\omega)} d_{E^*_{m^*}}(y^*, X(\omega))dP(\omega)\\ &\le \int_{B\setminus A} d_{H^*_{m^*}}(X,Y)dP.\end{aligned}$$

Finally, by combining these inequalities,

$$\int_B d_{H^*_{m^*}}(E^{\mathcal{B}}X, E^{\mathcal{B}}Y)dP \le \int_B d_{H^*_{m^*}}(X,Y)dP.$$

■

Here is a version of Lebesgue dominated convergence theorem for conditional expectations. Compare with Theorem 2.7 in Hiai [35] for the primal Banach space.

Theorem 3.6 *Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and let $(X_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ with $g := \sup_{\in\mathbb{N}\cup\{\infty\}}|X_n| \in L^1_{\mathbb{R}}(\mathcal{F})$. Assume that*

$$\lim_{n\to\infty} d_{H^*_b}(X_n(\omega), X_\infty(\omega)) = 0 \quad \forall\omega\in\Omega$$

then

$$\lim_{n\to\infty} d_{H^*_{m^*}}(E^{\mathcal{B}}X_n(\omega), E^{\mathcal{B}}X_\infty(\omega)) = 0 \quad a.s.\ \omega\in\Omega$$

Proof. Applying Theorem 3.5-(e) yields

$$d_{H^*_{m^*}}(E^{\mathcal{B}}X_n(\omega), E^{\mathcal{B}}X_\infty(\omega)) \leq E^{\mathcal{B}} d_{H_{m^*}}(X_n(\omega), X_\infty(\omega)) \quad \text{a.s.} \quad \omega \in \Omega.$$

As $d_{H^*_{m^*}}(A, B) \leq d_{H^*_b}(A, B)$ for $A, B \in cwk(E^*)$ we have that

$$d_{H^*_{m^*}}(X_n(\omega), X_\infty(\omega)) \leq d_{H^*_b}(X_n(\omega), X_\infty(\omega)) \leq 2g(\omega)$$

for all $\omega \in \Omega$, it follows that

$$d_{H^*_{m^*}}(E^{\mathcal{B}}X_n(\omega), E^{\mathcal{B}}X_\infty(\omega)) \leq E^{\mathcal{B}} d_{H^*_b}(X_n(\omega), X_\infty(\omega)) \to 0 \quad \text{a.s.} \quad \omega \in \Omega$$

when $n \to \infty$. ■

4 Law of Large Numbers in a dual space

Thanks to good measurability properties for convex weak*compact valued integrably bounded random sets and their conditional expectation developed in Section 3 we provide some convergence results in the law of large numbers for $\mathcal{K}^*$-valued integrably bounded random sets. Now we need to introduce some probabilistic notions and terminologies in the dual space E^* although these are somewhat similar to those given in the primal space E. Let $\mathcal{F}(E^*_{m^*})$ be the collection of nonempty m^*-closed subset of $E^*_{m^*}$. On $\mathcal{F}(E^*_{m^*})$ we consider the Effros tribe $\mathcal{E}$ generated by the sets of the form

$$\{K \in \mathcal{F}(E^*_{m^*}) : K \cap O \neq \emptyset\}$$

where O is the m^*-open sets in $E^*_{m^*}$ and we consider on $\mathcal{K}^*$ the tribe $\mathcal{B}(\mathcal{K}^*) := \mathcal{E}|\mathcal{K}^*$. Then a $\mathcal{K}^*$-valued random set can be viewed as a measurable mapping from the measurable space $(\Omega, \mathcal{F})$ into the measurable space $(\mathcal{K}^*, \mathcal{B}(\mathcal{K}^*))$. We denote by

$$\mathcal{F}_X := X^{-1}(\mathcal{B}(\mathcal{K}^*)) = \{X^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{B}(\mathcal{K}^*)\}$$

the smallest σ-algebra of $\mathcal{F}$ for which X is measurable. The distribution μ_X of X is the probability measure μ_X defined on $\mathcal{B}(\mathcal{K}^*)$ by

$$\mu_X(\mathcal{U}) = P(X^{-1}(\mathcal{U})), \quad \mathcal{U} \in \mathcal{B}(\mathcal{K}^*).$$

A sequence (X_n) of $\mathcal{K}^*$-valued random set is *independent* if the $\mathcal{F}_{X_n}$ are independent, *identically distributed* if all μ_{X_n} are identical and *i.i.d* if they are independent and identically distributed. On account of the above considerations we summarize some useful properties of $\mathcal{K}^*$-valued random sets using some arguments in the primal space given in [35]. At this point, compare with a similar result in the primal space [34, Proposition 2.6 and Remark 2.7]. The following results constitute a key tool in the study of law of large numbers in the dual space.

Proposition 4.1 *(1) Let X be an integrably bounded $\mathcal{K}^*$-valued random set, then $E[X, \mathcal{F}_X]$ is $\sigma(E^*, E)$ compact.*
(2) Let X and Y be two identically distributed integrably bounded $\mathcal{K}^*$-valued random sets. Then, for each $f \in \mathcal{S}^1_X(\mathcal{F}_X)$, there exists $g \in \mathcal{S}^1_Y(\mathcal{F}_Y)$ such that f and g are identically distributed.
(3) If X is an integrably bounded $ck(E^*_b)$-valued random set, $X \in \mathcal{L}^1_{ck(E^*_b)}(\mathcal{F})$ for short, then $E[X] = E[X, \mathcal{F}_X]$.
(4) Let X and Y be two identically distributed integrably bounded $ck(E^*_b)$-valued random sets, then $E[X, \mathcal{F}_X] = E[Y, \mathcal{F}_Y]$.

Proof. (1) Let $\mathcal{F}_X$ be the the smallest σ-algebra of $\mathcal{F}$ for which X is measurable. Then the convex m^*-compact convex valued mapping $X : \Omega \to E^*_{m^*}$ is $\mathcal{F}_X$-measurable, that is, for each m^*-open set in O in $E^*_{m^*}$, $X^- O \in \mathcal{F}_X$. Since $E^*_{m^*}$ is Lusin metrizable space, by classical measurable selection [17, Theorem III.8], X admits a $(\mathcal{F}_X, \mathcal{B}(E^*_{m^*}))$-measurable (equivalently $(\mathcal{F}_X, \mathcal{B}(E^*_s))$-measurable, (equivalently scalarly $\mathcal{F}_X$-measurable) selection $f : \Omega \to E^*$. Further the mapping $|X|$ is $\mathcal{F}_X$-measurable and integrable. Whence such a selection f belongs to $L^1_{E^*}[E](\mathcal{F}_X)$. Briefly the set $S^1_X(\mathcal{F}_X)$ of $L^1_{E^*}[E](\mathcal{F}_X)$-integrable selections of X is nonempty, and $S^1_X(\mathcal{F}_X)$ is convex $\sigma(L^1_{E^*}[E](\mathcal{F}_X), L^\infty_E(\mathcal{F}_X))$ compact [13, Corollary 6.5.10], consequently the expectation

$$E[X, \mathcal{F}_X] := \{E(f) : f \in S^1_X(\mathcal{F}_X)\}$$

is convex $\sigma(E^*, E)$ compact.
(2) Since $E^*_{m^*}$ is a Lusin space, $f : \Omega \to E^*_{m^*}$ is $(\mathcal{F}_X, \mathcal{B}(E^*_{m^*}))$-measurable, by a classical factorization argument in Lusin spaces, we find a $(\mathcal{B}(\mathcal{K}^*), \mathcal{B}(E^*_{m^*}))$-measurable function $\Phi : \mathcal{K}^* \to E^*_{m^*}$ satisfying $f(\omega) = \Phi(X(\omega))$ for every $\omega \in \Omega$. Define $g(\omega) = \Phi(Y(\omega)), \omega \in \Omega$. Since X and Y are identically distributed, f and g are also identically distributed. We have

$$\begin{aligned}\int_\Omega ||g(\omega)||_{E^*_b} dP(\omega) &= \int_{\mathcal{K}^*} ||\Phi(K)||_{E^*_b} d\mu_Y(K) \\ &= \int_{\mathcal{K}^*} ||\Phi(K)||_{E^*_b} d\mu_X(K) = \int_\Omega ||f(\omega)||_{E^*_b} dP(\omega) < \infty.\end{aligned}$$

Because for each $x^* \in E^*$ the function $K \to d_{E^*_{m^*}}(x^*, K)$ is $\mathcal{B}(\mathcal{K}^*)$-measurable, and for each $K \in \mathcal{K}^*$ the function $x^* \to d_{E^*_{m^*}}(x^*, K)$ is continuous on $E^*_{m^*}$, the function $(x^*, K) \to d_{E^*_{m^*}}(x^*, K)$ from $E^*_{m^*} \times \mathcal{K}^*$ into $\mathbb{R}$ is $\mathcal{B}(E^*_{m^*}) \otimes \mathcal{B}(\mathcal{K}^*)$-measurable, hence $d_{E^*_m}(f(.), X(.))$ and $d_{E^*_{m^*}}(g(.), Y(.))$ are identically distributed. Hence $d_{E^*_{m^*}}(f(\omega), X(\omega)) = 0$ a.s. implies that

$$d_{E^*_{m^*}}(g(\omega), Y(\omega)) = 0 \quad \text{a.s.} \quad \omega \in \Omega.$$

