AN EXISTENCE RESULT FOR A CLASS OF SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS *

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ABSTRACT

In this work, we study the semilinear degenerate Dirichlet problem,
\[ \sum_{j=1}^{m} X_j^* X_j u + cu + f(x, u, Xu) = 0, \text{ in } \Omega; \quad u = \phi, \text{ on } \partial \Omega, \]
where \( X = \{X_1, \ldots, X_m\} \) is a system of real smooth vector fields which satisfies the Hörmander’s condition. Assume that \( X_1, \ldots, X_m \) satisfies some supplementary conditions on the boundary \( \partial \Omega \), \( f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^m), \partial_z f(x, z, \xi) \geq 0, \text{sign} \, z f(x, z, 0) \geq \mu > -\infty, c(x) \geq c_0 > 0. \)
With some growth hypothesis of \( f(x, z, \xi) \) in the variables \( \xi \), we have proved the existence and the uniqueness of solution \( u \in C^\infty(\Omega) \) of above semilinear Dirichlet problem, if \( \phi \in C^\infty(\partial \Omega). \)

Key Words Semilinear degenerate elliptic equation, vector fields, “non-isotropic” Hölder’s space, Dirichlet problems.

Classification 35I, 35H.

1 Introduction

In this work, we study the following semilinear Dirichlet problem:

\[ \begin{cases} \quad L u \equiv \sum_{j=1}^{m} X_j^* X_j u + cu + f(x, u, Xu) = 0, & \text{in } \Omega \\ \quad u = \phi, & \text{on } \partial \Omega \end{cases} \]  

where \( X = \{X_1, \ldots, X_m\} \) is a system of real smooth vector fields defined in an open domain \( M \subset \mathbb{R}^n, n \geq 2, \Omega \) is a bounded open subdomain of \( M \) with \( \partial \Omega \) smooth, \( c(x) \geq c_0 > 0. \ X_j^* = -X_j + c_j \) is the adjoint of \( X_j \). We assume that the system of vector fields \( X = \{X_1, \ldots, X_m\} \) satisfies the following Hörmander’s condition:

\( X_1, \ldots, X_m \) together with their commutators \( X_\alpha = [X_{\alpha_1}, \ldots][X_{\alpha_{a-1}}, X_{\alpha_a}] \ldots \)
up to some fixed length \( r \) span the tangent space at each point of \( M \).

We will study the problem (1) and similar to the case of second order elliptic equations. The role of Laplaceian $-\Delta_x$ is substituted by the Hörmander’s operators $H = \sum_{j=1}^{m} X_j^* X_j + c$. Actually by using the geometry and the function spaces associated with the system of vector fields $X$, the operators $H$ seems to satisfy nearly all properties of Laplacian $-\Delta_x$ (see [1], [2], [3], [4], [5]). For example, we have proved in [10] the following linear Dirichlet problem:

\[
\begin{align*}
H u &= f, & \text{in } \Omega, \\
u &= \phi, & \text{on } \partial \Omega,
\end{align*}
\]

has a solution $u \in S^{k+2,\alpha}(\Omega)$, if $f \in S^{k,\alpha}(\Omega), \phi \in S^{k+2,\alpha}(\partial \Omega)$, and $\partial \Omega$ satisfies following additional conditions (S. E. $\partial \Omega$)

$\partial \Omega$ is non characteristic for the system $X$. And for all $1 \leq j \leq r$, we have $X_j^0 = X_j \cap T_x(\partial \Omega)$ for all $x \in \partial \Omega$, and the dimension of $X_j^0$ is constant in a neighborhood of $\overline{\Omega}$.

Where $X_1$ is the linear space spanned by the vector fields $X_1, \ldots, X_m$ with smooth real coefficients in $C^\infty(M)$, $X_j = [X_1, X_j-1]$. And for $x \in \partial \Omega, X_j^0 = X_j \cap T_x(\partial \Omega), X_j^0 = [X_1^0, X_j^0-1]$. $S^{k,\alpha}(\Omega)$ is the “non-isotropic” Hölder space associated with the system of vector fields $X$ (see [7] and Section 2).

Then, the Hörmander’s condition implies that $X_j(\xi) = T_x M$ for all $x \in M$. And the condition (S. E. $\partial \Omega$) implies that the bases of $X_j^0$ (vector fields defined on $\partial \Omega$) satisfies the Hörmander’s condition as well on the manifold $\partial \Omega$ at order $r$.

Using the results of [8] (Theorem 2), we prove in this work the following theorem:

**Theorem 1** Assume that the system of vector fields $X = \{X_1, \ldots, X_m\}$ and $\partial \Omega$ satisfies the Hörmander’s condition and (S. E. $\partial \Omega$). Let $f \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m)$, $\partial_\xi f (x, z, \xi) \geq 0, \sign z f (x, z, 0) \geq \mu > -\infty, f \in S^{k+2,\alpha}(\partial \Omega), k \in \mathbb{N}, 1 > \alpha > 0$, and if there is $\theta \in [1, 2[$, such that $|f|_{\alpha,K,M} \leq C(K^\theta + 1)$ for all $K > 0$ and $0 < M \leq M_0$. Then there exists a solution $u \in S^{k+2,\beta}(\Omega)$ of Dirichlet problem (1) for some $\beta > 0$.

Where we denote by

\[
\begin{align*}
M_0 &= \sup_{\partial \Omega} |\phi| + c_0^{-1} \mu, \\
\Omega_{K,M} &= \overline{\Omega} \times \{|z| \leq M\} \times \{\xi \leq K\}, \\
|f|_{\alpha,K,M} &= \sup_{\Omega_{K,M}} |f(x, z, \xi)| + \sup_{\Omega_{K,M} \times \Omega_{K,M}} \frac{|f(x, z, \xi) - f(x_0, z_0, \xi_0)|}{\rho(x, x_0)^\alpha + |z - z_0|^\alpha + K^{-\alpha} |\xi - \xi_0|^\alpha}.
\end{align*}
\]

Since the equation (1) is degenerate elliptic and subelliptic, we call equation (1) semilinear subelliptic. Using the properties of Hörmander’s operator $H$, we have proved the interior regularities for quasilinear second order subelliptic equation of form $\sum_{i,j=1}^{m} A_{ij}(x, u, X u)X_i X_j u + B(x, u, X u) = 0$, and the existence of weak solution for variational problems (see [6],[7],[8],[9],[10]).
2 Preliminary Lemmas And Notations

We define now the sub-unit metric on \( M \) associated with \( X \) as in [4] and [7].

**Definition 1** Let \( C(\delta) \) be a class of absolutely continuous mappings \( \phi : [0,1] \rightarrow M \) which almost everywhere satisfy the differential equation

\[
\phi'(t) = \sum |J| \leq r \ a_{J}(t)X_{J}(\phi(t))
\]

with \( |a_{J}(t)| < \delta |J| \), then we define

\[
\rho(x, y) = \inf \{ \delta > 0 \mid \exists \phi \in C(\delta) \text{ with } \phi(0) = x, \phi(1) = y \}.
\]

Then, \( \rho \) is a local metric on \( M \), and for any small compact subset \( K \subset M \), there exists a constant \( C > 0 \) such that

\[
C^{-1}|x - y| \leq \rho(x, y) \leq C|x - y|^{1/r}
\]

for any \( x, y \in K \).

We introduce now a class of “non-isotropic” Hölder continuous functions. For \( 1 > \alpha > 0 \), we define

\[
S^{\alpha}(\overline{\Omega}) = \left\{ f \in S^{0}(\overline{\Omega}); [f]_{\alpha, \overline{\Omega}}^{X} = \sup_{x,y \in \overline{\Omega}} \frac{|f(x) - f(y)|}{\rho(x, y)^{\alpha}} < +\infty \right\}
\]

and for \( k \in \mathbb{N} \), \( 1 > \alpha \geq 0 \), we define

\[
S^{k,\alpha}(\overline{\Omega}) = \{ u \in S^{\alpha}(\overline{\Omega}); X^{j}u \in S^{\alpha}(\overline{\Omega}), \forall |J| \leq k \}
\]

Set:

\[
[u]_{k,0,\overline{\Omega}}^{X} = \sup_{|J| = k} \sup_{x \in \overline{\Omega}} |X^{j}u(x)|
\]

and

\[
[u]_{k,\alpha,\overline{\Omega}}^{X} = \sup_{|J| = k} [X^{j}u(x)]_{k,\alpha,\overline{\Omega}}^{X}.
\]

The norms on \( S^{k,\alpha}(\overline{\Omega}) \) are given by

\[
\|u\|_{S^{k,\alpha}(\overline{\Omega})} = \sum_{j=0}^{k} [u]_{j,0,\overline{\Omega}}^{X} + [u]_{j,\alpha,\overline{\Omega}}^{X}.
\]

Then the norms of \( S^{k,\alpha}(\overline{\Omega}) \) is also convex, and \( S^{k,\alpha}(\overline{\Omega}) \) is a Banach space (see [7]). Hörmander’s condition implies that \( S^{k,\alpha}(\overline{\Omega}) \subset C^{k/r}(\overline{\Omega}) \) for all \( k \in \mathbb{N} \).