As $Y(\omega)$ is m^*-compact, $g \in S^1_Y(\mathcal{F}_Y)$.
(3) Now we will prove (3) by applying some arguments in the proof of Lemma 3.1 in Hiai [35] via the norm compactness condition on X and the conditional expectation $E^{\mathcal{F}_X} f = E(f|\mathcal{F}_X)$ of $f \in L^1_{E^*}[E](\mathcal{F})$. This needs a bit more details. Observe that X admits a countable norm dense set $\{f_i\}_{i\in\mathbb{N}}$ of $L^1_{E^*}[E](\mathcal{F}_X)$-integrable selections. Given $f \in S^1_X$ and $\varepsilon > 0$, imitating the construction in [36, (5.5)], there is a finite measurable partition $(A_i : i = 1, ..n)$ of Ω such that

$$|f - \sum_{i=1}^n 1_{A_i} f_i|_1 := \int_\Omega ||f(\omega) - \sum_{i=1}^n 1_{A_i} f_i(\omega)||_{E^*_b} dP(\omega) \le \varepsilon$$

therefore

$$|E^{\mathcal{F}_X}(f - \sum_{i=1}^n 1_{A_i} f_i)|_1 \le |f - \sum_{i=1}^n 1_{A_i} f_i|_1 \le \varepsilon.$$

By convexity we have

$$E^{\mathcal{F}_X}(\sum_{i=1}^n 1_{A_i} f_i) = \sum_{i=1}^n E^{\mathcal{F}_X}(1_{A_i} f_i) = \sum_{i=1}^n E^{\mathcal{F}_X}(1_{A_i}) f_i \in S^1_X(\mathcal{F}_X).$$

This shows that given $f \in S^1_X$, and $\varepsilon > 0$ there exist $g_\varepsilon \in S^1_X(\mathcal{F}_X)$ such that

$$|E^{\mathcal{F}_X} f - g_\varepsilon|_1 \le \varepsilon.$$

Therefore

$$\begin{aligned} |\int_A \langle x, E^{\mathcal{F}_X} f - g_\varepsilon \rangle dP| &\le \int_A |\langle x, E^{\mathcal{F}_X} f - g_\varepsilon \rangle| dP \\ &\le \int_\Omega ||E^{\mathcal{F}_X} f - g_\varepsilon||_{E^*_b} dP \le \varepsilon \end{aligned}$$

for all $A \in \mathcal{F}_X$ and for all $x \in \overline{B}_E$. In other words, for every $n \in \mathbb{N}$, there exists $g_n \in S^1_X(\mathcal{F}_X)$ such that

$$\begin{aligned} |\int_A \langle x, E^{\mathcal{F}_X} f - g_n\rangle dP| &\leq \int_A |\langle x, E^{\mathcal{F}_X} f - g_n\rangle| dP \\ &\leq \int_\Omega ||E^{\mathcal{F}_X} f - g_n||_{E^*_b} dP \leq \frac{1}{n} \end{aligned}$$

for all $A \in \mathcal{F}_X$ and for all $x \in \overline{B}_E$. Since $g_n \in S^1_X(\mathcal{F}_X)$ and $S^1_X(\mathcal{F}_X)$ is sequentially $\sigma(L^1_{E^*}[E](\mathcal{F}_X), L^\infty_E(\mathcal{F}_X))$ compact [13, Corollary 6.5.10], we may assume that $(g_n)_{n\in\mathbb{N}}$ converges to $h \in S^1_X(\mathcal{F}_X)$ with respect to this topology. Passing to the limit when $n \to \infty$ in the inequality $|\int_A \langle x, E^{\mathcal{F}_X} f - g_n\rangle dP| \leq \frac{1}{n}$ shows that

$$\langle x, E^{\mathcal{F}_X} f\rangle = \langle x, h\rangle \quad \text{a.s.}$$

for each $x \in E$, i.e. $E^{\mathcal{F}_X} f = h$ scalarly a.s. By separability, we may conclude that $E^{\mathcal{F}_X} f = h$ a.s. in E^*_s. This proves that

$$\{E^{\mathcal{F}_X} f : f \in S^1_X\} \subset S^1_X(\mathcal{F}_X).$$

Now (2) follows easily. Indeed we have

$$\begin{aligned} E[X] = \{E(f) : f \in \mathcal{S}^1_X\} &= \{E(E(f|\mathcal{F}_X)) : f \in \mathcal{S}^1_X\} \\ &\subset \{E(f) : f \in \mathcal{S}^1_X(\mathcal{F}_X)\} = E[X, \mathcal{F}_X]. \end{aligned}$$

(4) is immediate from (3). ■

When X is $cwk(E^*_b)$-valued, i.e. convex weakly compact valued, both the set of selections S^1_X and the expectation $E[X]$ enjoy good weak compactness properties, namely

Proposition 4.2 *Let X be a $cwk(E^*_b)$-valued scalarly measurable and integrably bounded mapping, $X \in \mathcal{L}^1_{cwk(E^*_b)}(\mathcal{F})$ for short. Then*
(1) S^1_X is convex $\sigma(L^1_{E^*}[E](\mathcal{F}), (L^1_{E^*}[E](\mathcal{F}))^*)$ compact, where $(L^1_{E^*}[E](\mathcal{F}))^*$ denotes the topological dual of the Banach space $L^1_{E^*}[E](\mathcal{F})$.
(2) $E[X]$ is weakly compact.

Proof. (1) is Corollary 4.2 in [3] and (2) follows easily. ■

Here is an easy consequence. We need the following definition. A uniformly integrable sequence $(u_n)_{n\in\mathbb{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ is *weakly tight* if for every

$\varepsilon > 0$ there is a scalarly measurable and integrably bounded weakly compact convex valued mapping $\Phi_\varepsilon : \Omega \Rightarrow E^*$ with $0 \in \Phi_\varepsilon(\omega)$ for all $\omega \in \Omega$ such that

$$\sup_{n\in\mathbb{N}} P(\{\omega \in \Omega : u_n(\omega) \notin \Phi_\varepsilon(\omega)\}) \leq \varepsilon.$$

By repeating the arguments in [5] we see that such a sequence is relatively weakly compact in $L^1_{E^*}[E](\mathcal{F})$. Indeed it is easily seen that u_n can be witten as $u_n = 1_{A_n}u_n + 1_{\Omega\setminus A_n}u_n$ with $A_n \in \mathcal{F}$ and $1_{A_n}u_n \in S^1_{\Phi_\varepsilon}$ and $||1_{\Omega\setminus A_n}u_n||_1 \leq \varepsilon$. By Proposition 4.2-(1) (or [4, Proposition 4.2]) $S^1_{\Phi_\varepsilon}$ is weakly compact in $L^1_{E^*}[E](\mathcal{F})$. In view of Grothendieck lemma [30], we conclude that $(u_n)_{n\in\mathbb{N}}$ is relatively weakly compact. ■

The following is useful in the law of large numbers for norm compact valued integrably bounded random sets. That is a dual version of Lemma 3.1-(1) in [35].

Proposition 4.3 *If X is an integrably bounded norm compact-valued random set, then*

$$\overline{co}\, E[X] \subset E[\overline{co}\, X] = E[\overline{co} X, \mathcal{F}_X] \subset \overline{co} E[X, \mathcal{F}_X]$$

so that $\overline{co}\, E[X] = \overline{co} E[X, \mathcal{F}_X]$.

Proof. *Step 1.* Since a norm compact set is a weak star compact set, and is also m^*-compact in $E^*_{m^*}$. Hence X is a fortiori a m^*-compact valued random set. Let $\mathcal{F}_X$ be the the smallest σ-algebra of $\mathcal{F}$ for which X is measurable. Then the m^*-compact valued mapping $X : \Omega \Rightarrow E^*_m$ is $\mathcal{F}_X$-measurable, that is, for each m^*-open set in O in E^*_m, $X^- O \in \mathcal{F}_X$. Since $E^*_{m^*}$ is Lusin metrizable space, by a classical measurable selection theorem [17], X admits a countable dense sequence $(f_i)_{i\in\mathbb{N}}$ of $(\mathcal{F}_X, \mathcal{B}(E^*_{m^*})$-measurable (equivalently $(\mathcal{F}_X, \mathcal{B}(E^*_s))$-measurable, (equivalently scalarly $\mathcal{F}_X$-measurable) selections. Further the mapping $|X|$ is $\mathcal{F}_X$-measurable and integrable so that $(f_i)_{i\in\mathbb{N}} \subset S^1_X(\mathcal{F}_X)$. Since X is norm compact valued X admits a norm dense sequence of selections $(f_i)_{i\in\mathbb{N}} \subset S^1_X(\mathcal{F}_X)$. It is not difficult to check that the associated convex norm compact valued mapping $\overline{co}\, X$ enjoys the same properties and the set $S^1_{\overline{co}\, X}(\mathcal{F}_X)$ of $L^1_{E^*}[E](\mathcal{F}_X)$-integrable selections of $\overline{co}\, X$ is nonempty and weakly compact in $L^1_{E^*}[E](\mathcal{F}_X)$ thanks to Proposition 4.2-(1), consequently the expectation $E[\overline{co}\, X]$ is convex weakly compact by Proposition 4.2-(2). By an appropriate modification of the proof of Theorem 1.5 in [36] we assert that