Using the hypotheses (S. E. \( \partial \Omega \)) on \( \partial \Omega \), we can also define the functions spaces \( S^{k,\alpha}(\partial \Omega) \) by the bases of \( X_{J}^{0} \) as in (9) (see [8], [10]).

As for the classical Hölder space, we also have the interpolation inequalities in the space \( S^{k,\alpha}(\Omega) \). For \( j + \beta < k + \alpha, j, k \in \mathbb{N}, 0 \leq \alpha, \beta \leq 1, u \in S^{k,\alpha}(\Omega) \), and any \( \varepsilon > 0 \), we have

\[
\|u\|_{S^{j,\beta}(\Omega)} \leq \varepsilon \|u\|_{S^{k,\alpha}(\Omega)} + C(\varepsilon, j, k, \Omega, r) \|u\|_{L^{\infty}(\Omega)}.
\]

In [8] and [10], we have proved a abstract existence results (see Theorem 2 of [8]). We rewrite in the form of this paper.
Theorem 2 Assume that the hypothesis \((S, E. \partial \Omega)\) is satisfied, and \(\phi \in S^{2,\alpha}(\partial \Omega)\) with \(0 < \alpha < 1\). If for some fixed \(0 < \beta < 1\), there exists a constant \(B\) such that for all solutions \(u \in S^{2,\alpha}(\Omega)\) of following Dirichlet problems \((0 \leq \sigma \leq 1)\):

\[
\begin{align*}
\sum_{j=1}^{m} X_j \ast X_j u + \sigma (\alpha u + f(x, u, Xu)) &= 0, \quad \text{in } \Omega, \\
u &= \alpha \phi, \quad \text{on } \partial \Omega.
\end{align*}
\]

we have a priori estimates

\[
\|u\|_{S^{1,\beta}(\Omega)} \leq B.
\]

Then the Dirichlet problem (1) has a solution in the class \(S^{2,\beta}(\Omega)\). Furthermore if \(\phi \in S^{k+2,\alpha}(\partial \Omega)\) with \(k \in \mathbb{N}\), then \(u \in S^{k+2,\beta}(\Omega)\).

Now we have transformed the nonlinear degenerate Dirichlet problems (1) to the the problem of construction of apriori estimates (13).

3 Schauder Estimates For The Hörmander Operators

We study in this section the following linear Dirichlet problem:

\[
Hu = f, \quad \text{in } \Omega; \quad u = \phi, \quad \text{on } \partial \Omega.
\]

with \(c(x) \geq c_0 > 0\). By [1], there exists Green’s kernel \(G(x, y)\) for the operators \(H\).

From [5] and [7] we have

Lemma 1 For \(n \geq 2, K \subset \subset \Omega, \) and \((x, y) \in K \times K\), we have

\[
|X^j G(x, y)| \leq C_J \rho(x, y)^{2-|J|} |B(x, \rho(x, y))|^{-1},
\]

where differential are taken in \(x\) or \(y\).

We shall use the inequality (15) to prove the Schauder estimate of Hörmander operators in the “non-isotropic” Hölder spaces \(S^{k,\alpha}\). Firstly, we have the maximum principle

Lemma 2 If \(u \in S^2(\Omega)\) is a solution of Dirichlet problem (14), \(c(x) \geq c_0 > 0\). Then we have

\[
\|u\|_{L^\infty(\Omega)} \leq c_0^{-1} \|f\|_{L^\infty(\Omega)}.
\]

If \(u \in S^2(\Omega), u \leq 0 \) on \(\partial \Omega\) verifies \(Hu \leq 0\) in \(\Omega\). Then \(u \leq 0\) in \(\Omega\)

This is just the results of J.-M. Bony [1]. We have also

Lemma 3 Let \(u \in S^{2,\alpha}(\Omega), u|_{\partial \Omega} = 0, \alpha > 0\), then there exists a constant \(C\) such that

\[
\|u\|_{S^{2,\alpha}(\Omega)} \leq C \|Hu\|_{S^{\alpha}(\Omega)}.
\]

The proof of this Lemma is in [10], so we have obtain Schauder type estimate in “non-isotropic” function spaces for degenerate elliptic operators. As in the elliptic case, we well use this Lemma to study nonlinear problems (1).
4 A Priori Estimate For Semilinear Equations

Using the maximum principle, we have the following comparison principle.