$$S^1_{\overline{co}\, X}(\mathcal{F}_X) = \overline{co}\, S^1_X(\mathcal{F}_X) \tag{4.1}$$

in the Banach space $L^1_{E^*}[E](\mathcal{F}_X)$. We give the details of this fact for convenience. Since $S^1_X(\mathcal{F}_X) \subset S^1_{\overline{co}\,X}(\mathcal{F}_X)$, we have

$$\overline{co}\, S^1_X(\mathcal{F}_X) \subset S^1_{\overline{co}X}(\mathcal{F}_X)$$

because $S^1_{\overline{co}X}(\mathcal{F}_X)$ is convex and weakly compact in $L^1_{E^*}[E](\mathcal{F}_X)$. Let us define

$$U = \{g : g = \sum_{i=1}^{m} = \alpha_i f_i, \quad \alpha_i \geq 0, \text{ rational } \quad \sum_{i=1}^{m} \alpha_i = 1, m \geq 1\}.$$

Then U is a countable dense subset of $S^1_{\overline{co}\,X}(\mathcal{F}_X)$ with

$$\overline{co}\, X(\omega) = \text{norm closure } \{g(\omega) : g \in U\}$$

for all $\omega \in \Omega$. Using this fact and arguing as in [36, Lemma 3.1] shows that, given $f \in S^1_{\overline{co}\,X}(\mathcal{F}_X)$ and $\varepsilon > 0$, there is a finite measurable partition $\{A_1, .., A_n\}$ and $g_1, ..., g_n \subset U$ such that

$$|f - \sum_{k=1}^{n} 1_{A_k} g_k|_1 < \varepsilon.$$

As in [36, Theorem 1.5] we have[1]

$$\sum_{k=1}^{n} 1_{A_k} g_k \in co\, S^1_X(\mathcal{F}_X).$$

We claim that $f \in \overline{co}\, S^1_X(\mathcal{F}_X)$. Note that $\overline{co}\, S^1_X(\mathcal{F}_X)$ is weakly compact in $L^1_{E^*}[E](\mathcal{F}_X)$. The preceding estimate shows that, for every $n \in \mathbb{N}$ there is $h_n \in co\, S^1_X(\mathcal{F}_X)$ such that

$$\begin{aligned} |\int_A \langle x, f - h_n\rangle dP| &\leq \int_A |\langle x, f - h_n\rangle| dP \\ &\leq \int_\Omega ||f - h_n||_{E^*_b} dP \leq \frac{1}{n} \end{aligned}$$

for all $A \in \mathcal{F}_X$ and for all $x \in \overline{B}_E$. Hence $E(f) = \lim_n E(h_n)$ in E^*_b with $E(h_n) \in coE[X, \mathcal{F}_X]$. Further, since $h_n \in coS^1_X(\mathcal{F}_X)$ and $\overline{co}S^1_X(\mathcal{F}_X)$ is weakly compact in $L^1_{E^*}[E](\mathcal{F}_X)$ we may assume that $(h_n)_{n\in\mathbb{N}}$ converges

[1] namely $\sum_{k=1}^{n} 1_{A_k} g_k$ is a convex combination with positive rational coefficients of functions in $S^1_X(\mathcal{F}_X)$

weakly in $L^1_{E^*}[E](\mathcal{F}_X)$ to $h \in \overline{co} S^1_X(\mathcal{F}_X)$. Passing to the limit when $n \to \infty$ in the inequality $|\int_A \langle x, f - h_n \rangle dP| \le \frac{1}{n}$ shows that

$$\langle x, f \rangle = \langle x, h \rangle \quad \text{a.s.}$$

for each $x \in E$, i.e. $f = h$ scalarly a.s. By separability, we may conclude that $f = h$ a.s. in E^*_s.
Step 2 and final conclusion. We have $\overline{co}\, E[X] \subset E[\overline{co} X]$ because $E[\overline{co} X]$ is convex weakly compact by Proposition 4.2-(2). By Proposition 4.1-(3) we have

$$\overline{co}\, E[X] \subset E[\overline{co} X] = E[\overline{co} X, \mathcal{F}_X] = \{E(f) : f \in S^1_{\overline{co}\, X}(\mathcal{F}_X)\}.$$

By (4.1) we have

$$\{E(f) : f \in S^1_{\overline{co}\, X}(\mathcal{F}_X)\} = \{E(f) : f \in \overline{co}\, S^1_X(\mathcal{F}_X)]\}.$$

It follows that

$$\overline{co}\, E[X] \subset E[\overline{co}\, X] = E[\overline{co} X, \mathcal{F}_X] \subset \overline{co}\, E[X, \mathcal{F}_X].$$

$\blacksquare$

We begin with an ergodic version for a stationary sequence of integrably bounded $ck(E^*_b)$-valued (i.e. convex norm compact valued) random sets and its application to weak star Kuratowski convergence for law of large numbers in the dual space. We will provide complete details of proof since our tools can be applied to other variants.

Theorem 4.4 *Let (X_n) be a stricly stationary sequence of integrably bounded $ck(E^*_b)$-valued random sets such that $g := \sup_{n \in \mathbb{N}} |X_n| \le \alpha$ is integrable, Let $\mathcal{I}$ denote the tribe of invariant events of (X_n). Then we have*

$$d_{H^*_{m^*}} \left(\frac{1}{n} \sum_{i=1}^n X_i, E^{\mathcal{I}}[X_1] \right) = 0 \quad a.s.$$

Furthermore, as the topologies m^* and w^* coincide on $g(\omega)\, \overline{B}_{E^*}$, we have

$$w^* K\text{-} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n X_i(\omega) = E^{\mathcal{I}}[X_1] \quad a.s.$$

The proof of Theorem 4.4 follows similar lines as in [23, 24]. We need a preliminary Vitali-type lemma:

Lemma 4.5 *Let (Q_n) be a sequence of Borel probability measures on the metric space $(ck(E^*_{m^*}), d_{H^*_{m^*}})$ which narrowly converges to a Borel probability measure Q on $(ck(E^*_{m^*}), d_{H^*_{m^*}})$ (that is, for any continuous bounded function $\varphi : ck(E^*_{m^*} \to \mathbb{R}$, the sequence $(Q_n(\varphi)) = (\int \varphi \, dQ_n)$ converges to $Q(\varphi)$). Assume that $|.|$ is uniformly integrable with respect to (Q_n) and Q. Then the Aumann-Gelfand expectations of Q_n converge for the Hausdorff distance $d_{H^*_{m^*}}$ to the Aumann-Gelfand expectation of Q:*

$$\lim_{n\to\infty} d_{H^*_{m^*}} \left(\int x^* dQ_n(x^*), \int x^* dQ(x^*) \right) = 0.$$

Proof. By [17, Theorem II-8], the metric space $(ck(E^*_{m^*}), d_{H^*_{m^*}})$ is separable. We can thus apply Jakubowski's version of Skorokhod's representation theorem [38, Theorem 2]: for any subsequence of (Q_n), we can find a further subsequence (which we denote by (Q_n) for simplicity of notations) and a sequence (Y_n) of $ck(E^*_{m^*})$-valued random sets defined on the Lebesgue interval $([0,1], \mathcal{B}_{[0,1]}, l)$ such that Y_n converges to Y a.s. for the Hausdorff distance $d_{H^*_{m^*}}$. Note that, by the integrability of $|.|$ with respect to Q_n and Q, the random sets Y_n and Y are integrably bounded. By Proposition 4.1-(4), we have $E[Y] = E[X]$ and $E[Y_n] = E[X_n]$ for every n.

Let us also observe that, if X is an integrably bounded $ck(E^*_{m^*})$-valued random set with distribution Q on the Borel tribe of $\mathcal{B}(ck(E^*_{m^*}))$, then

$$E[X] = \int_{ck(E^*_{m^*})} x \, dQ(x) \tag{4.2}$$

where $\int_{ck(E^*_{m^*})} x \, dQ(x)$ denotes the set of integrals of the form

$$\int_{ck(E^*_{m^*})} \varphi(x) \, dQ(x)$$

where $\varphi : ck(E^*_{m^*}) \to E^*_{m^*}$ satisfies Q-a.s. $\varphi(x) \in x$. Indeed, by Proposition 4.1-(3), since X is integrably bounded and $ck(E^*_{m^*})$-valued, we have that $E[X] = E[X, \mathcal{F}_X]$, and if $u \in \mathcal{S}^1_X(\mathcal{F}_X)$, by a well known theorem of Doob (see [26, page 603] or [25, page 18]) there exists a Borel measurable mapping $\varphi : ck(E^*_{m^*}) \to E^*_{m^*}$ such that $u = \varphi \circ X$, which proves that the right hand side of (4.2) is a subset of the left hand side. The converse inclusion is trivial.