**Lemma 4** Let $u, v \in S^2(\bar{\Omega})$, $Lu \leq Lv$ in $\Omega$, $u \leq v$ on $\partial \Omega$. Under the assumption of Theorem 1, we have $u \leq v$ in $\Omega$.

**Proof:** Set $w = u - v$, then $w \leq 0$ on $\partial \Omega$ and

$$H(w) + (f(x,u,Xu) - f(x,v,Xv)) \leq 0$$

Remark that

$$f(x,u(x),Xu(x)) - f(x,v(Xv(x)) = f(x,u(x),Xu(x)) - f(x,u(x),Xv(x)) + f(x,u(x),Xv(x)) - f(x,v(x),Xv(x)) = \sum_{j=1}^{m} \partial_{x_j} f(x,u(Xu(x)))X_jw(x) + \partial_z f(x,u(x),Xv(x))w(x).$$

and $\partial_z f(x,u(x),Xv(x)) \geq 0$. We have

$$\sum_{j=1}^{m} (X_j^*X_jw + b_jX_jw) + \bar{c}w \leq 0; \quad w|_{\partial \Omega} \leq 0.$$  

with $\bar{c}(x) = c(x) - \partial_{x_i} f(x,u(x)) \geq c_0 > 0$. and $b_j, c \in S^{1,\alpha}(\bar{\Omega})$. Then Lemma 2 implies that $w \leq 0$ in $\Omega$.

Using this Lemma, we get a priori estimates of $\|u\|_{L^\infty}$.

**Theorem 3** Under the assumptions of Theorem 1, if $u \in S^2(\bar{\Omega})$, $Lu = 0$ in $\Omega$, then

$$\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u| + c_0^{-1}|\mu|.$$  

**Proof:** Set

$$v(x) = \sup_{\partial \Omega} u^+ + c_0^{-1}|\mu|,$$

Since $u \leq v$ on $\partial \Omega$, $v \geq 0$ in $\Omega$, then

$$f(x,v,Xv) = f(x,v,0) \geq \mu.$$  

using the comparison principle, we have

$$Lv = c(x)v(x) + f(x,v,0) \geq c(x)\mu + \mu \geq 0 = Lu, \text{ in } \Omega,$$

by Lemma 2, we have proved $u \leq \sup_{\partial \Omega} u^+ + c_0^{-1}|\mu|$ in $\Omega$. In the other hand, set

$$v_1(x) = \inf_{\partial \Omega} u^- - c_0^{-1}|\mu|,$$
Since $u \geq v_1$ on $\partial \Omega$, $v_1 \leq 0$ in $\Omega$, then
\[ f(x, v_1, Xv_1) = f(x, v_1, 0) \leq -\mu. \]

using the comparison principle, we have
\[ Lv_1 = c(x)v(x)_1 + f(x, v_1, 0) \leq -c(x)c_0|\mu| - \mu \leq 0 = Lu, \text{ in } \Omega, \]
by Lemma 2, we have proved $u \geq \inf_{\partial \Omega} u^0 = \frac{c_0}{c_1}|\mu|$ in $\Omega$. Which prove the Theorem 3.

**Theorem 4** Let $u \in S^{2,\alpha}(\Omega)$, $1 > \alpha > 0$ be a solution of Dirichlet problem (12). Under the assumption of Theorem 1, we have
\[ \|u\|_{S^{2,\beta}(\Omega)} \leq B < +\infty, \]
with $\beta = \min\{\frac{\alpha}{2\theta-\alpha}, \frac{\theta-\alpha}{\alpha}\}$, $B = B(n, m, r, \alpha, \theta, c_0, \mu)$.