Now, applying Theorem 3.5-(e) with X and Y defined on $[0,1]$, $\mathcal{B} = \{\emptyset, [0,1]\}$, and $B = [0,1]$, we get

$$d_{H^*_{m^*}} \left(\int X \, dl, \int Y \, dl \right) \leq \int d_{H^*_{m^*}}(X, Y) \, dl.$$

We thus have

$$d_{H^*_{m^*}}\left(\int x^* dQ_n(x^*), \int x^* dQ(x^*)\right)$$
$$= d_{H^*_{m^*}}\left(\int Y_n\, dl, \int Y\, dl\right) \leq \int d_{H^*_{m^*}}(Y_n, Y)\, dl \longrightarrow 0$$

by Vitali theorem and the uniform integrability assumption.

Let us denote, for every n,

$$\rho_n = d_{H^*_{m^*}}\left(\int x^* dQ_n(x^*), \int x^* dQ(x^*)\right).$$

We have proved that, for every subsequence of (ρ_n) there is a further subsequence which converges to 0. This proves that (ρ_n) converges to 0. ∎

Proof of Theorem 4.4. By Proposition 4.1-(3), since each X_n is integrably bounded and $ck(E^*_{m^*})$-valued, we have that $E[X_n] = E[X_n, \mathcal{F}_{X_n}]$ and $E^{\mathcal{I}}[X_1] = \{E^{\mathcal{I}}[u];\, u \in \mathcal{S}^1_{X_1}\}$.

By [17, Corollary II-9], the metric space $(ck(E^*_{m^*}), d_{H^*_{m^*}})$ is Lusin, because m^* is Lusin. Let $\omega \mapsto Q_\omega$ denote a regular version of the conditional law of X_1 with respect to $\mathcal{I}$. For each integer $n \geq 1$ and each $\omega \in \Omega$, let $Q_{n,\omega}$ be the empirical distribution $\frac{1}{n}\sum_{i=1}^n \delta_{X_i(\omega)}$, where δ_x denotes the Dirac mass at x. By the assumption on g, the sequence $(Q_{n,\omega})$ is tight for almost every ω. Furthermore, by the ergodic theorem, for any continuous $f : (ck(E^*_{m^*}), d_{H^*_{m^*}}) \to \mathbb{R}$ such that $f(X_1)$ is integrable,

$$\lim_{n\to\infty} Q_{n,\omega}(f) - Q_\omega(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f(X_i(\omega)) - E^{\mathcal{I}}[f(X_1)](\omega)) = 0 \quad a.e.$$

This shows that a.s. $(Q_{n,\omega})$ has only one possible limit, which is Q_ω. Thus there exists a measurable subset Ω' of Ω such that $P(\Omega') = 1$ and the sequence $(Q_{n,\omega})$ narrowly converges to Q_ω for all $\omega \in \Omega'$. We deduce, by Lemma 4.5, for every $\omega \in \Omega'$,

$$d_{H^*_{m^*}}\left(\frac{1}{n}\sum_{i=1}^n X_i, E^{\mathcal{I}}[X_1]\right) = d_{H^*_{m^*}}\left(\int x^* dQ_{n,\omega}(x^*), \int x^* dQ_\omega(x^*)\right) \longrightarrow 0.$$

∎

Remark 4.6 In the case when (X_n) is i.i.d., the variable $g = \sup_n |X_n|$ is necessarily constant (with finite value). Indeed, the variable $\widetilde{g} = \limsup_{n\to\infty}$ is a tail r.v., thus by the zero-one law it is a.s. constant, say $\widetilde{g}(\omega) = R$ a.s. But, as (X_n) is identically distributed, if $P(|X_1| > r) > 0$ then $|X_n| > r$ infinitely often with probability 1 by the Borel-Cantelli lemma, a contradiction.

The same arguments as in the proof of Theorem 4.4 but replacing the ergodic theorem by Etemadi's strong law of large numbers [28] yield a strong law of large numbers for pairwise identically distributed $ck(E_b^*)$-valued random sets:

Theorem 4.7 *Let (X_n) be a pairwise independent identically distributed sequence of integrably bounded $ck(E_b^*)$-valued (i.e. convex norm compact valued) random sets such that $g := \sup_{n\in\mathbb{N}} |X_n| \le \alpha$ is integrable. Then we have*

$$d_{H^*_{m^*}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i, E[X_1]\right) = 0 \quad a.s.$$

Furthermore, as the topologies m^* and w^* coincide on $g(\omega)\,\overline{B}_{E^*}$, we have

$$w^*K\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i(\omega) = E[X_1] \quad a.s.$$

Using the above techniques it is not difficult to prove the following SLLN for pairwise independent identically distributed sequence of integrably bounded $ck(E)$-valued (i.e. convex norm compact valued) random sets in the primal Banach space E. We summarize this fact as follows.

Theorem 4.8 *Let E be a Banach space such its dual is strongly separable. Let $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ (resp. $\mathcal{L}^1_{ck(E)}(\mathcal{F})$) the set of all integrably bounded $cwk(E)$-valued (resp. $ck(E)$-valued (i.e. convex weakly compact valued) (resp. convex compact valued) random sets in E. Then the following hold*
(a) Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$, then for any $X, Y \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$, $E^{\mathcal{B}}X, E^{\mathcal{B}}Y \in \mathcal{L}^1_{cwk(E)}(\mathcal{B})$ and for any $B \in \mathcal{B}$

$$\int_B d_{H_E}(E^{\mathcal{B}}X, E^{\mathcal{B}}Y)dP \le \int_B d_{H_E}(X,Y)dP$$

(b) Let (X_n) be a pairwise independent identically distributed sequence of integrably bounded $ck(E)$-valued (i.e. convex norm compact valued) random

sets in E, such that $g := \sup_{n\in\mathbb{N}} |X_n| \le \alpha$ *is integrable, then*

$$E[X_n] = E[X_1] \in ck(E), \forall n \in \mathbb{N}$$

and

$$d_{H_E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i, E[X_1]\right) = 0 \quad a.s.$$

Now we proceed to further variants for SLLN in the dual space. We need the following definition.

Definition 4.9 The Banach space E is *weakly compactly generated (WCG)* if there exists a weakly compact subset of E whose linear span is dense in E.

Theorem 4.10 *Assume that E is WCG. Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of independent* $ck(E_b^*)$*-valued random sets satisfying:*
(i) $X_n \subset g\overline{B}_{E^*}$*, for all* $n \in \mathbb{N}$ *and for some* $g \in L^1_{\mathbb{R}}(\mathcal{F})$*.*
(ii) $\sum_{n=1}^{\infty} \frac{E(|X_n|^2)}{n^2} < \infty$*.*
(iii) There exists $M \in \mathcal{K}^*$ *such that*

$$w^*\text{-}ls\, E[X_n] \subset M \subset w^*\text{-}li\, E[X_n, \mathcal{F}_{X_n}].$$

Then we have

$$w^*\text{-}K\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i(\omega) = M \quad a.s.$$

Proof. *Step 1.* By (iii) and Proposition 4.1-(3) it is clear that

$$w^*\text{-}K\text{-}\lim_{n\to\infty} E[X_n] = M.$$

By (i) the sequence $(E[X_n])_{n\in\mathbb{N}}$ is uniformly bounded in E^*. Since E is WCG, by virtue of (ii) and [29, Theorem 4.11] we have equivalently

$$\lim_{n\to\infty} E(\delta^*(x, X_n)) = \lim_{n\to\infty} \delta^*(x, E[X_n]) = \delta^*(x, M) \quad \forall x \in E.$$

Recall that $D_1 = (e_k)_{k\in\mathbb{N}}$ is a dense sequence in $\overline{B}_E$. Then from the independence of $(X_n)_{n\in\mathbb{N}}$, for each $k \in \mathbb{N}$, the sequence $(\delta^*(e_k, X_n))_{n\in\mathbb{N}}$ is independent in $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ and by (ii) we have

$$\sum_{n=1}^{\infty} \frac{E(|\delta^*(e_k, X_n)|^2)}{n^2} \le \sum_{n=1}^{\infty} \frac{E(|X_n|^2)}{n^2} < \infty.$$

As $\mathbb{R}$ is of type 2, applying the law of large numbers to this sequence yields

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}[\delta^*(e_k, X_i) - E(\delta^*(e_k, X_i)] = 0 \quad \text{a.s.}$$

Consequently

$$\lim_{n\to\infty} \delta^*(e_k, \frac{1}{n}\sum_{i=1}^{n} X_i) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \delta^*(e_k, E[X_i]) = \delta^*(e_k, M) \quad \text{a.s.}$$

Using the preceding equality and standard argument, we see that

$$w^*\text{-}ls \frac{1}{n}\sum_{i=1}^{n} X_i \subset M \quad \text{a.s.} \tag{4.3}$$

Step 2. Let $x^* \in M \subset w^*\text{-}li\, E[X_n, \mathcal{F}_{X_n}]$. There is $f_n \in S^1_{X_n}(\mathcal{F}_{X_n})$ such that $x^* = w^*\text{-}\lim_{n\to\infty} E(f_n)$. For each $k \in \mathbb{N}$, let us write

$$|\langle e_k, x^* - \frac{1}{n}\sum_{i=1}^{n} f_i\rangle \leq |\langle e_k, x^* - \frac{1}{n}\sum_{i=1}^{n} E(f_i)\rangle| + |\langle e_k, \frac{1}{n}\sum_{i=1}^{n}[E(f_i) - f_i]\rangle|. \tag{4.4}$$

From the independence of (X_n), (f_n) is also independent, so for each $k \in \mathbb{N}$, the sequence $(\langle e_k, f_n\rangle)_{n\in\mathbb{N}}$ is independent and by (ii) we have

$$\sum_{n=1}^{\infty} \frac{E(|\langle e_k, f_n\rangle|^2)}{n^2} \leq \sum_{n=1}^{\infty} \frac{E(|f_n|^2)}{n^2} \leq \sum_{n=1}^{\infty} \frac{E(|X_n|^2)}{n^2} < \infty.$$

As $\mathbb{R}$ is of type 2, appplying the law of large numbers to the sequence $(\langle e_k, f_n\rangle)_{n\in\mathbb{N}}$ yields

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}[\langle e_k, f_i) - E(\langle e_k, f_i\rangle)] = 0 \quad \text{a.s.}$$

It is easy to see that

$$\lim_{n\to\infty} \langle e_k, x^* - \frac{1}{n}\sum_{i=1}^{n} E(f_i)\rangle = 0.$$

From the estimate (4.4) we conclude that

$$\lim_{n\to\infty} \langle e_k, x^*\text{-}\frac{1}{n}\sum_{i=1}^{n} f_i\rangle = 0 \quad \text{a.s.}$$

Consequently

$$x^* \in w^*\text{-}li\, \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{a.s.}$$

As M and $w^*\text{-}li\, \frac{1}{n}\sum_{i=1}^{n} X_i$ are weak star compact, we deduce that

$$M \subset w^*\text{-}li\, \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{a.s.} \tag{4.5}$$

Then the required result follows from (4.3) and (4.5). ■

Corollary 4.11 *Assume that E is WCG. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent $ck(E_b^*)$-valued random sets satisfying:*
(j)$X_n \subset g\overline{B}_{E^*}$, for all $n \in \mathbb{N}$ and for some $g \in L^2_{\mathbb{R}}(\mathcal{F})$.
(jj) There exists $M \in \mathcal{K}^*$ such that

$$w^*\text{-}ls\, E[X_n] \subset M \subset w^*\text{-}li\, E[X_n, \mathcal{F}_{X_n}].$$

Then we have

$$w^*\text{-}K\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i(\omega) = M \quad \textit{a.s.}$$

Remarks 4.12 1) Theorems 4.4-4.7-4.8-4.10 are a dual version of similar results obtained by [35, 41] in Banach spaces. Theorem 4.7 is even new in the context of primal Banach spaces. See also [14, 33, 34] for more results on the law of large numbers for random sets in Banach spaces. It is worth to mention that the techniques developed in Theorem 4.4 provide a convexification in the limit if we consider the compact valued integrably bounded random sets.
2) A law of large numbers for pairwise independent elements in $L^1_{E^*}[E]$ satisfying some tightness condition is available in [12, Corollary 2] which is a version of Etemadi's SLLN for elements in $L^1_{E^*}[E]$ in the topology E^*_c. See [12, Remark 3], and [43, Theorem 3.3].

5 Law of large numbers and ergodic theorem involving subdifferential operators

We need to recall and summarize some notions on the subdifferential mapping of local Lipchizean functions developed by L. Thibault [44]. Let f :

$E \to \mathbb{R}$ be a locally Lipchizean function. By Christensen [18, Theorem 7.5], there is a set D_f such that its complementary is Haar-nul (hence Df is dense in E) such that for all $x \in D_f$ and for all $v \in E$

$$r_f(x,v) = \lim_{\delta \to 0} \frac{f(x+\delta v) - f(x)}{\delta}$$

exists and $v \mapsto r_f(x,v)$ is linear and continuous. Let us set $\nabla f(x) = r_f(x,.) \in E^*$. Then $r_f(x,v) = \langle \nabla f(x), v \rangle$, $\nabla f(x)$ is the gradient of f at the point x. Let us set

$$\mathcal{L}_f(x) = \{\lim_{j \to \infty} \nabla f(x_j) | x_j \in D_f, x_j \to x\}.$$

By definition, the subdifferential $\partial f(x)$ in the sense of Clarke [20] at the point $x \in E$ is defined by

$$\partial f(x) = \overline{co}\, \mathcal{L}_f(x).$$

The generalized directional derivative of f at a point $x \in E$ in the direction $v \in E$ is denoted by

$$f^{\cdot}(x,v) = \limsup_{h \to 0, \delta \to 0} \frac{f(x+h+\delta v) - f(x+h)}{\delta}.$$

Proposition 5.1 *Let $f : E \to \mathbb{R}$ be a locally Lipchizean function. Then the subdifferential $\partial f(x)$ at the point $x \in E$ is convex weak star compact and*

$$f^{\cdot}(x,v) = \sup\{\langle \zeta^*, v \rangle | \zeta^* \in \partial f(x)\} \quad \forall v \in E$$

that is, $f^{\cdot}(x,.)$ is the support function of $\partial f(x)$.

Proof. See Thibault [44, Proposition I.12]. ∎

Here are some useful properties of the subdifferential mapping.

Proposition 5.2 *Let $f : E \to \mathbb{R}$ be a locally Lipchizean function. Then the convex weak star compact valued subdifferential mapping ∂f is upper semicontinuous with respect to the weak star topology.*

Proof. See [44, Proposition I. 17]. Indeed we have

$$\delta^*(v, \partial f(x)) = f^{\cdot}(x; v) = \limsup_{h \to 0, \delta \to 0} \frac{[f(x+h+\delta v) - f(x+h)]}{\delta}$$

As $f^{\cdot}(.; v)$ is upper semicontinuous and ∂f is convex compact valued in E^*_s, by [17] or [46], ∂f is upper semicontinuous in E^*_s. ∎

Proposition 5.3 *Let $(T, \mathcal{T})$ a measurable space and $f : T \times E \to \mathbb{R}$ such that*
$f(., \zeta)$ is $\mathcal{T}$-measurable, for every $\zeta \in E$.
$f(t, .)$ is locally Lipschitzean for every $t \in T$.
Let $f_t^{\cdot}(x; v)$ the directional derivative of $f(t, .) := f_t$ in the direction v for every fixed $t \in T$. Let x and v be two $\mathcal{T}$-measurable mappings from T to E. Then the following hold:
(a) the mapping $t \mapsto f_t^{\cdot}(x(t); v(t))$ is $\mathcal{T}$-measurable.
(b) the mapping $t \mapsto \partial f_t(x(t))$ is graph measurable, that is, its graph belongs to $\mathcal{T} \otimes \mathcal{B}(E_s^*)$.

Proof. See Thibault [44, Proposition I.20 and Corollary I. 21]. Note that the convex weak star compact valued mapping $t \mapsto \partial f_t(x(t))$ is scalarly $\mathcal{T}$-measurable, and so enjoys good measurability properties because E_s^* is a locally convex Lusin space. See in particular Proposition 3.1. ■

We end with two specific applications in the law of large numbers and ergodic theorem involving the subdifferential operators.

Theorem 5.4 *Assume that E is WCG. Let $f : E \to \mathbb{R}$ be a Lipschitzean mapping, i.e. there exists $\beta > 0$ such that for all $x, y \in E$, $|f(x) - f(y)| \leq \beta||x - y||$. Let $(u_n)_{n \in \mathbb{N}}$ be an i.i.d sequence in $L_E^1(\Omega, \mathcal{F}, P)$. Then we have*

$$w^*\text{-}K\text{-} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \partial f(u_i(\omega)) = \int_\Omega \partial f(u_1(\omega)) dP(\omega) \quad a.s.$$

where $\int_\Omega \partial f(u_1(\omega)) dP(\omega)$ is the Aumann-Gelfand multivalued integral of the convex weak star compact valued mapping $\partial f(u_1(.))$.