**Proof:** Set $K = \max\{1, |Xu|_{0,\Omega}\}$, $K_{\nu} = \|Xu\|_{S^\nu(\Omega)}$ for $u \in [0, 1]$, $f(x) = f(x, u(x), Xu(x))$. Since $\|u\|_{L^\infty} \leq M_0$, we have
\[ \|f\|_{S^{2,\alpha}(\Omega)} = \|f\|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{\rho(x,y)^{\alpha\nu}} \leq C(K^\theta + 1)\left(1 + K^{\alpha\nu} + K_{\nu} - \alpha K_{\nu}^\alpha\right) \leq C_1(K^{\theta+\alpha\nu} + K^{\theta-\alpha}K_{\nu}^\alpha). \]

Using the Schauder’s estimate (17) of linear Dirichlet problems
\[ Hu = -\bar{f}, \text{ in } \Omega, \quad u = \phi, \text{ on } \partial \Omega. \]
We have
\[ \|u\|_{S^{2,\alpha}(\Omega)} \leq C\left\{\|u\|_{L^\infty(\Omega)} + \|\phi\|_{S^{2,\alpha}(\partial \Omega)} + \|f\|_{S^{\alpha\nu}(\Omega)}\right\} \leq C\left\{M_1 + \|\phi\|_{S^{\nu}(\Omega)}\right\} \leq C_2\left\{K^{\theta+\alpha\nu} + K^{\theta-\alpha}K_{\nu}^\alpha\right\}. \]

We need now the following precise interpolation inequality:
\[ \|u\|_{S^{2,\alpha}(\Omega)} \leq \epsilon\|u\|_{S^{2,\alpha}(\Omega)} + C_\alpha \epsilon^{-2/\alpha}\|u\|_{L^\infty(\Omega)}; \]
\[ \|u\|_{S^{1,\beta}(\Omega)} \leq \epsilon\|u\|_{S^{1,\beta}(\Omega)} + C_\beta \epsilon^{-(1+\beta)/(1-\beta)}\|u\|_{L^\infty(\Omega)}; \]
\[ \|u\|_{S^1(\Omega)} \leq \epsilon\|u\|_{S^1(\Omega)} + C\epsilon^{-1}\|u\|_{L^\infty(\Omega)}, \]
for any $\alpha, \beta, \epsilon \in [0, 1]$. \[ 6 \]
Taking $\varepsilon = \frac{1}{C^2}$ in (20), we have
\[ \|u\|_{S^2(\Omega)} \leq K^{\theta+\alpha \nu} + K^{\theta-\alpha} K_{\nu}^\alpha + C(\nu) M_0, \]
and take $\varepsilon = K^{\alpha-\theta}/2$ in (21), then
\[ \|u\|_{S^1(\nu:\nu)(\Omega)} \leq \frac{1}{2} K^{\alpha+\alpha \nu} + \frac{1}{2} K_{\nu}^\alpha + C(\nu, M_0) K^{(\theta-\alpha)(1+\nu)/(1-\nu)}. \]
Take $\beta = \nu$, using $K_{\beta} \geq 1$, we obtain
\[ \|u\|_{S^1(\beta)(\Omega)} \leq C_3 K^\theta. \]
(23)
Which implies that
\[ \|f\|_{S^{\alpha\beta}(\Omega)} \leq C K^{\theta+\alpha\theta-\alpha} \leq C K^{2\theta}. \]
Hence, for all $\gamma, \varepsilon \in [0, 1]$, we have
\[ \|u\|_{S^1(\Omega)} \leq \varepsilon\|u\|_{S^1(\Omega)} + C \varepsilon^{-1} M_0 \leq \varepsilon (M_1 + \|f\|_{S^{\alpha\beta}(\Omega)}) + C \varepsilon^{-1} M_0 \leq \varepsilon\|f\|_{S^{\alpha\beta}(\Omega)} + C(M_0) \varepsilon^{-1}. \]
Now convexity of norms in $S^\alpha(\Omega)$ give that
\[ \|f\|_{S^{\alpha\beta\gamma}(\Omega)} \leq 4 \left( \|f\|_{S^\alpha(\Omega)} \right)^\gamma \left( \|f\|_{L^\infty(\Omega)} \right) \leq C K^{2\theta+\gamma} K^{\theta(1-\gamma)} = C K^{\theta(1+\gamma)}. \]
Take $\gamma = \frac{2-\theta}{2\theta}$, we have
\[ K \leq \|u\|_{S^1(\Omega)} \leq C(\theta, M_0) \varepsilon^{-1} + \varepsilon C_4 K^{1+\frac{\theta}{2}}. \]
Since $\frac{\theta}{2} < 1$, let $\varepsilon = \min\{\frac{1}{2}, (2C_4 K^{\theta/2})^{-1}\}$, we have
\[ K \leq C_5, \]
where $C_5$ is independent on $u$. So we have proved Theorem 3 by use (23).

**End of proof of Theorem 1**

The uniqueness of solution of the Dirichlet problem (1) is immediate from the comparison principle Lemma 4. The existence of solution give by abstract Theorem 2 and a priori estimates Theorem 3.
References