Proof. By the Lipschitz assumption it is clear that $|\partial f(x)| \leq \beta \quad \forall x \in E$ so that for each $n \in N$, $\omega \mapsto \partial f(u_n(\omega))$ is a convex weak star compact valued and integrably bounded, shortly $\partial f(u_n(.)) \in \mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$ with $|\partial f(u_n(\omega))| \leq \beta$ for all $n \in \mathbb{N}$ and for all $\omega \in \Omega$. Indeed by Proposition 5.3, $\omega \mapsto f^{\cdot}(u_n(\omega); v(\omega))$ is $\mathcal{F}$-measurable, for all $n \in \mathbb{N}$ and for all $\mathcal{F}$-measurable mapping $v : \Omega \to E$, in particular, using Proposition 5.2, the support function

$$\delta^*(v, \partial f(u_n(.))) = f^{\cdot}(u_n(\omega); v)$$

of the $cwk(E^*)$-valued mapping $\partial f(u_n(.))$ is $\mathcal{F}$-measurable, for every $v \in E$. Recall that for each $v \in E$, the function $f^{\cdot}(.; v)$ is upper semicontinuous on

E and is bounded because

$$|\delta^*(v, \partial f(x)| = |f^{\cdot}(x; v)| \leq \beta ||v||.$$

Now let $D_1 = (e_k)_{k\in\mathbb{N}}$ be a dense sequence in the closed unit ball $\overline{B}_E$. From the above consideration, it is clear that for each $k \in \mathbb{N}$, the sequence $(\delta^*(e_k, \partial f(u_n(.))))_n = (f^{\cdot}(u_n(.); e_k))_n$ is i.i.d in $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, P)$. According to the classical law of large numbers, we have for a.s. $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta^*(e_k, \partial f(u_i(\omega))) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f^{\cdot}(u_i(\omega); e_k)$$

$$= E(f^{\cdot}(u_1(.); e_k)) = E(\delta^*(e_k, \partial f(u_1(.)))) = \delta^*(e_k, \int_\Omega \partial f(u_1(\omega))dP(\omega)).$$

By density argument we may assert that

$$\lim_{n\to\infty} \delta^*(e, \frac{1}{n} \sum_{i=1}^{n} \partial f(u_i(\omega))) = \delta^*(e, \int_\Omega \partial f(u_1(\omega))dP(\omega)) \quad \text{a.s.}$$

for all $e \in \overline{B}_E$ [2]. Since $\partial f(u_n(\omega)) \subset \beta \overline{B}_{E^*}$ for all $n \in \mathbb{N}$ and for all $\omega \in \Omega$ and $\int_\Omega \partial f(u_1(\omega))dP(\omega) \subset \beta \overline{B}_{E^*}$ and the Banach space E is WCG, by [29, Theorem 4.11], we deduce that

$$w^*\text{-}K\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \partial f(u_i(\omega)) = \int_\Omega \partial f(u_1(\omega))dP(\omega) \quad \text{a.s.}$$

■

Theorem 5.5 *Assume that E is WCG. Let T be a $\mathcal{F}$-measurable transformation of Ω preserving P, $\mathcal{I}$ the σ algebra of invariant sets. Let $f : \Omega \times E \to \mathbb{R}$ be a mapping satisfying*
(a) For every $x \in E$, $f(., x)$ is $\mathcal{F}$-measurable on Ω.
(b) There exists $\beta \in L^1_{\mathbb{R}^+}(\Omega, \mathcal{F}, P)$ such that for all $\omega \in \Omega$, for all $x, y \in E$

$$|f(\omega, x) - f(\omega, y)| \leq \beta(\omega)||x - y||.$$

Then for any $u \in L^0_E(\Omega, \mathcal{I}, P)$ the following holds

$$w^*\text{-}K\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega)) = E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega) \quad a.s.$$

where $\partial f_{.}(u(.))$ is the mapping $\omega \mapsto \partial f_\omega(u(\omega))$ and $E^{\mathcal{I}}[\partial f_{.}(u(.))]$ is the conditional expectation of $\partial f_{.}(u(.))$ with respect to $\mathcal{I}$.

[2] For more details, one may consult the proof of Theorem 5.5 below

Proof. By Propositions 5.1-5.3 recall that

$$\begin{aligned}\delta^*(v, \partial f_\omega(x)) = f_\omega^{\cdot}(x; v) &= \limsup_{h \to 0,\, \delta \to 0} \frac{[f_\omega(x+h+\delta v) - f_\omega(x+h)]}{\delta} \\ &\leq \beta(\omega)||v|| = \delta^*(v, \beta(\omega)\overline{B}_{E^*})\end{aligned}$$

for every $\omega \in \Omega$, for every $v \in E$ and for every $x \in E$ and the mapping $\omega \mapsto f_\omega^{\cdot}(u(\omega); v(\omega))$ is $\mathcal{F}$-measurable for every $\mathcal{F}$-measurable mapping $u : \Omega \to E$ and $v : \Omega \to E$. Let $D_1 = (e_k)_{k \in \mathbb{N}}$ be a dense sequence in the closed unit ball $\overline{B}_E$.Then the mapping $f_\omega^{\cdot}(u(\omega); e_k)$ is $\mathcal{F}$-measurable for every $\mathcal{I}$-measurable mapping $u : \Omega \to E$ and since u is $\mathcal{I}$-measurable

$$f_{T^i\omega}^{\cdot}(u(\omega); e_k) = f_{T^i\omega}^{\cdot}(u(T^i\omega); e_k) \quad \forall \omega \in \Omega$$

so that by the classical ergodic theorem for real valued quasi-integrable functions, see e.g. [9, 47] we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} f_{T^i\omega}^{\cdot}(u(\omega); e_k) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} f_{T^i\omega}^{\cdot}(u(T^i\omega); e_k) = E^{\mathcal{I}} f_\omega^{\cdot}(u(\omega); e_k).$$

By Proposition 5.1 we have

$$\begin{aligned} f_\omega^{\cdot}(u(\omega); e_k) &= \delta^*(e_k, \partial f_\omega(u(\omega))) \quad \forall \omega \in \Omega, \\ f_{T^i\omega}^{\cdot}(u(\omega); e_k) &= \delta^*(e_k, \partial f_{T^i\omega}(u(\omega))) \quad \forall \omega \in \Omega. \end{aligned}$$

By the above computation we see that the mapping $\omega \mapsto \partial f_\omega(u(\omega))$ belongs to $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ because $\partial f_\omega(u(\omega)) \subset \beta(\omega)\overline{B}_{E^*}$ for all $\omega \in \Omega$. Further, by Theorem 3.4 (or Theorem 3.5) the conditional expectation $E^{\mathcal{I}}[\partial f_.(u(.))]$ belongs to $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{I})$ with

$$\delta^*(e_k, E^{\mathcal{I}}[\partial f_.(u(.))](\omega)) = E^{\mathcal{I}} \delta^*(e_k, \partial f_\omega(u(\omega))) = E^{\mathcal{I}} f_\omega^{\cdot}(u(\omega); e_k) \quad \text{a.s.}$$

Finally by combining these equalities we get

$$\lim_{n\to\infty} \delta^*(e_k, \frac{1}{n} \sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega))) = \delta^*(e_k, E^{\mathcal{I}}[\partial f_.(u(.))](\omega)) \quad \text{a.s.}$$

Since $\partial f_\omega(u(\omega)) \subset \beta(\omega)\overline{B}_{E^*}$ and $E^{\mathcal{I}}[\partial f_.(u(.))](\omega) \subset E^{\mathcal{I}}\beta(\omega)\overline{B}_{E^*}$, for all $\omega \in \Omega$, we deduce by denseness that

$$\lim_{n\to\infty} \delta^*(e, \frac{1}{n} \sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega))) = \delta^*(e, E^{\mathcal{I}}[\partial f_.(u(.))](\omega)) \quad \text{a.s.} \quad \forall e \in \overline{B}_E.$$

This need a careful look. Applying the classical Birkhoff ergodic theorem to β yields

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \beta(T^i\omega) = E^{\mathcal{I}}\beta(\omega) \quad \text{a.s.}$$

Consequently $\frac{1}{n}\sum_{i=0}^{n-1} \beta(T^i\omega)$ is pointwise bounded a.s., say

$$\gamma(\omega) := \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=0}^{n-1} \beta(T^i\omega) < \infty \quad \text{a.s.}$$

It follows that

$$\frac{1}{n}\sum_{i=0}^{n-1} \partial f_{T^i\omega}(u(\omega)) \subset [\frac{1}{n}\sum_{i=0}^{n-1} \beta(T^i\omega)]\overline{B}_{E^*} \subset \gamma(\omega)\overline{B}_{E^*} \quad \text{a.s.}$$

There is a negligible set N_0 such that for each $\omega \in \Omega \setminus N_0$

$$\gamma(\omega) := \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=0}^{n-1} \beta(T^i\omega) < \infty$$

and there is a negligible set N_k such that for each $\omega \in \Omega \setminus N_k$

$$\lim_{n\to\infty} \delta^*(e_k, \frac{1}{n}\sum_{i=0}^{n-1} \partial f_{T^i\omega}(u(\omega))) = \delta^*(e_k, E^{\mathcal{I}}[\partial f_.(u(.))](\omega)).$$

Then $N = \cup_{k\geq 0} N_k$ is negligible. Let $\omega \in \Omega \setminus N$, $e \in \overline{B}_E$ and $\varepsilon > 0$. Pick $e_j \in D_1$ such that

$$\max\{\delta^*(e - e_j, E^{\mathcal{I}}\beta(\omega)\overline{B}_{E^*}), \delta^*(e_j - e, E^{\mathcal{I}}\beta(\omega)\overline{B}_{E^*})\} < \varepsilon$$

and

$$\max\{\delta^*(e - e_j, \gamma(\omega)\overline{B}_{E^*}), \delta^*(e_j - e, \gamma(\omega)\overline{B}_{E^*})\} < \varepsilon.$$

For simplicity let us set

$$S_n(\omega) := \frac{1}{n}\sum_{i=0}^{n-1} \partial f_{T^i\omega}(u(\omega)), \quad \forall n \in \mathbb{N} \quad \forall \omega \in \Omega.$$

Let us write the estimate

$$\begin{aligned}
&|\delta^*(e, S_n(\omega)) - \delta^*(e, E^{\mathcal{I}}[\partial f_.(u(.))](\omega))| \\
&\qquad \leq |\delta^*(e, S_n(\omega)) - \delta^*(e_j, S_n(\omega))| \\
&\qquad\quad + |\delta^*(e_j, S_n(\omega)) - \delta^*(e_j, E^{\mathcal{I}}[\partial f_.(u(.))](\omega))| \\
&\qquad\quad + |\delta^*(e_j, E^{\mathcal{I}}[\partial f_.(u(.))](\omega)) - \delta^*(e, E^{\mathcal{I}}[\partial f_.(u(.))](\omega))|
\end{aligned}$$

As $S_n(\omega) \subset \gamma(\omega)\overline{B}_{E^*}$ and $E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega) \subset E^{\mathcal{I}}\beta(\omega)\overline{B}_{E^*}$ for all $n \in \mathbb{N}$ and for all $\omega \in \Omega \setminus N$, we have the estimates

$$|\delta^*(e, S_n(\omega)) - \delta^*(e_j, S_n(\omega))| \\ \leq \max\{\delta^*(e - e_j, \gamma(\omega)\overline{B}_{E^*}), \delta^*(e_j - e, \gamma(\omega)\overline{B}_{E^*})\} < \varepsilon$$

and

$$|\delta^*(e_j, E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega)) - \delta^*(e, E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega))| \\ \leq \max\{\delta^*(e - e_j, E^{\mathcal{I}}\beta(\omega)\overline{B}_{E^*}), \delta^*(e_j - e, E^{\mathcal{I}}\beta(\omega)\overline{B}_{E^*})\} < \varepsilon.$$

Finally we get

$$|\delta^*(e, S_n(\omega)) - \delta^*(e, E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega))| \\ < |\delta^*(e_j, S_n(\omega)) - \delta^*(e_j, E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega))| + 2\varepsilon.$$

As $|\delta^*(e_j, S_n(\omega)) - \delta^*(e_j, [E^{\mathcal{I}}\partial f_{.}(u(.))](\omega))| \to 0$, from the preceding estimate, it is immediate to see that for $\omega \in \Omega \setminus N$ and $e \in \overline{B}_E$ we have

$$\lim_{n\to\infty} \delta^*(e, \frac{1}{n}\sum_{i=0}^{n-1} \partial f_{T^i\omega}(u(\omega))) = \delta^*(e, E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega)).$$

In other words, $\frac{1}{n}\sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega))$ converges scalarly a.s. to $E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega)$. Since E is WCG, by [29, Theorem 4.11], we conclude that

$$w^*\text{-}K\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \partial f_{T^i\omega}(u(\omega)) = E^{\mathcal{I}}[\partial f_{.}(u(.))](\omega) \quad \text{a.s.}$$

■

Corollary 5.6 *With the hypothesis and notations of Theorem 5.5, if T is ergodic, then the following holds*

$$w^*\text{-}K\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \partial f_{T^i\omega}(u(\omega)) = E[\partial f_{.}(u(.))] \quad a.s.$$

where $\partial f_{.}(u(.))$ is the mapping $\omega \mapsto \partial f_\omega(u(\omega))$ and

$$E[\partial f_{.}(u(.))] := \int_\Omega \partial f_\omega(u(\omega))dP(\omega)$$

is the expectation of $\partial f_{.}(u(.))$.

We end the paper with an unusual closure type lemma arisen in evolution problems, see [4, 10, 27, 39, 40, 44, 45] and the references therein.

Theorem 5.7 *Assume that E is WCG. Let T be a $\mathcal{F}$-measurable transformation of Ω preserving P, $\mathcal{I}$ the σ algebra of invariant sets. Let $f : \Omega \times E \to \mathbb{R}$ be a mapping satisfying*
(a) For every $x \in E$, $f(.,x)$ is $\mathcal{F}$-measurable on Ω.
(b) There exists $\beta \in L^1_{\mathbb{R}^+}(\Omega, \mathcal{F}, P)$ such that for all $\omega \in \Omega$ and for all $x, y \in E$

$$|f(\omega, x) - f(\omega, y)| \leq \beta(\omega)||x - y||.$$

Let $g : \Omega \times E \to \mathbb{R}$ be a mapping satisfying
(c) For every $x \in E$, $g(.,x)$ is $\mathcal{F}$-measurable on Ω.
(d) There exists $\lambda \in L^1_{\mathbb{R}^+}(\Omega, \mathcal{F}, P)$ such that for all $\omega \in \Omega$ and for all $x, y \in E$

$$|g(\omega, x) - g(\omega, y)| \leq \lambda(\omega)||x - y||.$$

Let $u \in L^0_E(\Omega, \mathcal{I}, P)$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^0_E(\Omega, \mathcal{F}, P)$ which pointwise norm converges to $u_\infty \in L^0_E(\Omega, \mathcal{F}, P)$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence in $L^1_{E^*}[E](\Omega, \mathcal{F}, P)$ which $\sigma(L^1_{E^*}[E], L^\infty_E)$ converges to $v_\infty \in L^1_{E^*}[E](\Omega, \mathcal{F}, P)$. Assume that

$$0 \in v_n(\omega) + \frac{1}{n}\sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega)) + \partial g_\omega(u_n(\omega)) \quad \forall n \in \mathbb{N}, \quad \forall \omega \in \Omega$$

Then the following inclusion holds

$$-v_\infty(\omega) \in E^{\mathcal{I}}[\partial f_.(u(.))](\omega) + \partial g_\omega(u_\infty(\omega)) \quad a.s. \quad \omega \in \Omega.$$

Proof. Let $(e_k)_{k\in\mathbb{N}}$ be a dense sequence in the closed unit ball $\overline{B}_E$. From the inclusion

$$0 \in v_n(\omega) + \frac{1}{n}\sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega)) + \partial g_\omega(u_n(\omega)), \quad \forall n \in \mathbb{N}, \quad \forall \omega \in \Omega.$$

it follows that, for each $k \in \mathbb{N}$,

$$0 \leq \langle e_k, v_n(\omega)\rangle + \delta^*(e_k, \frac{1}{n}\sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega))) + \delta^*(e_k, \partial g_\omega(u_n(\omega))), \ \forall n \in \mathbb{N}, \ \forall \omega \in \Omega.$$

For $A \in \mathcal{F}$ and for $k \in \mathbb{N}$, we have by integrating this inequality

$$(5.1)\quad 0 \le \int_A \langle e_k, v_n(\omega)\rangle dP(\omega) + \int_A \delta^*(e_k, \frac{1}{n}\sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega)))dP(\omega) + \int_A \delta^*(e_k, \partial g_\omega(u_n(\omega)))dP(\omega).$$

It is clear that

$$(5.2)\qquad \lim_{n\to\infty} \int_A \langle e_k, v_n(\omega)\rangle dP(\omega) = \int_A \langle e_k, v_\infty(\omega)\rangle dP(\omega).$$

By Theorem 5.5

$$\lim_{n\to\infty} \delta^*(e_k, \frac{1}{n}\sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega))) = \delta^*(e_k, E^{\mathcal{I}}[\partial f_.(u(.))](\omega)) \quad \text{a.s.}$$

so that by Lebesgue-Vitali theorem
(5.3)

$$\lim_{n\to\infty} \int_A \delta^*(e_k, \frac{1}{n}\sum_{i=1}^{n} \partial f_{T^i\omega}(u(\omega)))dP(\omega) = \int_A \delta^*(e_k, E^{\mathcal{I}}[\partial f_.(u(.))](\omega))dP(\omega).$$

Let us examine the last integral

$$\int_A \delta^*(e_k, \partial g_\omega(u_n(\omega))dP(\omega).$$

We have clearly

$$\limsup_{n\to\infty} g^{\cdot}_\omega(u_n(\omega), e_k) \le g^{\cdot}_\omega(u_\infty(\omega), e_k) = \delta^*(e_k, \partial g_\omega(u_\infty(\omega))) \le \lambda(\omega)$$

so that

$$(5.4)\quad \limsup_{n\to\infty} \int_A \delta^*(e_k, \partial g_\omega(u_n(\omega)))dP(\omega) \le \int_A g^{\cdot}_\omega(u_\infty(\omega), e_k)dP(\omega) = \int_A \delta^*(e_k, \partial g_\omega(u_\infty(\omega)))dP(\omega).$$

By passing to the limit when n goes to ∞ in (5.1) and using (5.2) , (5.3), and (5.4), we get

$$0 \le \int_A \langle e_k, v_\infty(\omega)\rangle + \int_A \delta^*(e_k, E^{\mathcal{I}}[\partial f_.(u(.))](\omega))dP(\omega) + \int_A \delta^*(e_k, \partial g_\omega(u_\infty(\omega)))dP(\omega)$$

$$= \int_A [\langle e_k, v_\infty(\omega)\rangle + \delta^*(e_k, E^{\mathcal{I}}[\partial f_.(u(.))](\omega)) + \delta^*(e_k, \partial g_\omega(u_\infty(\omega)))]dP(\omega)$$

which implies

$$0 \le \langle e_k, v_\infty(\omega)\rangle + \delta^*(e_k, E^{\mathcal{I}}[\partial f_.(u(.))](\omega)) + \delta^*(e_k, \partial g_\omega(u_\infty(\omega))) \quad \text{a.s.}$$

which by density and weak star compactness of conditional expectation and subdifferential yields

$$-v_\infty(\omega) \in E^{\mathcal{I}}[\partial f_.(u(.))](\omega) + \partial g_\omega(u_\infty(\omega)) \quad \text{a.s.}$$

■

Remark 5.8 In the particular case when E is reflexive, $L^1_{E^*}[E](\Omega, \mathcal{F}, P)$ coincides with the usual Lebesgue-Bochner space $L^1_{E^*_b}(\Omega, \mathcal{F}, P)$, and Theorem 5.7 is even new when E is reflexive separable.

At this point we present a variant of Theorem 5.7 in a separable Hilbert space. Recall that, for a given $\rho \in]0, +\infty]$, a nonempty subset S of a Hilbert space H is ρ-prox-regular or equivalently ρ-proximally smooth [21, 45] if and only if every nonzero proximal normal to S can be realized by a ρ-ball. This is equivalent to say that for every $x \in S$, and for every $v \neq 0,\ v \in N^p(S; x)$,

$$\langle \frac{v}{||v||}, x' - x\rangle \le \frac{1}{2}\rho||x' - x||^2$$

for all $x' \in S$ where $N^p_S(x)$ is the proximal normal cone of S at the point $x \in S$ defined by

$$N^p_S(x) = \{\xi \in H : \exists r > 0, x \in \ \mathrm{Proj}_S(x + r\xi)\}.$$

We make the convention $\frac{1}{\rho} = 0$ for $\rho = +\infty$ and recall that for $\rho = +\infty$, the ρ-proximal regularity of S is equivalent to the convexity of S. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ a proper function and $x \in \mathit{dom} f$ with $f(x) < +\infty$, the *proximal subdifferential* of f at x is the set $\partial^p f(x)$ of all elements $v \in H$ for which there exists $\varepsilon > 0$ and $r > 0$ such that

$$f(y) \ge f(x) + \langle v, y - x\rangle - r||y - x||^2$$

for all $y \in \overline{B}_H(x, \varepsilon)$.

The following proposition summarizes some important consequences of proximally regular sets, for the proofs we refer to [6, 27].

Proposition 5.9 *For any nonempty ρ-prox-regular closed subset S of H and $x \in S$, the following hold*

1) $\partial^p d_S(x) = N_S^p(x) \bigcap \overline{B}_H(0,1)$ *where* $\overline{B}_H(0,1)$ *is the closed unit ball in* H*, and* $\partial^p d_S(x)$ *is the proximal subdifferential of the distance function* $d_S : x \mapsto d(x,S)$ *at the point* x.
(2) The proximal subdifferential $\partial^p d_S(x)$ *coincides with the Clarke subdifferential* $\partial^c d_S(x)$ *at all points* $x \in S$ *satisfying* $d(x,S) < \rho$.
3) *For all* $x \in H$ *with* $d_S(x) \leq \rho$*, the projection* $\mathrm{Proj}_S(x)$ *is single-valued.*
4) *Let* $C : [0,T] \times H \Rightarrow H$ *be a* ρ*-prox regular closed valued mapping satisfying*

$$|d(u, C(t,x)) - d(v, C(s,y))| \leq ||u - v|| + v(t) - v(s) + L||x - y||$$

for all u, x, v, y *in* H *and for all* $s \leq t$ *in* $[0,T]$*, where* $v : [0,T] \to \mathbb{R}^+$ *is a nondecreasing absolutely continuous function and* L *is a positive constant. Then the convex weakly compact valued mapping* $(t,x,y) \to \partial^p d_{C(t,x)}(y)$ *satisfies the upper semicontinuity property: Let* (t_n, x_n) *be a sequence in* $[0,T] \times H$ *converging to some* $(t,x) \in [0,T] \times H$*, and* (y_n) *be a sequence in* H *with* $y_n \in C(t_n, x_n)$ *for all* n*, converging to* $y \in C(t,x)$*, then, for any* $z \in H$*,*

$$\limsup_n \delta^*(z, \partial^p d_{C(t_n,x_n)}(y_n)) \leq \delta^*(z, \partial^p d_{C(t,x)}(y)).$$

We finish with a variant of Theorem 5.7.

Theorem 5.10 *Assume that* H *is a separable Hilbert space. Let* T *be a* $\mathcal{F}$*-measurable transformation of* Ω *preserving* P*,* $\mathcal{I}$ *the* σ *algebra of invariant sets. Let* $C : H \Rightarrow H$ *be a* ρ*-prox regular closed valued mapping satisfying*

$$|d(u, C(x)) - d(v, C(y))| \leq ||u - v|| + L||x - y||$$

for all u, x, v, y *in* H *where* L *is a positive constant. Let* $f : \Omega \times H \to \mathbb{R}$ *be a mapping satisfying*
(a) For every $x \in H$*,* $f(.,x)$ *is* $\mathcal{F}$*-measurable on* Ω.
(b) There exists $\beta \in L^1_{\mathbb{R}^+}(\Omega, \mathcal{F}, P)$ *such that for all* $\omega \in \Omega$ *and for all* $x, y \in H$

$$|f(\omega, x) - f(\omega, y)| \leq \beta(\omega)||x - y||.$$

Let $u \in L^0_H(\Omega, \mathcal{I}, P)$*, and let* $(x_n)_{n\in\mathbb{N}}$*,* $(y_n)_{n\in\mathbb{N}}$ *be two sequences in* $L^0_H(\Omega, \mathcal{F}, P)$ *which pointwise norm converge to* x_∞ *and* y_∞ *in* $L^0_H(\Omega, \mathcal{F}, P)$ *with* $y_n(\omega) \in C(x_n(\omega))$ *for all* $n \in \mathbb{N}$ *and for all* $\omega \in \Omega$ *and* $(v_n)_{n\in\mathbb{N}}$ *be a sequence in* $L^1_H(\Omega, \mathcal{F}, P)$ *which* $\sigma(L^1_H, L^\infty_H)$ *converges to* $v_\infty \in L^1_H(\Omega, \mathcal{F}, P)$*. Assume that*

$$0 \in v_n(\omega) + \frac{1}{n}\sum_{i=1}^{n-1} \partial f_{T^i\omega}(u(\omega)) + \partial^p d_{C(x_n(\omega))}(y_n(\omega)) \quad \forall n \in \mathbb{N}, \quad \forall \omega \in \Omega.$$

Then the following inclusion holds

$$-v_\infty(\omega) \in E^{\mathcal{I}}[\partial f_.(u(.))](\omega) + \partial^p d_{C(x_\infty(\omega))}(y_\infty(\omega)) \quad a.s. \quad \omega \in \Omega.$$

Proof. For shortness we omit the proof which is a direct application of Proposition 5.9 and Theorem 5.7. ■

We conclude this paper with a problem and remarks.

Problem and remarks 5.11 (1) In the context of Banach spaces, fairly general versions of law of large numbers for double array of independent (or pairwise independent) unbounded closed valued random sets in a separable Banach space are obtained in [14, Theorems 4.5-4.6].
(2) The usual embedding method for the law of large numbers for convex compact valued random sets in separable Banach space seems unavailable in the framework of dual space and the use of Bochner integration involving the Borel tribe $\mathcal{B}(E^*_b)$ is irrelevant in this context. For more properties of subdifferential of locally Lipschitzean functions defined on separable Banach space involving the use of Haar measure and the Suslin property of the weak star dual space, we refer to [27, 44].
(3) The present study is a step forward in the convergence problem for both the law of large numbers and ergodic theorem for *integrably bounded* convex weak star compact valued Gelfand-integrable mappings, several open problems will appear when this integrability assumption is no longer true, even the existence of conditional expectation for these mappings is available. In particular, the law large numbers for double array of independent (or pairwise independent) weak star compact valued random set is an open problem, even for $L^1_{E^*}[E]$ elements. Compare with the SLLN of Csorgo and al [22], Etemadi [28] and Castaing-Raynaud de Fitte [12, Corollary 2]. In particular, a.s. convergence in E^*_c for pairwise independent i.i.d sequences in $L^1_{E^*}[E]$ is provided in [12]. In view of applications, it is worthwhile to present a study of independence and distribution for unbounded random sets in the dual space. At this point one may consult the papers by Hess dealing with Banach separable spaces or more generally complete separable metric spaces [31, 32, 33, 34] where further related results can be found.

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